ARCHIVUM MATHEMATICUM (BRNO)
Tomus 49 (2013), 275–293

JORDAN- AND LIE GEOMETRIES

Wolfgang Bertram

Abstract. In these lecture notes we report on research aiming at understanding the relation beween *algebras* and *geometries*, by focusing on the classes of *Jordan algebraic* and of *associative structures* and comparing them with *Lie structures*. The geometric object sought for, called a *generalized projective*, resp. an *associative geometry*, can be seen as a combination of the structure of a *symmetric space*, resp. of a *Lie group*, with the one of a *projective geometry*. The text is designed for readers having basic knowledge of Lie theory – we give complete definitions and explain the results by presenting examples, such as *Grassmannian geometries*.

Introduction

The following question will serve as guideline for this series of lectures: *is there a correspondence between classes of algebraic structures and classes of geometric structures?* At present, there is no general theory answering this question. But there are two "archetypical" examples:

(A) *The correspondence between (certain) associative commutative algebras and (certain) manifolds or varieties.* This correspondence is of basic importance for *algebraic geometry*; it does not, in this form, generalize to arbitrary associative algebras – the corresponding geometry, called *non-commutative geometry*, is not a geometry in the usual sense (cf. [20]).

(B) *The Lie functor.* This correspondence comes in two versions:

(B1) Lie algebras correspond to Lie groups,

(B2) Lie triple systems correspond to symmetric spaces.

The principles underlying (A), respectively (B), are quite different from each other. The following example is closer in the spirit to (B) than to (A):

The Jordan functors. Recall that Lie algebras come from skew-symmetrizing associative products via $[X,Y] = XY - YX$. In a similar way, *Jordan algebras*

2010 *Mathematics Subject Classification*: primary 16W10; secondary 17C37, 20N10, 22A30, 51B25, 51P05, 81P05.

Key words and phrases: Jordan algebra (triple system, pair), associative algebra (triple systems, pair), Lie algebra (triple system), graded Lie algebra, symmetric space, torsor (heap, groud, principal homogeneous space), homotopy and isotopy, Grassmannian, generalized projective geometry.

DOI: 10.5817/AM2013-5-275

come from symmetrizing them:

$$X \bullet Y = (XY + YX)/2 . \tag{1}$$

During the second half of the 20th century, it turned out that *Jordan algebraic structures [algebras, triple systems, pairs]* correspond to geometric structures, in a similar way as Lie algebras correspond to Lie groups. Here is a rough and by no means complete table of types of Jordan algebraic structures and their geometries. The horizontal line separates the Riemannian cases from the non-Riemannian ones. (Abbreviations: JA: Jordan algebra, JP: Jordan pair, JTS: Jordan triple system, s.s.: semisimple, f.d.: finite dimensional, eucl.: euclidean)

type of Jordan structure	corresponding geometry	work of...
f.d. positive (=eucl.) JA	symmetric cones	Koecher, Vinberg (cf. [22])
	and their tube domains	Koecher (cf. [22])
positive Hermitian JTS	bounded symmetric domains	Loos [31], Kaup (cf. [37])
f.d. positive real JTS	symmetric R-spaces	Loos [31], [32]
Jordan Hilbert triples	(Riem.) Jordan symmetric sp.	Chu [19]
(real) s.s. f.d. JP	parahermitian symmetric spaces	Kaneyuki [25]
case (V, V), V eucl.JA	Cayley-type symmetric spaces	Koufany [27]
(real) s.s. f.d. JA	ω-domains	Koecher [26]
(real f.d.) JA	quadratic prehomog. sym. sp.	B. [4], Ch. II
(real f.d.) JTS	symmetric spaces with twist	B. [4]
(general) JP	generalized projective geometries	B. [5]
(general) JTS	generalized polar geometries	B. [5]
(general) JA	self-dual gen. proj. geometries	B. [6]

Other non-associative examples. There is a huge zoology of classes of non-associative algebraic structures: *alternative algebras, flexible algebras, Malcev algebras, structurable algebras, left symmetric algebras, Lie-admissible algebras, Leibniz algebras, Zinbiel algebras...* Jean-Louis Loday coined the term “coquecigrue” for the mythological beast (the hypothetic geometric structure) that should correspond to one of these (Leibniz algebras). Research is ongoing in this domain.

Associative algebras. An associative algebra can be seen as a Jordan algebra with an additional structure, or as a Lie algebra with an additional structure. Therefore, following the paradigm (B), then we are led to ask: *what is the geometric object corresponding to an associative algebra?* Asking this is not in contradiction with non-commutative geometry – it just means that paradigms (A) and (B) are different; of course, it raises the question to better understand the relation between them.

In these lectures, I will give a basic introduction first to Jordan algebraic structures (lecture 1), then to various geometries corresponding to them (lecture 2), and finally come back to associative structures (lecture 3). The approach presented here is very general: we do not assume that the base field $\mathbb{K}$ is the real or complex number field nor that the dimension over $\mathbb{K}$ is finite (in fact, we allow $\mathbb{K}$ to be a commutative ring, so there need not be a dimension at all). The point of view is thus more algebraic and synthetic-geometric than most readers will be used to; however, everything is nicely compatible with the differential-geometric point of view – I will give some remarks and references on this in the last section.

Acknowledgement. These notes follow closely in style and contents the series of lectures I have given at the 33rd Winter School "Geometry and Physics" in Srní, January 12th to January 18th, 2013, and I thank the organizers for inviting me to give these lectures in such pleasant surroundings.

1. First lecture: "There are no Jordan algebras, there are only Lie algebras"

This dictum is attributed to Isaiah Kantor: he discovered a construction nowadays often called the *Kantor-Koecher-Tits* (KKT) *construction*. This construction permits to understand Jordan algebras in terms of (graded) Lie algebras. The first aim of this lecture is to explain what Jordan algebras are, second, why there are no Jordan algebras but only Lie algebras, and third, why the converse seems to be almost true as well.

1.1. **What is a Jordan algebra (Jordan triple system, Jordan pair)?** There are three main categories of Jordan algebraic structures. To get a first impression, here are the typical examples for these categories, obtained by *symmetrizing associative structures*:

- (A) *Jordan algebra* of square matrices $M(n,n;\mathbb{K})$ with $X \bullet Y = (XY + YX)/2$,
- (B) *Jordan triple system* of rectangular matrices $M(p,q;\mathbb{K})$ with $T(X,Y,Z) = XY^tZ + ZY^tX$,
- (C) *Jordan pair* $(V^+, V^-) := \big(M(p,q;\mathbb{K}), M(q,p;\mathbb{K})\big)$ with $T^{\pm}(X,Y,Z) = XYZ + ZYX$.

Definition 1.1. A special Jordan algebra is a subspace V of an associative algebra $\mathbb{A}$ closed under the Jordan product $x \bullet y := (xy + yx)/2$. Equivalently: V is closed under squaring: $x \in V \Rightarrow x^2 \in V$.

Lemma 1.2. *In every special Jordan algebra the Jordan identity holds:*

$$x \bullet (x^2 \bullet y) = x^2 \bullet (x \bullet y).$$

Proof. In an associative algebra, let $L_x(z) = xz = R_z(x)$. By associativity, the operators L_x and R_y commute: $[L_x, R_y] = 0$, and $[L_x, L_{x^2}] = 0$, whence

$$[L_x + R_x, L_{x^2} + R_{x^2}] = 0$$

which applied to some y this gives the Jordan identity. □

Definition 1.3. A (linear) Jordan algebra is a commutative $\mathbb{K}$-algebra satisfying the Jordan identity. A (linear) Jordan triple system (JTS) is a $\mathbb{K}$-vector space V together with a trilinear map $T\colon V^3 \to V$ satisfying

(JT1) symmetry in the outer variables: $T(u,v,w) = T(w,v,u)$,
(JT2) an identity in 5 variables to be given below.

A (linear) Jordan pair is a pair of $\mathbb{K}$-vector spaces (V^+, V^-) together with two trilinear maps $T^{\pm}\colon V^{\pm} \times V^{\mp} \times V^{\pm} \to V^{\pm}$ which are symmetric in the outer variables and satisfy an identity (LJP2) to be given below. Here $\mathbb{K}$ is a commutative field or ring such that 2 and 3 are invertible in $\mathbb{K}$. As usual, *morphisms* of such structures are bilinear (resp. trilinear) maps which are compatible with the binary, resp. ternary, "product" maps.

Remark. We assume throughout that 2 and 3 are invertible in the base ring $\mathbb{K}$. For all three Jordan categories, the more sophisticated algebraic theory, valid for completely general $\mathbb{K}$, is *quadratic* in nature (cf. [30, 33]). For instance, observe that squares x^2 are the same in the associative and in a special Jordan algebra. Indeed, the Jordan product is equivalent to the *squaring map* $\mathbb{A} \to \mathbb{A}$, $x \mapsto x^2$ (and essentially also to the *inversion map* $\mathbb{A}^\times \to \mathbb{A}$, $x \mapsto x^{-1}$, cf. [35]).

1.2. **"There are only graded Lie algebras."** Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and Γ an abelian group. The cases $\Gamma = \mathbb{Z}/2\mathbb{Z}$ et $\Gamma = \mathbb{Z}$ will be most important later on.

Definition 1.4. We say that $\mathfrak{g}$ is $\overline{\Gamma\text{-graded}}$ if $\mathfrak{g}$ comes with a direct sum decomposition $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ such that, for all $\alpha, \beta \in \Gamma$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. $\underline{\text{Morphisms}}$ of Γ-graded Lie algebras are Lie algebra homomorphisms respecting the gradings.[1]

Typically, gradings arise in the following way: let $f : \mathfrak{g} \to \mathfrak{g}$ be diagonalizable, and $\mathfrak{g} = \oplus_\lambda \mathfrak{g}_\lambda$ be the eigenspace decomposition. Then:
– if f is an automorphism, then $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda\mu}$: we have $\Gamma \subset (\mathbb{K}^\times, \cdot)$,
– if f is a derivation, then $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$: we have $\Gamma \subset (\mathbb{K}, +)$.
Conversely, given such bracket rules, one can define a *grading derivation* (resp. *grading automorphism*). Example: for eigenvalues ± 1, this gives

Lemma 1.5. *There is a bijection between $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebras and* $\underline{\textit{symmetric pairs}}$*, i.e., pairs $(\mathfrak{g}, \sigma)$ of a Lie algebra with an involution σ (automorphism of order 2). The grading is given by the eigenspaces of σ.*

The -1-eigenspace $\mathfrak{m} := \mathfrak{g}_{-1}$ of an involution σ is stable under the ternary Lie bracket $[X, Y, Z] := [[X, Y], Z]$. This ternary bracket has the following properties:

(LT1) $[X, Y, Z] = -[Y, X, Z]$
(LT2) $[X, Y, Z] + [Y, Z, X] + [Z, X, Y] = 0$ (Jacobi identity)
(LT3) the endomorphism $R(X, Y) : [X, Y, \cdot] : \mathfrak{m} \to \mathfrak{m}$ is a *derivation*:

$$\boxed{[X, Y, [U, V, W]] = [[X, Y, U], V, W] + [U, [X, Y, V], W] + [U, V, [X, Y, W]]}.$$

Definition 1.6. A $\underline{\text{Lie triple system}}$ (LTS) is a $\mathbb{K}$-vector space $\mathfrak{m}$ together with a trilinear ternary product $[X, Y, Z]$ satisfying the identities (LT1)–(LT3).

Lemma 1.7 (Standard Imbedding)**.** *Every LTS arises as -1-eigenspace in an involutive Lie algebra. More precisely, we have a bijection between Lie triple systems and $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebras that are* $\underline{\textit{minimal}}$*, i.e., generated by the -1-eigenspace of the involution.*

Proof. Given a LTS $\mathfrak{m}$, define $\mathfrak{g} := \mathfrak{m} \oplus R(\mathfrak{m} \otimes \mathfrak{m}) \subset \mathfrak{m} \oplus \mathrm{Der}(\mathfrak{m})$ together with the "obvious" Lie bracket such that $\mathfrak{m} = \mathfrak{g}_{-1}$ and $R(\mathfrak{m} \otimes \mathfrak{m}) = \mathfrak{g}_1$ (cf. [28]). □

[1]Remark: one should not mix up this with Lie *super*algebras – even if $\Gamma = \mathbb{Z}/2\mathbb{Z}$, we speak here of ordinary, graded (and not super-graded) Lie algebras!

Remark. This bijection is in general *not* an equivalence of categories!

1.2.1. *The category JP of Jordan pairs, and* 3*-graded Lie algebras.* Now we are ready to define the Jordan categories:

Definition 1.8. A 3-graded Lie algebra is a $\mathbb{Z}$-graded Lie algebra such that $\mathfrak{g}_n = 0$ for $n \neq 0, 1, -1$. Thus

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad [\mathfrak{g}_1, \mathfrak{g}_1] = 0 = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}], \quad [\mathfrak{g}_1, \mathfrak{g}_{-1}] \subset \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_i] \subset \mathfrak{g}_i.$$

The endomorphism D which is $n\,\mathrm{id}$ on $\mathfrak{g}_n$ for $n = -1, 0, 1$, is then a derivation of $\mathfrak{g}$, called the grading derivation. Obviously, $\mathfrak{m} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is then a LTS, and it is *polarized* in the following sense:

Definition 1.9. A polarized LTS is a LTS of the form $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_1$ such that $[\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k] \subset \mathfrak{m}_{i+j+k}$. Equivalently, the restriction of the grading map $I := D|_{\mathfrak{m}}$ is an invariant polarization ($I^2 = \mathrm{id}$) which is twisted (i.e., a LTS-derivation).

Theorem 1.10 (KKT: first version)**.** *If $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_{-1}$ is a polarized LTS, let $V^\pm := \mathfrak{m}_{\pm 1}$ and*

$$T^\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm, \quad (u, v, w) \mapsto T^\pm(u, v, w) := [[u, v], w]\,.$$

Then $(V^\pm, T^\pm)$ is a linear Jordan pair: the trilinear maps $T^\pm$ satisfy
(LJP1): $T^\pm(u, v, w) = T^\pm(w, v, u)$, and
(LJP2):

$$\boxed{\begin{aligned} T^\pm\big(u, v, T^\pm(x, y, z)\big) = {} & T^\pm\big(T^\pm(u, v, x), y, z\big) - T^\pm\big(x, T^\mp(v, u, y), z\big) \\ & + T^\pm\big(x, y, T^\pm(u, v, w)\big) \end{aligned}}$$

Conversely, given a linear Jordan pair (V^+, V^-), we may reconstruct the polarized LTS by these formulae, and using the standard imbedding, we get back a 3*-graded Lie algebra. Summing up, there is a bijection between linear Jordan pairs and* 3*-graded Lie algebras that are minimal (i.e., generated by $\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$).*

The proof is a simple consequence of the standard imbedding: note that (LJP2) is a rewriting of (LT3) by using the new notation. Jordan pairs form, in a sense, the "most primitive" or "most basic" Jordan category. Note that, as above, the theorem describes a bijection of objects, but in general not an equivalence of categories. In general, these objects are not supposed to be semi-simple, nor even finite dimensional. If they are, and $\mathbb{K} = \mathbb{R}$, then *classification* is relatively easy ([34]). The following table shows the 6 main types of Jordan pairs; in the first line, E and F are arbitrary $\mathbb{K}$-modules (in finite dimension: $E = \mathbb{K}^p$, $F = \mathbb{K}^q$), in the second line, $E = F$ $(= \mathbb{K}^n)$, in the third and following lines, E, resp. V carry a non-degenerate quadratic form, and the last two lines are related to octonion algebras:

family name	Jordan pair (V^+,V^-)	$T^+(x,y,z)=$	3-graded Lie algebra $\mathfrak{g}=$
$I_{p,q}$: rectangular	$(\mathrm{Hom}(E,F),\mathrm{Hom}(F,E))$	$xyz+zyx$	$\mathbb{P}\mathfrak{gl}(E\oplus F)$ general linear
$I_{n,n}$: quadratic	case $E=F$		
II: symmetric	$(\mathrm{Sym}(E),\mathrm{Sym}(E))$	$xyz+zyx$	$\mathfrak{sp}(E,E)$ symplectic
III: skew	$(\mathrm{Asym}(E),\mathrm{Asym}(E))$	$xyz+zyx$	$\mathfrak{o}(E,E)$ orthogonal
IV: spin factor	(V,V)	see below	$\mathfrak{o}(V\oplus\mathbb{K}^2)$ orthogonal
V: Albert algebra	$V^+=V^-=\mathrm{Herm}(3,\mathbb{O}_\mathbb{K})$	xy^tz+zy^ty	exceptional of type E_7
VI: octonion plane	$(M(1,2;\mathbb{O}_\mathbb{K}),M(2,1;\mathbb{O}_\mathbb{K}))$	$xyz+zyx$	exceptional of type E_6

where in case IV we have $T^+(x,y,z)=\langle x,y\rangle z+\langle z,y\rangle x-\langle x,z\rangle y$, for any symmetric bilinear form $\langle\cdot,\cdot\rangle$. For $\mathbb{K}=\mathbb{C}$ and finite dimension, this gives a complete list of simple pairs. For other fields, such as $\mathbb{K}=\mathbb{R}$, the list becomes longer: e.g., spaces of (skew-)Hermitian matrices are simple real Jordan structures.

1.2.2. *The category of Jordan triple systems, and involutions of graded Lie algebras.*

Definition 1.11. An involution of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ is an automorphism $\tau:\mathfrak{g}\to\mathfrak{g}$ of order 2 such that $\tau(\mathfrak{g}_i)=\mathfrak{g}_{-i}$.

Theorem 1.12 (KKT: second version)**.** *An involution of a 3-graded Lie algebra induces an involution of the polarized LTS $\mathfrak{m}=\mathfrak{m}_{-1}\oplus\mathfrak{m}_1$ and of the linear Jordan pair (V^+,V^-). The vector space $V:=\mathfrak{m}_1$, together with the trilinear product*

$$T(u,v,w):=[[u,\tau(v)],w]$$

is a Jordan triple system, i.e., $T\colon V^3\to V$ is a trilinear map satisfying the identies (JT1) and (JT2) obtained from (LJP1) and (LJP2) by omitting the indices $\pm$. Conversely, given a JTS, one may reconstruct a linear Jordan pair with involution and the 3-graded Lie algebra with involution. Thus we have bijections between:

- *Jordan triple systems T,*
- *Jordan pairs (V^+,V^-) with involution,*
- *twisted polarized LTS $\mathfrak{m}=\mathfrak{m}^+\oplus\mathfrak{m}^-$ with involution $\tau|_\mathfrak{m}$,*
- *3-graded Lie algebras $\mathfrak{g}$ with involution τ.*

Definition 1.13. The 3-graded Lie algebra associated to a JTS (T,V) is called the KKT-algebra of T, and (V^+,V^-) its underlying Jordan pair.

Going through the proof of the preceding theorem, one notices that the LTS of the symmetric pair $(\mathfrak{g}^\tau,\mathfrak{h})$, where $\mathfrak{h}=\mathfrak{g}^\tau\cap\mathfrak{g}_0$, is given by the following

Lemma 1.14 (The Jordan-Lie Functor)**.** *For any JTS (V,T), we have a LTS on V defined by*

$$[x,y,z]:=R_T(x,y)z:=T(x,y,z)-T(y,x,z).$$

This defines a functor JTS $\to$ LTS, $T \mapsto R_T$, which we call the Jordan-Lie functor.

The Jordan pairs from the table given above all admit several different involutions. For $\mathbb{K} = \mathbb{R}$, the list of JTS is quite long (classification by Erhard Neher), and comparing it with the list of LTS (Marcel Berger [1]) reveals a surprising fact (also due to E. Neher): *the Jordan-Lie functor is not far from being a bijection on simple real finite-dimensional objects* – see [4] for references, and for the list of classical objects. As far as we know, there is no general conceptual explanation of this fact – the only work we know of is on the compact case (O. Loos, [32]): Loos describes the Jordan-Lie functor in terms of *cubic unit lattices* of root systems of compact LTS.

1.2.3. *The category of Jordan algebras.* It turns out that binary Jordan algebras form the "most complicated" of the three Jordan categories. The proofs of the following results are not just a simple check of definitions, as it was the case for the other results given so far:

Theorem 1.15 ("Meyberg's Theorem")**.** *Assume (V^+, V^-) is a linear Jordan pair and $y \in V^-$. Then V^+ with bilinear product $\bullet$ is a Jordan algebra, called the local or homotope Jordan algebra at y:*

$$x \bullet z := T^+(x, y, z)\,.$$

Theorem 1.16 (KKT: third version)**.** *The local algebra has a unit element, if, and only if, the operator*

$$Q(y) := Q^-(y)\colon V^+ \to V^-\,, \qquad x \mapsto Q(y)x := \frac{1}{2}T^+(y, x, y)$$

is bijective. If this is the case, then $(Q(y))^{-1}y$ is the unit element in the homotope algebra, and the JTS T has the same underlying Jordan pair as the JTS $\tilde{T}$ given by

$$\tilde{T}(x, y, z) = x \bullet (y \bullet z) - y \bullet (x \bullet z) + (x \bullet y) \bullet z\,.$$

Definition 1.17. We say that a Jordan pair (V^+, V^-) is of the first kind, or that it has invertible elements, if some of its local Jordan algebras have a unit element. (In the table above, this is true for $I_{n,n}$, II, III with $\dim E$ even, IV and V. This is related to the list of Hermitian symmetric spaces *of tube type.*)

Theorem 1.18 ("Fundamental Formula")**.** *For all $x \in V^+$ and $y \in V^-$,*

$$Q(x)Q(y)Q(x) = Q(Q(x)y)\,.$$

"There are no Jordan algebras..." Summing up, as Kantor noticed, a Jordan algebra is "is nothing but" one of the local algebras attached to a 3-graded Lie algebra. However, this is not the end of the story: in [33], Kevin McCrimmon answers to Kantor (loc. cit., p. 14): "Of course, this can be turned around: nine times out of ten, when you open up a Lie algebra you find a Jordan algebra inside which makes it tick." This is indeed quite close to being true, but it is rather hard to explain why this is so. On the one hand, to a single Jordan algebra, we can associate several Lie algebras: the KKT-algebra is, in a sense, the "biggest" of

all of them; there are two other ones, the so-called structure algebra $\mathfrak{g}_0$, and the derivation algebra $\mathfrak{g}_0 \cap \mathfrak{g}^\tau$. But in general none of these constructions is functorial. On the other hand, we have the Jordan-Lie functor: as mentioned above, the fact that it is quite close to being a bijection remains rather mysterious.

2. Second lecture: "Social symmetric spaces"

In the first lecture we explained that a JTS T can be interpreted as an LTS R with an additional structure (namely a realization as $R = R_T$: the Jordan-Lie functor). Since LTS correspond to symmetric spaces, JTS shall correspond to "Jordan symmetric spaces", i.e., to symmetric spaces with some additional structure. We are going to explain this in two steps:

(A) First, *what is a symmetric space?*
(B) and second *what is the additional structure?* To name it, we will call it a "social structure".

2.1. Symmetric spaces.

In Lie theory, a symmetric space is usually defined as a homogeneous space $M = G/H$ such that G is a Lie group and H an open subgroup of the fixed point group G^θ of some non-trivial involution θ of G. In [28], O. Loos has developed an equivalent, but much more conceptual approach:

Definition 2.1. A symmetric space is a manifold M equipped with a smooth binary "product map" $\mu\colon M \times M \to M$, $(x, y) \mapsto \mu(x, y) =: s_x(y)$ such that

(M1) $\mu(x, x) = x$, i.e., x is a fixed point of s_x,
(M2) $\mu(x, \mu(x, y)) = y$, i.e., $s_x^2 = \mathrm{id}$,
(M3) $\mu(x, \mu(y, \mu(x, z))) = \mu(\mu(x, y), \mu(x, z))$, i.e., s_x is an automorphism of μ,
(M4) the fixed point x of s_x is *isolated*, equivalently: $T_x(s_x) = -\mathrm{id}_{T_x M}$.

We will use both approaches, but Loos' approach is our favorite one (see [4], Chapter I, or [7] for an account on the general theory of symmetric spaces). One of the conceptual advantages of Loos' approach is that it is *base point free*: "all points are created equal". There is a base-point free approach to groups as well – not so well-known, possibly because there is no universally accepted terminology:

Definition 2.2. A torsor or heap or groud or principal homogeneous space (phos) is a set G together with a map denoted by $G^3 \to G$, $(x, y, z) \mapsto (xyz)$ such that

(T1) idempotency: $(xxy) = y = (yxx)$,
(T2) para-associativity: $(x(uvw)z) = ((xwv)uz) = (xw(vuz))$.

Exercise: (a) if $(G, e, \cdot)$ is a group, show that G with $(xyz) = xy^{-1}z$ is a torsor, (b) if G is a torsor and $y \in G$, show that $(G, y, \cdot_y)$ with $x \cdot_y z = (xyz)$ is a group, (c) show that both constructions are inverse to each other and are functorial (morphisms being suitably defined).

It is well-known that every Lie group G can be viewed as symmetric space: using base points, we consider a Lie group as homogeneous space $G \cong G \times G/dia(G \times G)$, and in a base point free way, we consider G with product

$$\mu(x,y) = (xyx) = xy^{-1}x\,. \tag{2.1}$$

In other words, we restrict the ternary product (xyz) to the diagonal $x = z$. (Note: in the same way an associative algebra with product xz is turned into a Jordan algebra: one restricts to the diagonal $x = z$.)

2.2. **Social structure: historical.** The following concept was introduced in [36]:

Definition 2.3. A symmetric R-space is a flag manifold G/P (G real semisimple Lie group, P a parabolic subgroup) which at the same time is a symmetric space $G/P = U/K$, with K open in U^θ and U a maximal compact subgroup U of G.

In the late 60ies and early 70ies, Solodovnikov, Rivillis, and Makarevič remarked that symmetric R-spaces contain (as open orbits) a lot of other symmetric spaces, besides the compact (total) U-orbit, and studied these realizations (see [4] for references). At the same time, O. Loos realized that symmetric R-spaces are in bijection with *compact real Jordan triple systems* (announced in [29] and proved in [32]). More generally, we will see that *generalized projective geometries* have in common with symmetric R-spaces a rich "social life" – there are large families of symmetric spaces living harmonically together, on an underlying set G/P where G is a big, "projective" or "conformal", overgroup, while all assumptions on finite-dimensionality or semisimplicity of G can be dropped.

2.2.1. *Grassmann geometries.* Before coming to the general construction, let us study this important example: for a vector space V over a field $\mathbb{K}$ (or even a module over a ring $\mathbb{K}$), we denote by $\mathcal{X} := \mathrm{Gras}(V)$ the full Grassmannian (set of all subspaces of V), and by $\mathcal{X}^+ := \mathrm{Gras}_E^F(V)$ the Grassmannian of type E and co-type F (set of all subspaces that are isomorphic to a given "model space" $E \subset V$ and admit a complementary subspace isomorphic to another given model space $F \subset V$). Denote by $\mathcal{X}^- := \mathrm{Gras}_F^E(V)$ the dual Grassmannian of $\mathcal{X}^+$. Recall:

Lemma 2.4. *The geometry $\mathcal{X}^+$ is homogeneous under the general linear group $G = \mathrm{Gl}(V)$. If $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\dim E = p$, $\dim F = q$, then the stabilizer of the base point $o = \mathbb{K}^p \subset \mathbb{K}^{p+q}$ is a (maximal) parabolic subgroup of the (semisimple) Lie group $\mathbb{P}\mathrm{Gl}(V)$. Moreover, any choice of scalar product on $\mathbb{K}^{p+q}$ induces on $\mathcal{X}^+$ a (compact) symmetric space structure, isomorphic to $\mathrm{O}(p+q)/(\mathrm{O}(p)\times\mathrm{O}(q))$, resp. $\mathrm{U}(p+q)/(\mathrm{U}(p)\times\mathrm{U}(q))$, resp. $\mathrm{Sp}(p+q)/(\mathrm{Sp}(p)\times\mathrm{Sp}(q))$.*

If we look at *open orbits in $\mathcal{X}$* which are symmetric spaces, then we get a much bigger class of not necessarily compact spaces. Most classically, among them the "affine cells" of the Grassmannian, viewed as flat symmetric spaces:

Lemma 2.5. *Let us call $a, x \in \mathcal{X}$ transversal if $V = a \oplus x$; notation: $a \top x$. Then, for all $a \in \mathcal{X}$, the set $V_a := \{x \in \mathcal{X} \mid a \top x\}$ of complements of a, is, in a natural way, an affine space over $\mathbb{K}$.*

This classical fact can be proved in various ways. For our purposes, a convenient proof is as follows: if $a \top x$, let $P^a_x : V \to V$ be the linear projection operator with kernel a and image x. If $x, z \top a$, and $r \in \mathbb{K}$, define two other linear operators by

$$M^a_{xz} := P^a_x - P^z_a = P^a_x - \mathrm{id}_V + P^a_z , \tag{2.2}$$

$$\Pi^a_{r;x} := P^a_x + rP^x_a = (1-r)P^a_x + r\mathrm{id}_V \, .$$

Choose an origin $y \in V_a$ (so y is a subspace transversal to a). Then V_a with

$$x + z := x +_y z := M^a_{xz}(y), \qquad rz := \Pi^a_{r;y}(z) \tag{2.3}$$

is a $\mathbb{K}$-module with origin y. If $\mathbb{K} = \mathbb{R}$, then $(V, y, +_y)$ is an abelian Lie group, and it is a symmetric space with symmetries $s_z(x) = z - x + z = z +_x z = M^a_{zz}(x) = \Pi^a_{-1;z}(x)$. Next, some non-flat symmetric spaces:

Lemma 2.6. *For any non-degenerate quadratic or symplectic form $\beta \colon V \times V \to \mathbb{K}$, let $p \colon \mathcal{X} \to \mathcal{X}$, $x \mapsto p(x) = x^\perp$ the corresponding orthocomplementation map, and*

$$M := M^p := \{ x \in \mathcal{X} \mid V = x \oplus x^\perp \}$$

the set of β-non-degenerate subspaces. Then M^p becomes a symmetric space with

$$s_z(y) := \Pi^{p(z)}_{-1;z}(y) = M^{p(z)}_{z,z}(y) \, .$$

If β is a scalar product, as in Lemma 2.4, then this gives the symmetric spaces named in that lemma; if β is indefinite, we get, e.g., symmetric orbits of the type $\mathrm{O}(p,q)/\mathrm{O}(p) \times \mathrm{O}(q)$, and similar spaces. Finally, for $p = q$ (i.e., $\mathrm{Gras}_E(E \oplus E)$), there is another particular space: the group case $\mathrm{Gl}(E)$, see next lecture.

2.2.2. *General construction of a geometry associated to a 3-graded Lie algebra.* In [16, 17], the following very general construction of a geometry out of a Jordan pair or -triple system has been given. The key principle is a systematic use of the relation between <u>gradings</u> and <u>filtrations</u> of a Lie algebra $\mathfrak{g}$:

(a) Start with a JTS or with a Jordan pair, and let $\mathfrak{g}$ its 3-graded KKT algebra. Without loss of generality, one may assume that the center of $\mathfrak{g}$ is trivial.

(b) We look at the *space of all inner 3-gradings of* $\mathfrak{g}$ (<u>Euler operators</u>), that is, the set of all inner derivations having 3 eigenvalues $-1, 0, 1$:

$$\mathcal{G} := \{ E \in \mathfrak{g} \mid (\mathrm{ad}(E))^3 - \mathrm{ad}(E) = 0 \} \tag{2.4}$$

This set can be defined for any Lie algebra, thus, strictly speaking, step (a) is not really necessary: it serves just to ensure that $\mathcal{G}$ contains a non-trivial element, which can be taken as base point in $\mathcal{G}$, if needed.

(c) For any Euler operator E, the map $S_E := \mathrm{id}_{\mathfrak{g}} - 2(\mathrm{ad}(E))^2 : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of order 2 having eigenvalues $(-1, 1, -1)$ where $\mathrm{ad}(E)$ has eigenvalues $(-1, 0, 1)$. The set $\mathcal{G}$, with product

$$\mu(E, F) = s_E(F) = F - 2[E, [E, F]] \tag{2.5}$$

satisfies the algebraic axioms (M1), (M2), (M3) of a symmetric space. The following steps show that it is *polarized* (and it also gives an algebraic analog of (M4)):

(d) For any Euler operator E, there is a positive and a negative flag of eigenspaces:

$$\mathfrak{f}^+(E) : (\mathfrak{g}_{-1} \subset \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \subset \mathfrak{g}), \qquad \mathfrak{f}^-(E) : (\mathfrak{g}_1 \subset \mathfrak{g}_1 \oplus \mathfrak{g}_0 \subset \mathfrak{g}) . \tag{2.6}$$

Let $\mathcal{F}$ be the space of "short flags" thus obtained: it plays the role of the Grassmannian $\mathcal{X}$ from the preceding example. There is an analogue of the affine cells:

Lemma 2.7. *For any flag $\mathfrak{f} = (\mathfrak{f}_0 \subset \mathfrak{f}_1 \subset \mathfrak{g}) \in \mathcal{F}$, the set $\mathfrak{f}^\top$ of all transversal flags of $\mathfrak{f}$ is an affine space over $\mathbb{K}$, modelled on $\mathfrak{f}_0$. Here, a flag $\mathfrak{e} = (\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \mathfrak{g})$ is called transversal to $\mathfrak{f}$ if they are "crosswise complementary": $\mathfrak{g} = \mathfrak{f}_0 \oplus \mathfrak{e}_1 = \mathfrak{e}_0 \oplus \mathfrak{f}_1$.*

The affine parts $\mathfrak{f}^\top$ of $\mathcal{F}$ are called "Jordan charts". They should be seen as a class of priviledged coordinates, in which most objects have particularly nice expressions. For instance, in a Jordan chart, elements of $\mathrm{Aut}(\mathfrak{g})$ act by *quadratic fractional maps.*

Theorem 2.8. *A pair $(\mathfrak{e}, \mathfrak{f}) \in \mathcal{F} \times \mathcal{F}$ is transversal, if and only if, there is an Euler operator $E \in \mathcal{G}$ such that $\mathfrak{e} = \mathfrak{f}^+(E)$ and $\mathfrak{f} = \mathfrak{f}^-(E)$. In other words, the image of the map $\mathcal{G} \to (\mathcal{F} \times \mathcal{F})$, $E \mapsto (\mathfrak{f}^+(E), \mathfrak{f}^-(E))$ is precisely the set of transversal pairs.*

Since the set of transversal pairs may be considered as "Zariski-dense" in $\mathcal{F} \times \mathcal{F}$, this imbedding defines a local product structure on $\mathcal{G}$.

(e) Now let us add an involution τ to the whole set-up (that is, in point (a), we start with a JTS T). The map τ acts naturally on $\mathcal{G}$ (by automorphisms of μ) and on $\mathcal{F} \times \mathcal{F}$ (by the exchange map). The fixed point set $\mathcal{G}^\tau$ is a symmetric subspace, identified with a part of $\mathcal{F}$, namely with the set of flags such that $\tau(\mathfrak{f})$ is transversal to $\mathfrak{f}$. (This generalizes Lemma 2.6.)

(f) There is a description in terms of *homogeneous spaces*: the whole space $\mathcal{F}$ is of course in general not homogeneous under $\mathrm{Aut}(\mathfrak{g})$, but certain "connected components" are (exactly as in the Grassmannian example, where $\mathrm{Gras}(V)$ is not homogeneous, but $\mathrm{Gras}_E^F(V)$ is). The connected components are then homogeneous spaces G/P. A good choice for G is the group generated by all $S_E S_F$ with $E, F \in \mathcal{G}$, called the projective elementary group.

(g) An abstract approach to spaces like $\mathcal{G}$ and $\mathcal{F}$ has been given in [5]: they are called generalized projective geometries. As seen above (item (d)), $\mathcal{F}$ and $\mathcal{X}$ have in common that they are covered by affine charts. The abstract approach features this: the affine parts "interact with each other" according to certain algebraic laws formulated in loc. cit., generalizing, in some sense, the symmetric space law (M3). See also [2] for new results in this direction.

(h) Up to know, everything is purely algebraic: there is no topology around. But, under very general and natural assumptions on $\mathbb{K}$ and on $\mathfrak{g}$, one can define smooth manifold structures on $\mathcal{G}$ and $\mathcal{F}$ – see Section 4 below.

2.3. **Family life.** By the preceding construction, we have constructed *one* symmetric space associated to a JTS, that is, to every involution of $\mathfrak{g}$ we have constructed one symmetric space structure on $\mathcal{F}$. Since there are many involutions, this gives indeed a family of symmetric space structures on $\mathcal{F}$ which we call isotopes of each other. More surprisingly, this family "degenerates smoothly", it contracts, e.g., to flat spaces. We will investigate some examples in the next chapter. On a Jordan algebraic level, the corresponding phenomenon is homotopy:

Lemma 2.9. *Given a Jordan pair (V^+, V^-), a homotopy is a linear map $\alpha\colon V^+ \to V^-$ such that*

$$\forall\, x, y, z \in V^+ : \qquad \alpha T^+(x, \alpha y, z) = T^-(\alpha x, y, \alpha z)\,.$$

Then the formula $T_\alpha(x, y, z) := T^+(x, \alpha y, z)$ defines a JTS structure on $V = V^+$.

The proof is by a direct, two-line computation (cf. [4], Lemma III.4.5). If α is invertible, we call it an isotopy, and for a JTS, these notions are defined in the same way by omitting superscripts $\pm$. For instance, any automorphism of order two of a JTS is an isotopy.

Definition 2.10. The structure variety of a Jordan pair is the set of all homotopies $\mathrm{Svar}(V^+, V^-) \subset \mathrm{Hom}_{\mathbb{K}}(V^+, V^-)$, and T_α is called the α-homotope. Also, the family

$$R_\alpha(x, y)z := R_{T_\alpha}(x, y)z = T_\alpha(x, y, z) - T_\alpha(y, x, z) = T^+(x, \alpha y, z) - T^+(y, \alpha x, z)$$

of Lie triple systems will be called homotopes of each other.

Note that $(R_\alpha)_{\alpha \in \mathrm{Svar}(V^+, V^-)}$ is a family of LTS all living on the same underlying vector space $V = V^+$, parametrized by the structure variety. This variety is an algebraic variety; it is a cone (stable under taking multiples), hence every element contracts to 0. Correspondingly, R_α and T_α describe deformations, resp. contractions of LTS, resp. of JTS. For $\alpha = 0$, we always get the most singular contraction, the flat LTS $R_0 = 0$.

The classification problem may now be refined as follows:

1. Classify (simple...) Jordan pairs (see list in first lecture),
2. Classify their isotopes, i.e., classifiy (simple...) JTS,
3. Classify the whole structure variety, i.e., all homotopes.

For real JTS, see [4] for a list of JTS, organized according to the point of view of isotopy, thus answering to 2. For special Jordan pairs, the last item has been achieved in [11, 12], by using the link with *associative* structures:

3. Third lecture : "A non-associative view on associative structures"

In this third lecture we come back to associative structures. From the point of view of Jordan theory, associative structures are "Jordan structures with additional structure" ("special structure"), just as JTS are LTS with additional structure. We summarize this by the "Jordan-Lie triangle":

$$\boxed{\begin{array}{ccccc} & & \text{associative: } xy & & \\ & \swarrow & & \searrow & \\ \text{Jordan: } x \bullet y & & & & \text{Lie: } [x, y] \end{array}}$$

When looking for a global, geometric version of this triangle, we are led to ask: *what is the "meaning" of associativity?* Put differently: *what kind of geometric* *structure corresponds to associative algebras?* This leads to further questions: *how do the "exceptional" structures arise?* Either of these topics would deserve a whole series of lectures on its own. We will give some first answers by looking at examples. First of all, the "Jordan Lie triangle" becomes a true triangle (including a bottom arrow) if we look at ternary structures. We start with an easy observation:

Lemma 3.1. *Assume $\mathbb{A}$ is an associative algebra. Then, for all $a \in \mathbb{A}$, the product $(x,z) \mapsto xaz$ is again bilinear and associative. Thus $[x,y]_a := xay - yax$ is a Lie bracket on $\mathbb{A}$.*

Slightly more generally, this holds also for *ternary* associative systems:

Definition 3.2. An associative pair is a pair of $\mathbb{K}$-modules $(\mathbb{A}^+, \mathbb{A}^-)$ with two ternary trilinear products $\mathbb{A}^\pm \times \mathbb{A}^\mp \times \mathbb{A}^\pm \to \mathbb{A}^\pm$, $(x,y,z) \mapsto \langle xyz \rangle^\pm$ such that

(A) $\langle xy\langle uvw\rangle\rangle = \langle\langle xyu\rangle vw\rangle = \langle x\langle vuy\rangle w\rangle$.

With suitably defined morphisms, associative pairs form a category AP. Obviously, for any fixed element $y \in \mathbb{A}^\mp$ we have an associative algebra on $\mathbb{A}^\pm$ $x \cdot_y z := \langle xyz\rangle^\pm$, called the local algebra at y or y-homotope algebra. Finally, associative triple systems are $\mathbb{K}$-modules with a trilinear map satisfying the same identity, where superscripts are omitted.

Lemma 3.3. *There is a functor $AP \to JP$ defined by symmetrizing: $T^\pm(x,y,z) := \langle xyz\rangle^\pm + \langle zyx\rangle^\pm$. A Jordan pair obtained in this way is called special.*

Similarly there is a functor ATS $\to$ JTS, and we complete with JTS $\to$ LTS to get

$$\begin{array}{ccccc} & & \text{associative TS: } \langle xyz\rangle & & \\ & \swarrow & & \searrow & \\ \text{Jordan TS: } T(x,y,z) & & \longrightarrow & & \text{Lie TS: } R(x,y)z \end{array}$$

Main example: rectangular matrices, $(\mathbb{A}^+, \mathbb{A}^-) = (M(p,q;\mathbb{K}), M(q,p;\mathbb{K}))$ with products $\langle XYZ\rangle^+ = XYZ$, $\langle XYZ\rangle^- = ZYX$. Thus the Jordan pair of rectangular matrices is special. For $p = q = n$, we are back in the case of the associative algebra of square matrices. More generally, consider $(\mathrm{Hom}(E,F), \mathrm{Hom}(F,E))$ with the same product.

3.1. **Associative geometries.** The prototype of these is the Grassmann geometry considered in the preceding lecture, which corresponds to the Jordan pair of rectangular matrices. The following result from [14] gives the geometric analog of the homotope-algebras of an associative pair. Analogously to (2.2), we define, for $a \top x$ and $b \top z$, an operator

$$M^{ab}_{xz} := P^a_x - P^z_b = P^a_x - \mathrm{id}_V + P^b_z \,. \tag{3.1}$$

Theorem 3.4. *Let $\mathcal{X} = \mathrm{Gras}(V)$ be a Grassmann geometry. Then, for all $a, b \in \mathcal{X}$, the set $U_{ab} := V_a \cap V_b$ of common complements of a and b, is a torsor with respect to the ternary product*

$$(xyz)_{ab} := M^{ab}_{xz}(y) \,.$$

This torsor is commutative if $a = b$; more generally, we have, for all $a, b \in \mathcal{X}$,

$$(xyz)_{ab} = (zyx)_{ba} .$$

Exercise: prove this – the proof uses only elementary linear algebra, see [14] for hints. The pentary map $\Gamma\colon (x, a, y, b, z) \mapsto (xyz)_{ab}$ has "wonderful" properties, which are studied in [14, 15, 3]. One of the most surprising features is that the map Γ extends, in a rather canonical way, to a map $\Gamma\colon \mathcal{X}^5 \to \mathcal{X}$, such that, for any fixed couple (a, b), the partial map of the remaining three arguments is a semi-torsor (it satisfies the property (T2), and hence, for fixed y, defines families of semigroups on $\mathcal{X}$). Moreover, Γ has a nice behavior under permutations, which remind of the classical *cross-ratio* on a projective line.

3.2. **Homotopy revisited.** The preceding theorem has two special cases which are, in a sense, opposite of each other:

(a) $a = b$: we get back the affine cell $U_{aa} = V_a$

(b) the transversal case $a \top b$: then $V = a \oplus b$, and U_{ab} is the set of common complements of a and b; each such complement is the graph of an invertible linear operator $a \to b$; hence U_{ab} can be identified with the set of linear isomorphisms from a onto b. This set carries a natural torsor structure, and it is not hard to show that this torsor structure agrees with the one from the preceding theorem. Therefore, as a group, U_{ab} is isomorphic to the general linear group $\mathrm{Gl}_{\mathbb{K}}(a)$.

Now, the general case of U_{ab} can be understood as a *deformation of (a) in direction of (b).* In an affine chart, the groups U_{ab} are explicitly described as follows:

Lemma 3.5. *For any matrix $A \in M(n, n; \mathbb{F})$, the product $X \cdot_A Y = X + Y - XAY$ on $M(n, n; \mathbb{F})$ is associative, has neutral element 0, and an element X is invertible iff $(1 - XA)$ is an invertible matrix. The inverse is then $j_A(X) = -(1 - XA)^{-1}X$.*

Proof. By direct computation: associativity:

$$X \cdot_A (Y \cdot_A Z) = X + Y + Z - (XAY + YAZ) + XAYAZ = (X \cdot_A Y) \cdot_A Z$$

invertibility: let $\phi(X) := 1 - AX$, then ϕ is a homomorphism: $\phi(0) = 1$,

$$\phi(X \cdot_A Y) = 1 - AX - AY + AXAY = (1 - AX)(1 - AY) = \phi(X)\phi(Y)$$

inverse: $0 = X \cdot_A Y$ iff $0 = X + Y - XAY$ iff $Y = -(1 - XA)^{-1}X$. $\square$

This construction generalizes to all other classical groups: orthogonal, unitary and symplectic groups all live in suitable Lagrangian geometries ([15]; in symplectic geometry, the symplectic case is referred to as canonical relations). We will not explain here the geometric construction from [15], but just the analog of the lemma above (whose proof is again elementary):

Lemma 3.6. *The following sets with product $X \cdot_A Y$ as above are* (*algebraic*) *groups:*

label	*underlying set*	*parameter space*	*product*
$\mathrm{Gl}_n(A;\mathbb{F})$	$:= \{X \in M(n,n;\mathbb{F}) \mid 1 - AX \text{ invertible}\}$	$A \in M(n,n;\mathbb{F})$	$X \cdot_A Y$
$\mathrm{Gl}_{p,q}(A;\mathbb{F})$	$:= \{X \in M(p,q;\mathbb{F}) \mid 1 - AX \text{ invertible}\}$	$A \in M(q,p;\mathbb{F})$	$X \cdot_A Y$
$\mathrm{O}_n(A;\mathbb{K})$	$:= \{X \in \mathrm{Gl}_n(A,\mathbb{K}) \mid X + X^t = X^t AX\}$	$A \in \mathrm{Sym}(n;\mathbb{K})$	$X \cdot_A Y$
$\mathrm{Sp}_{n/2}(A;\mathbb{K})$	$:= \{X \in \mathrm{Gl}_n(A,\mathbb{K}) \mid X - X^t = X^t AX\}$	$A \in \mathrm{Asym}(n;\mathbb{K})$	$X \cdot_A Y$
$\mathrm{U}_n(A;\mathbb{C})$	$:= \{X \in \mathrm{Gl}_n(A,\mathbb{C}) \mid X + \overline{X}^t = \overline{X}^t AX\}$	$A \in \mathrm{Herm}(n;\mathbb{C})$	$X \cdot_A Y$
$\mathrm{U}_n(A;\mathbb{H})$	$:= \{X \in \mathrm{Gl}_n(A,\mathbb{H}) \mid X + \overline{X}^t = \overline{X}^t AX\}$	$A \in \mathrm{Herm}(n;\mathbb{H})$	$X \cdot_A Y$
$\mathrm{U}_n(A;\widetilde{\mathbb{H}})$	$:= \{X \in \mathrm{Gl}_n(A,\mathbb{H}) \mid X + \widetilde{X}^t = \widetilde{X}^t AX\}$	$A \in \mathrm{Herm}(n;\widetilde{\mathbb{H}})$	$X \cdot_A Y$

Their Lie algebras are the following spaces with Lie bracket $[X,Y]_A = XAY - YAX$*:*

family name	*label and underlying space*	*parameter space*	*Lie bracket*
general linear (square)	$\mathfrak{gl}_n(A;\mathbb{F}) := M(n,n;\mathbb{F})$	$A \in M(n,n;\mathbb{F})$	$[X,Y]_A$
general linear (rectan.)	$\mathfrak{gl}_{p,q}(A;\mathbb{F}) := M(p,q;\mathbb{F})$	$A \in M(q,p;\mathbb{F})$	$[X,Y]_A$
orthogonal	$\mathfrak{o}_n(A;\mathbb{K}) := \mathrm{Asym}(n;\mathbb{K})$	$A \in \mathrm{Sym}(n;\mathbb{K})$	$[X,Y]_A$
[half-] symplectic	$\mathfrak{sp}_{n/2}(A;\mathbb{K}) := \mathrm{Sym}(n;\mathbb{K})$	$A \in \mathrm{Asym}(n;\mathbb{K})$	$[X,Y]_A$
$\mathbb{C}$-*unitary*	$\mathfrak{u}_n(A;\mathbb{C}) := \mathrm{Aherm}(n;\mathbb{C})$	$A \in \mathrm{Herm}(n;\mathbb{C})$	$[X,Y]_A$
$\mathbb{H}$-*unitary*	$\mathfrak{u}_n(A;\mathbb{H}) := \mathrm{Aherm}(n;\mathbb{H})$	$A \in \mathrm{Herm}(n;\mathbb{H})$	$[X,Y]_A$
$\mathbb{H}$-*unitary split*	$\mathfrak{u}_n(A;\widetilde{\mathbb{H}}) := \mathrm{Aherm}(n;\widetilde{\mathbb{H}})$	$A \in \mathrm{Herm}(n;\widetilde{\mathbb{H}})$	$[X,Y]_A$

Notation used here: we consider the associative algebra $M(n,n;\mathbb{F})$ (where $\mathbb{F}$ is any ring with unit) and the usual involutions (transposed or adjoint matrix); concerning their eigenspaces, Sym denotes symmetric, Asym skewsymmetric, Herm denotes Hermitian and Aherm skew-Hermitian matrices, where $\mathbb{K} = \mathbb{C}$ is always equipped with its usual complex conjugation, and for $\mathbb{K} = \mathbb{H}$ we use the following conventions: if nothing else is specified, we use the "usual" conjugation $\lambda \mapsto \overline{\lambda}$ (minus one on the imaginary part im$\mathbb{H}$ and one on the center $\mathbb{R} \subset \mathbb{H}$). If we consider $\mathbb{H}$ with its "split" involution $\lambda \mapsto \widetilde{\lambda} := j\overline{\lambda}j^{-1}$, then we write $\widetilde{\mathbb{H}}$. Note: in all cases, for invertible A we get a usual (i.e., semisimple or reductive) classical Lie algebra, whereas for non-invertible A we get "new" Lie algebras, which are contractions of their "usual" counterparts.

Exercise. *The set* $\widehat{\mathrm{O}}_n(A;\mathbb{K}) := \{X \in M(n,n;\mathbb{K}) \mid X + X^t = X^t AX\}$ *is stable under the product* $X \cdot_A Y$ *and forms a semigroup with unit* 0*. Thus we get a* <u>*semigroup hull*</u> *containing the orthogonal group as open dense subset. Similarly for all other classical groups.* (More in [15].)

Classification of homotopes. The families of classical groups just given are indeed associative counterparts of the homotopes introduced at the end of the preceding lecture. In [11], elementary constructions of a similar kind are given for symmetric spaces coming from Jordan structures:

Lemma 3.7. *Let* $\mathbb{A}$ *an associative algebra with involution* τ *(anti-automorphism of order* 2*), and let* $A \in \mathbb{A}$*.*

(i) *If A belongs to the eigenspace $\mathbb{A}^{\pm\tau}$, then the "opposite" eigenspace $A^{\mp\tau}$ is stable under the Lie bracket $[X,Y]_A$ (note: this has implicitly been used in the preceding construction!).*

(ii) *If A belongs to the eigenspace $\mathbb{A}^{\pm\tau}$, then* both *eigenspaces are LTS with respect to $[[X,Y]_A,Z]_A$.*

Item (i) leads to 2 types of Lie algebras, denoted by $\mathfrak{sp}_{\mathbb{A},A,\tau}$ and $\mathfrak{u}_{\mathbb{A},A,\tau}$, and item (ii) leads to 4 cases of LTS, two of them being "group cases, considered as LTS", and the other two corresponding to symmetric spaces of type "general linear/unitary", resp. "general linear/symplectic", according to the following table

	$A \in \mathbb{A}^{\tau}$	$A \in \mathbb{A}^{-\tau}$
LTS $\mathbb{A}^{\tau}$	$(\mathfrak{g}_{\mathbb{A},A}, \mathfrak{u}_{\mathbb{A},A,\tau})$	$\mathfrak{sp}_{\mathbb{A},A,\tau}$
LTS $\mathbb{A}^{-\tau}$	$\mathfrak{u}_{\mathbb{A},A,\tau}$	$(\mathfrak{g}_{\mathbb{A},A}, \mathfrak{sp}_{\mathbb{A},A,\tau})$

Next, given *two commuting* involutions τ, τ' of $\mathbb{A}$, let the joint eigenspace decomposition be written in the form

$$\mathbb{A} = \mathbb{A}^{(1,1)} \oplus \mathbb{A}^{(-1,1)} \oplus \mathbb{A}^{(1,-1)} \oplus \mathbb{A}^{(-1,-1)} .$$

Then, applying the lemma to each of the two involutions, we see that, if A belongs to one of these eigenspaces, all four eigenspaces are LTS with respect to $[[X,Y]_A,Z]_A$. This leads to 16 cases for combination of choices (eigenspace for the parameter, eigenspace for the underlying set of the LTS). Of course, this observation generalizes for an arbitrary finite number of commuting involutions, but, as a matter of fact, it turns out that *all classical symmetric spaces* can be obtained by choosing two commuting involutions on a suitable matrix algebra, and looking at one of the common eigenspaces. In [12], we have worked out the explicit form of the most relevant of these situations – the resulting tables contain a lot of information on the "family organization" of symmetric spaces. For instance, we see how some well-known classical symmetric spaces, such as, e.g., the *Siegel upper half plane*, contract to "degenerate" spaces. A long term project, with P. Bieliavsky, is to use such information in order to develop a quantization on such semi simple symmetric spaces, by "following" quantization theories from the almost flat case up to the non-degenerate cases.

4. Further topics, and afterhoughts

At the end of this series of lectures, let us add a list of some mathematical and philosophical remarks, concerning further topics and open problems.

(1) *Differential geometry, and infinite dimensional parabolic geometries.* The approach outlined in these lectures is rather algebraic in nature, including spaces of infinite dimension, whereas classical text books such as [22, 31, 26], and my lecture notes [4], deal with finite-dimensional real or complex geometries and use tools of classical differential geometry. Moreover, these texts mainly focus on the simple or semi-simple case, so that the geometry $\mathcal{X}$ is a *symmetric R-space G/P*, which is electable as a model space for, possibly curved, *parabolic geometries*. We would

like to propose as a challenge and as a possible direction of new research to define and to study *general parabolic geometries of Jordan type* (step 3 below):

1. First step: develop a general algebraic and geometric theory of the model spaces (spaces accociated to 3-graded Lie algebras: cf. these lectures),

2a. Second step: in presence of topology, define a smooth manifold structure on the model space and show that all structure maps are smooth ([16]),

2b. For the preceding item to make sense, one has to define the basic notions of differential calculus and differential geometry in a sufficiently general framework (this has been done in [13, 7]),

3. Combining the preceding points, define *Cartan geometries (over general base fields or -rings...)*, modelled on spaces defined in Item 2a.

The flavor of this approach will certainly be more algebraic than the one by functional analytic methods, used in Kaup's approach to infinite dimensional bounded symmetric domains (cf. [37]). It should be possible to extend these methods to 5-graded, or even longer $\mathbb{Z}$-graded model geometries, but the theory then becomes much more involved (cf. [18]).

(2) *Link with representation theory and harmonic analysis.* It turned out that the Jordan theoretic formulation is a remarkably effective tool in *harmonic analysis* on symmetric cones and on Hermitian symmetric spaces, see [22, 38]. It has been used in the analysis of *minimal representations* and of a generalized *Maslov index* (Clerc, Hilgert, Kobayashi, Koufany, Neeb, Ørsted, Pevzner and others).

(3) *Associative geometries.* When defining the notion of associative geometries, our original motivation was to understand the geometry of Jordan-Lie algebras (which include associative algebras *and* Hermitian matrices), see item (5) below. A list of open problems and further topics is given at the end of [14, 3].

(4) *Exceptional and semi-exceptional geometries.* They correspond to the Jordan pairs IV, V, and VI from the list given in the first lecture. The point of view of projective geometry builds a bridge from associative geometries to type VI, the <u>octonion</u>, or <u>Moufang projective plane</u> ([15]). Moreover, octonion geometries (type V and VI) are related to constructions of exceptional Lie groups related to "Freudenthal's magic square". On the other hand, spaces of type IV (spin factors) are "semi-exceptional": they correspond to <u>conformal geometry</u> of projective quadrics. Certainly they would deserve a series of lectures in their own right.

(5) *Link with physics?* Until today, Jordan theory does not belong to the "mathematical mainstream" and lives somewhere near the border of mathematics and physics. For instance, there is an interesting concept of <u>Jordan-Lie algebra</u>, which (to our knowledge) appears only in physics litterature (see [21]) and nowhere in mathematics – this concept seems to go back to a question by Niels Bohr asking what algebraic structures admit tensor products (see [23], cf. [10]). One reason why Jordan structures are still considered as somewhat "exotic" by many mathematicians may be that Pascual Jordan was a physicist and not a mathematician. Certainly, his approach to foundations of quantum mechanics did not hold sway; but this does

not mean that the last word on this issue has already been said. Some speculations on this can be found in [9, 10].

References

[1] Berger, M., *Les espaces symétriques non–compacts*, Ann. Sci. ENS (1957).

[2] Bertram, W., *Jordan geometries by inversions*, Preprint, 2013, `http://arxiv.org/abs/1308.5888`.

[3] Bertram, W., *The projective geometry of a group*, arXiv: `math.GR/1201.6201`.

[4] Bertram, W., *The Geometry of Jordan and Lie Structures*, Lecture Notes in Math., vol. 1754, Springer, Berlin, 2000.

[5] Bertram, W., *Generalized projective geometries: General theory and equivalence with Jordan structures*, Advances in Geometry **3** (2002), 329–369.

[6] Bertram, W., *The geometry of null systems, Jordan algebras and von Staudt's Theorem*, Ann. Inst. Fourier **53** fasc. 1 (2003), 193–225.

[7] Bertram, W., *Differential geometry, Lie groups and symmetric spaces over general base fields and rings*, Mem. Amer. Math. Soc. **192** (900) (2008), x+202, arXiv: `math.DG/0502168`.

[8] Bertram, W., *Homotopes and conformal deformations of symmetric spaces*, J. Lie Theory **18** (2008), 301–333, `math.RA/0606449`.

[9] Bertram, W., *Is there a Jordan geometry underlying quantum physics?*, Int. J. Theor. Phys. **47** (2) (2008), 2754—2782, arXiv: `math-ph/0801.3069`.

[10] Bertram, W., *On the hermitian projective line as a home for the geometry of quantum theory.*, AIP Conference Proceedings 1079, p. 14–25 (XXVII Workshop on Geometrical Methods in Physics, Bialowieza 2008), American Institute of Physics, New York, 2008.

[11] Bertram, W., Bieliavsky, P., *Homotopes of symmetric spaces. I : Construction by algebras with two involutions*, arXiv: `math.DG/1011.2923`.

[12] Bertram, W., Bieliavsky, P., *Homotopes of symmetric spaces. II : Structure Variety and Classification*, arXiv: `math.DG/1011.3161`.

[13] Bertram, W., Glöckner, H., Neeb, K.-H., *Differential calculus over general base fields and rings*, Exposition. Math. **22** (2004), 213–282, arXiv: `math.GM/030330`.

[14] Bertram, W., Kinyon, M., *Associative geometries. I: Torsors, linear relations and grassmannians*, J. Lie Theory **20** (2) (2010), 215–252, arXiv: `math.RA/0903.5441`.

[15] Bertram, W., Kinyon, M., *Associative geometries. II: Involutions, the classical torsors, and their homotopes*, J. Lie Theory **20** (2) (2010), 253–282, arXiv: `math.RA/0909.4438`.

[16] Bertram, W., Neeb, K.-H., *Projective completions of Jordan pairs. I: The generalized projective geometry of a Lie algebra*, J. Algebra **277** (2) (2004), 193–225, arXiv: `math.RA/0306272`.

[17] Bertram, W., Neeb, K.-H., *Projective completions of Jordan pairs. Part II: Manifold structures and symmetric spaces* (avec K.-H. Neeb), Geom. Dedicata **112** (1) (2005), 73–113, arXiv: `math.GR/0401236`.

[18] Chenal, J., *Generalized flag geometries and manifolds associated to short Z-graded Lie algebras in arbitrary dimension*, C. R. Math. Acad. Sci. Paris **347** (2009), 21–25, arXiv: `1007.4076v1[math.RA]`.

[19] Chu, Ch.-H., *Jordan Structures in Geometry and Analysis*, Cambridge University Press, 2012.

[20] Connes, A., *Non–commutative Geometry*, Academic Press, 1994.

[21] Emch, G., *Mathematical and Conceptual Foundations of 20th Century Physics*, North Holland, 1985.

[22] Faraut, J., Koranyi, A., *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

[23] Grgin, E., Petersen, A., *Algebraic implications of composability of physical systems*, Comm. Math. Phys. **50** (1976), 177–188.

[24] *Jordan archive* (Jordan preprint server) `http://molle.fernuni-hagen.de/~loos/jordan/index.html`.

[25] Kaneyuki, S., *On classification of parahermitian symmetric spaces*, Tokyo J. Math. **8** (1985), 473–482.

[26] Koecher, M., *The Minnesota Notes on Jordan Algebras and Their Applications* (reprint), Lecture Notes in Mat., vol. 1710, Springer, Berlin, 1999.

[27] Koufany, K., *Réalisation des espaces symétriques de type Cayley*, C. R. Math. Acad. Sci. Paris **318** (1994), 425–428.

[28] Loos, O., *Symmetric Spaces I*, Benjamin, New York, 1969.

[29] Loos, O., *Jordan triple systems, R-spaces and bounded symmetric domains*, Bull. Amer. Math. Soc. **77** (1971), 558–561.

[30] Loos, O., *Jordan Pairs*, Lecture Notes in Math., vol. 460, Springer, Berlin, 1975.

[31] Loos, O., *Bounded Symmetric Domains and Jordan Pairs*, University of California, Irvine, 1977, [See `http://molle.fernuni-hagen.de/~loos/jordan/index.html`].

[32] Loos, O., *Charakterisierung symmetrischer R–Räume durch ihre Einheitsgitter*, Math. Z. **189** (1985), 211–226.

[33] McCrimmon, K., *A Taste of Jordan Algebras*, Springer-Verlag, New York, 2004.

[34] Nagano, Tadashi, *Transformation groups on compact symmetric spaces*, Trans. Amer. Math. Soc. **118** (1965), 428–453.

[35] Springer, T. A., *Jordan Algebras and Algebraic Groups*, Classics in Mathematics, Springer-Verlag, 1998, Reprint of the 1973 edition.

[36] Takeuchi, Masaru, *Cell decompositions and Morse equalities on certain symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect. I **12** (1965), 81–192.

[37] Upmeier, H., *Symmetric Banach Manifolds and Jordan C^*-algebras*, North-Holland Math. Stud., North-Holland Publishing Co., Amsterdam, 1985.

[38] Upmeier, H., *Jordan algebras in analysis, operator theory, and quantum mechanics*, CBMS Regional Conference Series in Mathematics, 67. Published for the Conference Board of the Mathematical Sciences, Washington, DC, American Mathematical Society, Providence, RI, 1987.

Institut Élie Cartan de Lorraine (IECL),
Université de Lorraine at Nancy, CNRS, INRIA,
Boulevard des Aiguillettes, B.P. 239,
F-54506 Vandœuvre-lès-Nancy, France
E-mail: `wolfgang.bertram@univ-lorraine.fr`