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MONOTONICITY PROPERTIES OF OSCILLATORY SOLUTIONS OF DIFFERENTIAL EQUATION $(a(t)|y'|^{p-1}y')' + f(t,y,y') = 0$

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Abstract. We obtain monotonicity results concerning the oscillatory solutions of the differential equation $(a(t)|y'|^{p-1}y')' + f(t,y,y') = 0$. The obtained results generalize the results given by the first author in [1] (1976). We also give some results concerning a special case of the above differential equation.

1. Introduction

In this paper, we consider the differential equation

$$\big(a(t)|y'|^{p-1}y'\big)' + f(t,y,y') = 0 \tag{1.1}$$

where $p > 0$, a is positive continuous function on $J = [\bar{a},\infty) \subset \mathbb{R}_+ = [0,\infty)$, the function f is continuous on $D = \{(t,u,v) : t \in J, -\infty < u, v < \infty\}$ and

$$f(t,u,v)\,u > 0 \quad \text{for} \quad u \neq 0\,.$$

A function $y\colon [a_y, b_y) \to \mathbb{R} = (-\infty,\infty)$ is called a solution of (1.1) if $I = [a_y, b_y) \subset J$, $y \in C^1(I)$, $a|y'|^{p-1}y' \in C^1(I)$ and (1.1) is valid on I. A solution y is oscillatory if there exists an increasing sequence $\{t_k\}_{k=1}^\infty$ of zeros of y such that $a_y \le t_k < b_y$, $k = 1,2,\dots$, $\lim\limits_{k\to\infty} t_k = b_y$ and y is nontrivial in any left neighbourhood of b_y.

Note that solutions of (1.1) will be sometimes studied on subintervals of their maximal definition intervals.

We study solutions of (1.1) also on finite intervals since (1.1) may have solutions defined on such intervals that cannot be defined on J (so called noncontinuable solutions, singular solutions of the 2-nd kind, see e.g. [5], [7], [10], [12] and the references therein).

The structure of zeros of a solution of (1.1) can be complicated, see [4].

Let $z\colon [a_z, b_z) \subset J \to \mathbb{R}$ be a continuous function. According to [1] a point $C \in [a_z, b_z)$ is called an H-point of z if there exist sequences $\{\tau_k\}_{k=1}^\infty$ and $\{\bar{\tau}_k\}_{k=1}^\infty$

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of numbers of $[a_z, b_z)$ tending to C such that

$$z(\tau_k) = 0\,, \quad z(\bar{\tau}_k) \neq 0\,, \quad (\tau_k - C)(\bar{\tau}_k - C) > 0\,, \qquad k = 1, 2, \dots\,.$$

Clearly, if C is an H-point of z, then $z(C) = z'(C) = 0$.

Denote by $\mathcal{O}$ the set of oscillatory solutions of (1.1) defined on $[a_y, b_y) \subset J$ that have no H-points in $[a_y, b_y)$.

The aim of the paper is to study monotonicity properties of oscillatory solutions in the set $\mathcal{O}$.

Note, that H-points do not exist if there exists $\varepsilon > 0$ such that

$$\big|f(t, u, v)\big| \le r(t)\big(|u| + |v|\big) \quad \text{for} \quad t \in J, |u| \le \varepsilon,\ |v| \le \varepsilon$$

where $r \in C^0(J)$, see [10]. On the other hand there exists an equation of the form (1.1) that has a solution y with infinitely many H-points tending to ∞, see [3]. Thus, this solution defined in a neighbourhood of ∞ does not belong to $\mathcal{O}$; but the restriction of y to a suitable bounded definition interval belongs to $\mathcal{O}$.

Lemma 1.1. *Let $y \in \mathcal{O}$ be defined on $[a_y, b_y)$. Then there are sequences $\{t_k\}_{k=1}^{\infty}$ and $\{\tau_k\}_{k=1}^{\infty}$ such that $a_y \le t_k < \tau_k < t_{k+1} < b_y$, $y(t_k) = 0$, $y'(\tau_k) = 0$, $y(t) \neq 0$ if $t \neq t_k$, $y'(t) \neq 0$ if $t \ge t_1$ and $t \neq \tau_k$, $k = 1, 2, \dots$. Moreover,*

$$f\big(t, y(t), y'(t)\big)\ y'(t) > 0 \quad \textit{for} \quad t \in (t_k, \tau_k)\,,$$

$$f\big(t, y(t), y'(t)\big)\ y'(t) < 0 \quad \textit{for} \quad t \in (\tau_k, t_{k+1})\,.$$

Proof. See [4, Theorem 2] and its proof. □

Note, that according to (1.1), $\{t_k\}_{k=1}^{\infty}$ ($\{\tau_k\}_{k=1}^{\infty}$) is the sequence of all extremants of $a|y'|^{p-1}\,y'$ on $[t_1, b_y)$ (of y on $[t_1, b_y)$); zeros t_k and τ_k are simple and isolated.

A special case of (1.1) is the differential equation with p-Laplacian

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)f(y) = 0 \tag{1.2}$$

where $r(t)$ is a positive continuous on $\mathbb{R}_+$, f is continuous on $\mathbb{R}$ and $f(x)x > 0$ for $x \neq 0$. The study of oscillatory solutions of the differential equation (1.2) is an interest subject of many papers also in our days, see e.g. [4, 5, 6, 9, 12, 13, 14].

In [1] and [2] (see also references therein), some monotonicity results concerning the oscillatory solutions of the differential equation

$$y'' + f(t, y, y') = 0 \tag{1.3}$$

have been proved. Sufficient conditions for the monotonicity of the sequences $\big\{|y(\tau_k)|\big\}_{k=1}^{\infty}$ and $\big\{|y'(t_k)|\big\}_{k=1}^{\infty}$ for a solution y of (1.3) are derived. This problem has a long history which was initiated by P. Hartman, L. Lorch and M. Muldon for Bessel functions and higher monotonicity problem for second order linear equation, see [8, 11].

In [14], the existence of an oscillatory solution with decreasing amplitudes for (1.2) with $p = 1$ is proved.

In Section 2 we give analogous monotonicity properties of oscillatory solutions belonging to $\mathcal{O}$. Our results generalize the ones e.g. of [1] (see also references therein) and [8, 11] for (1.3). The obtained results coincide with the known results of [1] for $p = 1$ and $a(t) \equiv 1$. The same problem is solved for (1.2) in Section 3.

Notation. Let $D_1 = \big\{(t,u,v) : (t,u,v) \in D, v \geq 0\big\}$, $D_2 = \big\{(t,u,v) : (t,u,v) \in D, u \geq 0\big\}$. For a solution y of (1.1), we introduce the quasiderivation of y by

$$y^{[1]}(t) = a(t)|y'(t)|^{p-1}y'(t)\,.$$

2. Main results

Theorem 2.2. *Let $f(t,u,v) = f(t,u,-v)$ in D, $\big|f(t,u,v)\big|$ be non-increasing with respect to t in D and non-increasing with respect to v in D_1, and a be non-increasing on J. Let $y \in \mathcal{O}$ be defined on $[a_y, b_y)$ and let $\{t_k\}_{k=1}^{\infty}$ ($\{\tau_k\}_{k=1}^{\infty}$) be the sequence of all zeros of y (of y') given by Lemma 1.1. Let $k \in \mathbb{N}$, $z \in \big[0, |y(\tau_k)|\big]$ and let $s_1 \in [t_k, \tau_k]$ and $s_2 \in \big[\tau_k, t_{k+1}\big]$ be such that*

$$\big|y(s_1)\big| = \big|y(s_2)\big| = z\,.$$

Then

$$|y^{[1]}(s_1)| \geq |y^{[1]}(s_2)|\,. \tag{2.1}$$

Hence, the sequence $\{|y^{[1]}(t_k)|\}_{k=1}^{\infty}$ is non-increasing. If, moreover, $a \equiv 1$, then

$$\tau_k - t_k \leq t_{k+1} - \tau_k\,. \tag{2.2}$$

Proof. First, we multiply equation (1.1) by $-\frac{p+1}{p}(a(t)|y'|^p)^{1/p}$, so we obtain

$$-\frac{p+1}{p}\big(a(t)|y'|^p\big)^{1/p}\big(a(t)|y'|^{p-1}y'\big)' = \frac{p+1}{p}\big(a(t)|y'|^p\big)^{1/p} f(t,y,y')$$

and hence

$$-\big((a(t)|y'|^p)^{(p+1)/p}\big)' = \frac{p+1}{p}\big(a(t)|y'|^p\big)^{1/p} f(t,y,y')\operatorname{sgn} y'\,. \tag{2.3}$$

Let $k \in \mathbb{N}$. Then we integrate (2.3) from t to τ_k and obtain

$$\big(a(t)|y'(t)|^p\big)^{(p+1)/p} = \frac{p+1}{p}\int_t^{\tau_k} a^{1/p}(s)y'(s)f(s,y(s),y'(s))\,ds \tag{2.4}$$

for $t \in [t_k, t_{k+1}]$. Let $y(t)$ be positive on (t_k, t_{k+1}). In the case that $y(t)$ is negative, the proof is similar. Then, the function $f(t,y,y')$ will be positive on the same interval. Also the derivative of y will be positive for $t \in [t_k, \tau_k)$ and negative for $t \in (\tau_k, t_{k+1}]$, see Lemma 1.1. Since $y'(t) \neq 0$ for $t \neq \tau_k$ and $t \in [t_k, t_{k+1}]$ there exists the inverse function of $y(t)$ in each subintervals of $[t_k, t_{k+1}]$, so we denote by $s_1(y)$ the inverse function to $y(t)$ for $t \in [t_k, \tau_k]$ and $s_2(y)$ the inverse function to $y(t)$ for $t \in [\tau_k, t_{k+1}]$.

The equation (2.4) can be rewritten into the form

$$\big(a(s_i)|y'(s_i)|^p\big)^{(p+1)/p} = \frac{p+1}{p}\int_y^{y(\tau_k)} a^{1/p}(s_i(z))f(s_i(z),y(z),y'(s_i(z)))dz$$

for $y \in [0, y(\tau_k)]$, $i = 1,2$. From this

$$
\begin{aligned}
&\frac{d}{dy}\Big\{\big(a(s_1)|y'(s_1)|^p\big)^{(p+1)/p} - \big(a(s_2)|y'(s_2)|^p\big)^{(p+1)/p}\Big\} \\
&\quad = -\frac{p+1}{p}\big\{a^{1/p}(s_1)f(s_1,y,y'(s_1)) - a^{1/p}(s_2)f(s_2,y,y'(s_2))\big\} \\
&\quad = -\frac{p+1}{p}\big\{a^{1/p}(s_1)[f(s_1,y,y'(s_1)) - f(s_2,y,y'(s_1))] \\
&\qquad + [a^{1/p}(s_1) - a^{1/p}(s_2)]f(s_2,y,y'(s_1)) \\
&\qquad + a^{1/p}(s_2)[f(s_2,y,y'(s_1)) - f(s_2,y,y'(s_2))]\big\} \\
&\quad \le -\frac{p+1}{p}a^{1/p}(s_2)\big\{f(s_2,y,|y'(s_1)|) - f(s_2,y,|y'(s_2)|)\big\}\,.
\end{aligned}
\tag{2.5}
$$

As $a(s_1)\big|y'(s_1)\big|^p \le a(s_2)\big|y'(s_2)\big|^p$ implies $\big|y'(s_1)\big| \le \big|y'(s_2)\big|$ it follows from (2.5)

$$
\begin{aligned}
&a(s_1)\big|y'(s_1)\big|^p - a(s_2)\big|y'(s_2)\big|^p \le 0 \\
&\qquad\qquad \Rightarrow \frac{d}{dy}\big\{a(s_1)\big|y'(s_1)\big|^p - a(s_2)\big|y'(s_2)\big|^p\big\} \le 0\,.
\end{aligned}
\tag{2.6}
$$

We assume, contrarily, that there exists a number $y_1 \in \big[0, y(\tau_k)\big]$ such that for $y = y_1$

$$a(s_1)\big|y'(s_1)\big|^p < a(s_2)\big|y'(s_2)\big|^p\,.$$

From this and from (2.6) we have

$$a(s_1)\big|y'(s_1)\big|^p < a(s_2)\big|y'(s_2)\big|^p$$

for $y \ge y_1$. But it is a contradiction because for $y = y(\tau_k)$ we have

$$a(s_1(y))\big|y'(s_1(y))\big|^p - a(s_2(y))\big|y'(s_2(y))\big|^p = 0\,.$$

So, finally, the inequality

$$a(s_1)|y'(s_1)|^p \ge a(s_2)|y'(s_2)|^p\,, \quad y \in [0, y(\tau_k)]$$

holds and the desired result (2.1) is obtained.

Let $a \equiv 1$, $f_1(y) = \tau_k - s_1(y) \ge 0$ and $f_2(y) = s_2(y) - \tau_k \ge 0$ for $y \in \big[0, y(\tau_k)\big]$. Then it follows from the proved part (2.1) of the theorem that

$$\frac{d}{dy}\big[f_1(y) - f_2(y)\big] = -\frac{1}{y'(s_1(y))} - \frac{1}{y'(s_2(y))} \ge 0$$

for $y \in \big[0, y(\tau_k)\big]$. Hence $f_1 - f_2$ is nondecreasing and with regard to $f_1(y) = f_2(y) = 0$ for $y = y(\tau_k)$ we can conclude $f_1 \le f_2$, i.e.

$$\tau_k - s_1(y) \le s_2(y) - \tau_k\,, \qquad y \in \big[0, y(\tau_k)\big]\,.$$

The statement (2.2) now follows from the last formula where we substitute $y = y(\tau_k)$. □

Theorem 2.3. *Let $f(t,u,v)=f(t,u,-v)$ in D, $\big|f(t,u,v)\big|$ be non-decreasing with respect to t in D and non-decreasing with respect to v in D_1, and a be non-decreasing on J. Let $y\in\mathcal{O}$ be defined on $[a_y,b_y)$ and let $\{t_k\}_{k=1}^{\infty}$ $(\{\tau_k\}_{k=1}^{\infty})$ be the sequence of all zeros of y (of y') given by Lemma 1.1. Let $k\in\mathbb{N}$, $z\in\big[0,|y(\tau_k)|\big]$ and let $s_1\in[t_k,\tau_k]$ and $s_2\in\big[\tau_k,t_{k+1}\big]$ be such that*

$$\big|y(s_1)\big|=\big|y(s_2)\big|=z\,.$$

Then

$$|y^{[1]}(s_1)|\le|y^{[1]}(s_2)|\,.$$

Hence, the sequence $\big\{|y^{[1]}(t_k)|\big\}_{k=1}^{\infty}$ is nondecreasing. If, moreover, $a\equiv1$, then

$$\tau_k-t_k\ge t_{k+1}-\tau_k\,.$$

Proof. The proof is analogous as in Theorem 2.2. □

Theorem 2.4. *Let $f(t,u,v)=-f(t,-u,v)$ in D, $f(t,u,v)$ be non-increasing with respect to t in D_2, $f(t,u,v)$ be non-decreasing (non-increasing) with respect to v in D_2, $v\ge0$ (in D_2, $v\le0$), and a be non-increasing on J. Let $y\in\mathcal{O}$ be defined on $[a_y,b_y)$ and let $\{t_k\}_{k=1}^{\infty}$ $(\{\tau_k\}_{k=1}^{\infty})$ be the sequence of all zeros of y (of y') given by Lemma 1.1. Let $k\in\mathbb{N}$, $z\in\big[0,|y(\tau_k)|\big]$, $s_1\in[\tau_k,t_{k+1}]$, and $s_2\in\big[t_{k+1},\tau_{k+1}\big]$ be such that*

$$\big|y(s_1)\big|=\big|y(s_2)\big|=z\,.$$

Then the inequality

$$|y^{[1]}(s_1)|\le|y^{[1]}(s_2)|$$

holds, and the sequence $\{|y(\tau_k)|\}_{k=1}^{\infty}$ is non-decreasing.

Proof. We integrate (2.3) from t to t_{k+1} and we obtain

$$\tag{2.7} \begin{aligned}\big(a(t)|y'(t)|^p\big)^{(p+1)/p}-\big(a(t_{k+1})|y'(t_{k+1})|^p\big)^{(p+1)/p}\\ =\frac{p+1}{p}\int_t^{t_{k+1}}a^{1/p}(s)y'(s)f(s,y(s),y'(s))\,ds\,,\end{aligned}$$

for $t\in[\tau_k,\tau_{k+1}]$. Let $s_1(y)$ be the inverse function to $y(t)$ for $t\in[\tau_k,t_{k+1}]$ and let $s_2(y)$ be the inverse function to $y(t)$ for $t\in[t_{k+1},\tau_{k+1}]$. The equation (2.7) can be rewritten by

$$\tag{2.8} \begin{aligned}\big(a(s_i)|y'(s_i)|^p\big)^{(p+1)/p}-\big(a(t_{k+1})|y'(t_{k+1})|^p\big)^{(p+1)/p}\\ =-\frac{p+1}{p}\int_0^{y}a^{1/p}(s_i(z))f(s_i(z),z,y'(s_i(z)))\,dz\end{aligned}$$

for $i=1,2$ and $|y|\in\big[0,\min\big(|y(\tau_k)|,|y(\tau_{k+1})|\big)\big]=I$. Differentiating equation (2.8) we can obtain for $|y|\in I$

$$\tag{2.9} \begin{aligned}\frac{d}{d|y|}\big\{\big(a(s_1)|y'(s_1)|^p\big)^{(p+1)/p}-\big(a(s_2)|y'(s_2)|^p\big)^{(p+1)/p}\big\}\\ =-\frac{p+1}{p}\big\{a^{1/p}(s_1)f(s_1,|y|,y'(s_1))-a^{1/p}(s_2)f(s_2,|y|,y'(s_2))\big\}\,.\end{aligned}$$

Following the same procedure as in the proof of Theorem 2.2, we obtain

$$\begin{aligned}(2.10)\quad &\frac{d}{d|y|}\big\{\big(a(s_1)|y'(s_1)|^p\big)^{(p+1)/p} - \big(a(s_2)|y'(s_2)|^p\big)^{(p+1)/p}\big\}\mathit{multline}\\ &\qquad\le -\frac{p+1}{p}a^{1/p}(s_2)\big\{f(s_2,|y|,y'(s_1)) - f(s_2,|y|,y'(s_2))\big\}.\end{aligned}$$

For $y=0$ equation (2.8) gives

$$a(s_1)|y'(s_1)|^p = a(s_2)|y'(s_2)|^p = a(t_{k+1})|y'(t_{k+1})|^p\,. \tag{2.11}$$

Assume that there exists $y_1 \in I$, $y_1 \neq 0$, such that

$$a(s_1)|y'(s_1)|^p > a(s_2)|y'(s_2)|^p \tag{2.12}$$

for $|y| = y_1$. Then according to (2.9) and (2.11) there exists an interval $I_1 = (\bar z, y_1]$, $\bar z \ge 0$, such that the inequality (2.12) holds on I_1 and for $|y| = \bar z$

$$a(s_1)|y'(s_1)|^p = a(s_2)|y'(s_2)|^p.$$

This means that there exists a number $\xi \in I_1$ such that

$$\frac{d}{d|y|}\big\{\big(a(s_1)|y'(s_1)|^p\big)^{(p+1)/p} - \big(a(s_2)|y'(s_2)|^p\big)^{(p+1)/p}\big\}_{|y|=\xi} > 0$$

which is not correct because of (2.10) and (2.12). Thus

$$a(s_1)|y'(s_1)|^p \le a(s_2)|y'(s_2)|^p, \quad |y| \in I\,. \tag{2.13}$$

Suppose that $|y(\tau_k)| > |y(\tau_{k+1})|$. Then $I = [0, |y(\tau_{k+1})|]$ and for $|y| = |y(\tau_{k+1})|$ we have $|y'(s_1)| > 0$ and $|y'(s_2)| = 0$, so for $|y| = |y(\tau_{k+1})|$ we obtain $a(s_1)|y'(s_1)|^p > a(s_2)|y'(s_2)|^p$ which is not valid because of (2.13). So it is proved that $|y(\tau_k)| \le |y(\tau_{k+1})|$ for $k = 1, 2, \dots$. □

Theorem 2.5. *Let $f(t,u,v) = -f(t,-u,v)$ in D, $f(t,u,v)$ be non-decreasing with respect to t in D_2, $f(t,u,v)$ be non-increasing (non-decreasing) with respect to v in D_2, $v \ge 0$ (in D_2, $v \le 0$), and a be non-decreasing on J. Let $y \in \mathcal{O}$ be defined on $[a_y, b_y)$ and let $\{t_k\}_{k=1}^\infty$ ($\{\tau_k\}_{k=1}^\infty$) be the sequence of all zeros of y (of y') given by Lemma 1.1. Let $k \in \mathbb{N}$, $z \in \big[0, |y(\tau_{k+1})|\big]$, $s_1 \in [\tau_k, t_{k+1}]$, and $s_2 \in \big[t_{k+1}, \tau_{k+1}\big]$ be such that*

$$\big|y(s_1)\big| = \big|y(s_2)\big| = z\,.$$

Then the inequality

$$|y^{[1]}(s_1)| \ge |y^{[1]}(s_2)|$$

holds and the sequence $\{|y(\tau_k)|\}_{k=1}^\infty$ is non-increasing.

Proof. The proof is analogous as that of Theorem 2.4. □

3. Special case

Concerning equation (1.2), the results of Section 2 can be proved under weaker assumptions on $a(t)$ and $r(t)$. We define an auxiliary function

$$R(t) = a^{1/p}(t)r(t)\,, \quad t \in J\,.$$

Theorem 3.6. *Let $R \in C^1(J)$, y be an oscillatory solution of (1.2) on J and $f(y)$ a continuous odd function on $\mathbb{R}$. Then $y \in \mathcal{O}$. If, moreover, the function R is non-decreasing (non-increasing) and $\{\tau_k\}_{k=1}^{\infty}$ is the sequence of all extremants of y given by Lemma 1.1, then the sequence $\{|y(\tau_k)|\}_{k=1}^{\infty}$ is non-increasing (non-decreasing).*

Proof. Since $R \in C^1(J)$, according to [5, Theorems 2 and 3], every solutions y of (1.2) can be defined on $\mathbb{R}_+$ and it has no H-points, so $y \in \mathcal{O}$. Let y be a solution of (1.2) on J and consider

$$Y(t) = a(t)|y'(t)|^p = |y^{[1]}(t)| \tag{3.1}$$

and

$$Z(t) = \frac{Y^{(p+1)/p}(t)}{R(t)} + \frac{p+1}{p}\int_0^{y(t)} f(s)\,ds\,. \tag{3.2}$$

Then

$$Z'(t) = -\frac{R'(t)}{R^2(t)}Y^{(p+1)/p}(t)\,. \tag{3.3}$$

For the zeros τ_k, $k = 1, 2, \dots$ of y' we get from (3.1) $Y(\tau_k) = 0$, thus according to (3.2) and the fact that f is odd

$$Z(\tau_k) = \frac{p+1}{p}\int_0^{y(\tau_k)} f(s)ds = \frac{p+1}{p}\int_0^{|y(\tau_k)|} f(s)\,ds \tag{3.4}$$

and so

$$Z(\tau_{k+1}) = \frac{p+1}{p}\int_0^{|y(\tau_{k+1})|} f(s)\,ds\,. \tag{3.5}$$

Since $R(t)$ is non-decreasing (non-increasing), it follows from (3.3) $Z'(t) \le 0$ ($Z'(t) \ge 0$), so the function $Z(t)$ is non-increasing (non-decreasing), thus combining (3.4) and (3.5) we obtain the desired result. □

Theorem 3.7. *Let $R \in C^1(J)$, y be an oscillatory solution of (1.2) on J and $f(y)$ be a continuous function on $\mathbb{R}$. Then $y \in \mathcal{O}$. If, moreover, the function $R(t)$ is non-decreasing (non-increasing) and $\{t_k\}_{k=1}^{\infty}$ is the sequence of all extremants of $y^{[1]}$ given by Lemma 1.1, then the sequence $\{|y^{[1]}(t_k)|\}_{k=1}^{\infty}$ is non-decreasing (non-increasing).*

Proof. Let y be a solution of (1.2) on $\mathbb{R}_+$. Similarly to the proof of Theorem 3.6, $y \in \mathcal{O}$.

Now we consider the function $Y(t)$ given by (3.1) and the function

$$Z_1(t) = Y^{(p+1)/p}(t) + \frac{p+1}{p} R(t) \int_0^{y(t)} f(s)\, ds\,. \tag{3.6}$$

It is obvious that

$$Z_1'(t) = \frac{p+1}{p} R'(t) \int_0^{y(t)} f(s)\, ds$$

and according to $f(x)x > 0$ for $x \neq 0$ the functions $R(t)$ and $Z_1(t)$ have the same kind of monotonicity. From (3.6) for $t = t_k$ and $t = t_{k+1}$ and taking in account the assumptions of the theorem, we obtain the desired result. $\square$

Remark 3.8. Functions (3.2) and (3.6) are defined and investigated in [5].

4. Application

We apply our results to the quasilinear equation

$$\left(|y'|^{p-1}y'\right)' + r(t)|y|^{\lambda-1}y = 0 \tag{4.1}$$

where $p > 0$, $\lambda > 0$, $r > 0$ and r is a positive continuous function on $\mathbb{R}_+$.

Corollary 4.9. *Let y be an oscillatory solution of* (4.1) *defined on $I = [a_y, \infty) \subset \mathbb{R}_+$ that has no H-points in I. Denote by $\{t_k\}_{k=1}^{\infty}$ $(\{\tau_k\}_{k=1}^{\infty})$ all extremants of y' (of y) on I (on $[t_1, \infty)$).*

(i) *If r is non-increasing on I, then $\left\{|y(\tau_k)|\right\}_{k=1}^{\infty}$ is non-decreasing and $\left\{|y'(t_k)|\right\}_{k=1}^{\infty}$ is non-increasing.*

(ii) *If r is non-decreasing on I, then $\left\{|y(\tau_k)|\right\}_{k=1}^{\infty}$ is non-increasing and $\left\{|y'(t_k)|\right\}_{k=1}^{\infty}$ is non-decreasing.*

(iii) *If $r \in C^1(\mathbb{R}_+)$, then y is defined on $\mathbb{R}_+$ and it has no H-points.*

Proof. Case (i) $\big($(ii)$\big)$ follows from Theorems 2.2 and 2.4 (from Theorems 2.3 and 2.5). Case (iii) follows from [5, Theorems 2 and 3]. $\square$

Remark 4.10. Let $r \in C^1(\mathbb{R}_+)$. Then all solutions of (4.1) are oscillatory if and only if

$$\int_0^{\infty} t^{\lambda} r(t)\, dt = \infty \quad \text{in case} \quad \lambda < p$$

and

$$\int_0^{\infty} \left(\int_t^{\infty} r(s)\, ds \right)^{\frac{1}{p}} dt = \infty \quad \text{in case} \quad \lambda > p\,,$$

see [5, Theorem 2] and [12, Theorems 6.1, 11.3 and 11.4].

Example. Consider (4.1) with $p = 1$, $\lambda = 1$ and $r(t) \equiv C > 0$ on $\mathbb{R}_+$. Then the sequences $\{|y(\tau_k)|\}_{k=1}^{\infty}$ and $\{|y'(t_k)|\}_{k=1}^{\infty}$ are constant. This result was proved in [11, Lemma 1].

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