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ON THE NATURAL TRANSFORMATIONS OF WEIL BUNDLES

Ivan Kolář

Abstract. First we deduce some general results on the covariant form of the natural transformations of Weil functors. Then we discuss several geometric properties of these transformations, special attention being paid to vector bundles and principal bundles.

1. Introduction

All manifolds and maps are assumed to be infinitely differentiable and the manifolds are paracompact. Unless otherwise specified, we use the terminology and notation from the book [5].

In 1953, A. Weil used the concept of local algebra A (today called Weil algebra) to introduce the bundle $T^A M$ of infinitely near points of type A over a manifold M, [5], [6]. By definition, A is a finite dimensional, commutative, associative and unital algebra of the form $\mathbb{R} \times N$, where $\mathbb{R}$ are the real multiples of the unit of A and N is the ideal of all nilpotent elements of A. In [6], Weil defined

$$T^A M = \operatorname{Hom}(C^\infty M, A) \tag{1}$$

as the set of all algebra homomorphisms of the algebra $C^\infty M$ of all smooth functions on M into A. Every map $f\colon M \to \bar{M}$ induces $f^*\colon C^\infty \bar{M} \to C^\infty M$. For $\varphi \in \operatorname{Hom}(C^\infty M, A)$, one sets

$$T^A f(\varphi) = \varphi \circ f^* \in \operatorname{Hom}(C^\infty \bar{M}, A)\,. \tag{2}$$

This determines a bundle functor $T^A\colon \mathcal{M}f \to \mathcal{FM}$ called Weil functor. One verifies easily that T^A preserves products. If B is another Weil algebra and $\mu \in \operatorname{Hom}(A, B)$ is an algebra homomorphism, the rule

$$\mu_M(\varphi) = \mu \circ \varphi \in \operatorname{Hom}(C^\infty M, B) \tag{3}$$

defines a natural transformation $\mu_M\colon T^A M \to T^B M$.

The simpliest example of a Weil algebra is

$$\mathbb{D}^r_k = \mathbb{R}[x_1, \dots, x_k] / \langle x_1, \dots, x_k\rangle^{r+1} \tag{4}$$

where $\mathbb{R}[x_1, \dots, x_k]$ is the algebra of all polynomials in k undetermined. In particular, $\mathbb{D}^1_1 = \mathbb{D}$ is the classical algebra of dual (or Study) numbers.

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About 1986, the following fundamental result was deduced, see [5] for details. Let $m\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the multiplication of reals.

Theorem 1. *Let F be a product preserving bundle functor on $\mathcal{M}f$. Then $F\mathbb{R}$ is a Weil algebra with respect to the multiplication Fm and F concides with the Weil functor $T^{F\mathbb{R}}$. The natural transformations $t\colon T^A\to T^B$ are in bijection with the algebra homomorphisms $t_{\mathbb{R}}\colon A\to B$.*

One finds easily that $T^{\mathbb{D}^r_k}$ coincides with the functor T^r_k of (k,r)-velocities by C. Ehresmann, [1]. In particular, $T^{\mathbb{D}} = T$ is the classical tangent functor.

Since $A=\mathbb{R}\times N$ is finite dimensional, there is an integer r such that $N^{r+1}=0$. The smallest r with this property will be called the order $\mathit{ord}\, A$ of A. On the other hand, the dimension wA of the vector space N/N^2 is said to be the width of A. A Weil algebra A of width k and order r will be called Weil (k,r)-algebra. In [3], we deduced

Lemma. *Every Weil (k,r)-algebra is a factor algebra of $\mathbb{D}^r_k$. If π, $\varrho\colon \mathbb{D}^r_k\to A$ are two surjective algebra homomorphisms, then there is an algebra isomorphism $\sigma\colon \mathbb{D}^r_k\to\mathbb{D}^r_k$ satisfying $\pi=\varrho\circ\sigma$.*

Having in mind (1) and (2), we can say that the original Weil's approach is contravariant. In [3], we developed systematically the following covariant approach. Since π is determined up to an isomorphism $\mathbb{D}^r_k\to\mathbb{D}^r_k$, the following definition is independent of the choice of π.

Definition. We say that two maps γ, $\delta\colon \mathbb{R}^k\to M$ determine the same A-velocity $j^A\gamma = j^A\delta$, if for every smooth function $\varphi\colon M\to\mathbb{R}$,

$$\pi\big(j^r_0(\varphi\circ\gamma)\big)=\pi\big(j^r_0(\varphi\circ\delta)\big)\,. \tag{5}$$

In [3], we deduced

Proposition 1. *The space $\{j^A\gamma;\gamma\colon \mathbb{R}^k\to M\}$ of all A-velocities on M coincides with T^AM. For every $f\colon M\to\bar M$, we have*

$$T^Af(j^A\gamma)=j^A(f\circ\gamma)\,. \tag{6}$$

2. The covariant form of natural transformations

Consider an algebra homomorphism $\mu\colon A\to B$, $wB=p$. Applying μ to a map $\gamma\colon \mathbb{R}^k\to M$, we obtain a commutative diagram

$$\begin{tikzcd} T^A\mathbb{R}^k \arrow[r, "T^A\gamma"] \arrow[d, "\mu_{\mathbb{R}^k}"'] & T^AM \arrow[d, "\mu_M"] \\ T^B\mathbb{R}^k \arrow[r, "T^B\gamma"] & T^BM \end{tikzcd} \tag{7}$$

Consider $j^A\,\mathrm{id}_{\mathbb{R}^k}\in T^A\mathbb{R}^k$. Then $\mu_{\mathbb{R}^k}(j^A\,\mathrm{id}_{\mathbb{R}^k})\in T^B\mathbb{R}^k$ is of the form $j^B(\bar\mu)$, where $\bar\mu\colon \mathbb{R}^p\to\mathbb{R}^k$. Hence (7) implies

Proposition 2. *For every natural transformation $\mu\colon T^A \to T^B$ there exists a polynomial map $\bar{\mu}\colon \mathbb{R}^p \to \mathbb{R}^k$ such that*

$$\mu_M(j^A\gamma) = j^B(\gamma \circ \bar{\mu})\,. \tag{8}$$

□

Conversely, let $j^B(\bar{\mu}) \in T^B\mathbb{R}^k$, $\bar{\mu}(0) = 0$, determines a natural transformation $T^A \to T^B$ by (8). We may write $A = \mathbb{R}[x_1, \dots, x_k]/I$, where I is an ideal satisfying $\langle x_1, \dots, x_k\rangle^{r+1} \subset I$. Let $P_h(x_1, \dots, x_k)$ be some polynomials generating I, $h = 1, \dots, s$. Then $b_i = \mu(\pi(x_i))$ determine an algebra homomorphism $A \to B$ if and only if $P_h(b_1, \dots, b_k) = 0$. This describes all algebra homomorphisms $A \to B$.

Remark. It is worth mentioning that the class of Weil functors is closed even with respect to the existence of underlying lower order functors. For every $q \le r$, N_A^{q+1} is an ideal in A. Write $A_q = A/N_A^{q+1}$ for the factor algebra and $\pi_q\colon A \to A_q$ for the factor projection. Since $\mu(N_A) \subset N_B$, every natural transformation $\mu\colon T^A \to T^B$ is projectable over a natural transformation $\mu_q\colon T^{A_q} \to T^{B_q}$, $q \le r$. In the case $q = r - 1$, we proved in [2] that $T^AM \to T^{A_{r-1}}M$ is an affine bundle, whose associated vector bundle is the pullback of $TM \otimes N_A^r$ over $T^{A_{r-1}}M$.

In what follows, we study the geometric properties of the Weilian prolongations of certain geometric structures by using the covariant approach. Special attention is paid to the geometric properties of the natural transformations.

3. Prolongation of vector and affine bundles

Consider a vector bundle $p\colon E \to M$. The vector addition in E and the multiplication of vectors by real numbers are two maps

$$E \times_M E \to E\,, \quad \mathbb{R} \times E \to E\,. \tag{9}$$

Applying T^A, we construct

$$T^AE \times_{T^AM} T^AE \to T^AE\,, \quad A \times T^AE \to T^AE\,, \tag{10}$$

If we restrict the second map to $\mathbb{R} \subset A$, we obtain

Proposition 3. *$T^Ap\colon T^AE \to T^AM$ is also a vector bundle.*

Proof. This can be deduced by discussing the prolongation of all corresponding diagrams. However, our concept of A-velocity offers a more geometric proof, that is of jet-like character. We have

$$T^AE = \{j^A\gamma, \gamma\colon \mathbb{R}^k \to E\}\,.$$

For $c \in \mathbb{R}$, we define $c(j^A\gamma(\tau)) = j^A(c\gamma(\tau))$, $\tau \in \mathbb{R}^k$. If $p \circ \gamma_1 = p \circ \gamma_2$, we set

$$j^A\big(\gamma_1(\tau)\big) + j^A\big(\gamma_2(\tau)\big) = j^A\big(\gamma_1(\tau) + \gamma_2(\tau)\big)$$

with addition in the individual fibers of E. Then we verify easily that T^AE is a vector bundle. □

Let $\bar{E} \to \bar{M}$ be another vector bundle and $f\colon E \to \bar{E}$ be a $\mathcal{VB}$-morphism over $\underline{f}\colon M \to \bar{M}$. Analogously to Proposition 3, we deduce

Proposition 4. *$T^A f\colon T^A E \to T^A \bar{E}$ is a $\mathcal{VB}$-morphism over $T^A \underline{f}\colon T^A M \to T^A \bar{M}$.*

Consider an algebra homomorphism $\mu\colon A \to B$. Applying μ to a vector bundle $p\colon E \to M$, we obtain a commutative diagram

$$\begin{tikzcd} T^A E \arrow[r, "\mu_E"] \arrow[d, "T^A p"'] & T^B E \arrow[d, "T^B p"] \\ T^A M \arrow[r, "\mu_M"] & T^B M \end{tikzcd} \tag{11}$$

Proposition 5. (11) *is a $\mathcal{VB}$-morphism.*

Proof. By (8), we have $\mu_E(j^A\gamma_1 + j^A\gamma_2) = j^B((\gamma_1+\gamma_2)\circ\bar{\mu}) = j^B(\gamma_1\circ\bar{\mu}) + j^B(\gamma_2\circ\bar{\mu}) = \mu_E(j^A\gamma_1) + \mu_E(j^A\gamma_2)$. □

Let $q\colon H \to M$ be an affine bundle with the associated vector bundle $p\colon E \to M$.

Proposition 6. *$T^A q\colon T^A H \to T^A M$ is an affine bundle with the associated vector bundle $T^A p\colon T^A E \to T^A M$.*

Proof. Every two points $h_1, h_2 \in H_x$ determine a vector $h_1 - h_2 \in E_x$, $x \in M$. For two maps δ_1, $\delta_2\colon \mathbb{R}^k \to H$ satisfying $q\circ\delta_1 = q\circ\delta_2$, we have $j^A\delta_1$, $j^A\delta_2 \in T^A H$ and $j^A(\delta_1 - \delta_2) \in T^A E$ with the required properties. □

Proposition 7. *Let $\mu\colon A \to B$ be an algebra homomorphism. Then*

$$\begin{tikzcd} T^A H \arrow[r, "\mu_H"] \arrow[d, "T^A q"'] & T^B H \arrow[d, "T^B q"] \\ T^A M \arrow[r, "\mu_M"] & T^B M \end{tikzcd} \tag{12}$$

is an affine bundle morphism, whose associated $\mathcal{VB}$-morphism is (11)*.*

4. The flow natural exchange and connnections

In general, the Weil algebra corresponding to the iteration $T^B \circ T^A$ is $A\otimes B$, [6], [3]. The exchange algebra homomorphism $\mathrm{ex}\colon A\otimes B \to B\otimes A$ induces the following natural transformation $\mathrm{ex}_M\colon T^A T^B M \to T^B T^A M$. Let $t \in \mathbb{R}^k$ and $\tau \in \mathbb{R}^p$. Hence every $Z \in T^A T^B M$ is of the form

$$Z = j^A\big(t \mapsto j^B(\tau \mapsto \delta(t,\tau))\big)\,,$$

where $\delta\colon \mathbb{R}^k \times \mathbb{R}^p \to M$. Then

$$\mathrm{ex}_M(Z) = j^B\big(\tau \mapsto j^A(t \mapsto \delta(t,\tau))\big)\,. \tag{13}$$

Consider the case $B = \mathbb{D}$, i.e. $T^B = T$. Hence $T^A T M \to T^A M$ and $T T^A M \to T^A M$ are vector bundles. Write $\varkappa^A$ for $\mathrm{ex}\colon A\otimes\mathbb{D} \to \mathbb{D}\otimes A$. Using (13), one deduces easily that

$$\varkappa^A_M\colon T^A T M \to T T^A M \tag{14}$$

is a $\mathcal{VB}$-morphism over $T^A M$. Consider a vector field $X\colon M \to TM$ and write $\mathcal{T}^A X\colon T^A M \to TT^A M$ for its flow prolongation. On the other hand, we have $T^A X\colon T^A M \to T^A TM$. By (13), we deduce

$$\mathcal{T}^A X = \varkappa^A_M \circ T^A X\,. \tag{15}$$

That's why $\varkappa^A$ is called the flow natural exchange.

Consider a general connection Γ on a fibered manifold $p\colon Y \to M$ in the form of a lifting map

$$\Gamma\colon Y \times_M TM \to TY\,. \tag{16}$$

In [5], we constructed the induced general connection $\mathcal{T}^A\Gamma$ on $T^A p\colon T^A Y \to T^A M$ by a commutative diagram

$$\begin{tikzcd} T^A Y \arrow[d, equal] & \times_{T^A M} & T^A TM \arrow[r, "T^A\Gamma"] \arrow[d, "\varkappa^A_M"] & T^A TY \arrow[d, "\varkappa^A_Y"] \\ T^A Y & \times_{T^A M} & TT^A M \arrow[r, "\mathcal{T}^A\Gamma"] & TT^A Y \end{tikzcd} \tag{17}$$

Analogously to Proposition 5, we deduce

Proposition 8. *Let $\mu\colon A \to B$ be an algebra homomorphism. Then $\mathcal{T}^A\Gamma$ and $\mathcal{T}^B\Gamma$ are μ-related, i.e. the following diagram commutes*

$$\begin{tikzcd} T^A Y \arrow[d, "\mu_Y"'] & \times_{T^A M} & TT^A M \arrow[r, "\mathcal{T}^A\Gamma"] \arrow[d, "T\mu_M"] & TT^A Y \arrow[d, "T\mu_Y"] \\ T^B Y & \times_{T^B M} & TT^B M \arrow[r, "\mathcal{T}^B\Gamma"] & TT^B Y \end{tikzcd} \tag{18}$$

5. Principal and associated bundles

By [3], for a Lie group G, $T^A G$ and $T^B G$ are also Lie groups and $\mu_G\colon T^A G \to T^B G$ is a group homomorphism. For a principal bundle $P(M,G)$, $T^A P(T^A M, T^A G)$ is also a principal bundle.

Proposition 9. *$\mu_P\colon T^A P \to T^B P$ is a $\mathcal{PB}$-morphism with the associated group homomorphism $\mu_G\colon T^A G \to T^B G$.*

Proof. Write $(u,g) \mapsto u\cdot g$ for the right action of G on P. Consider $v(\tau)\colon \mathbb{R}^k \to P$, $\gamma(\tau)\colon \mathbb{R}^k \to G$. Then $\mu_P(j^A v \cdot j^A\gamma) = j^B\big((v\cdot\gamma)\circ\bar{\mu}\big) = \mu_P(j^A v)\circ\mu_G(j^A\gamma)$. $\square$

Consider a left action $l\colon G \times S \to S$ of a Lie group G on a manifold S.

Proposition 10. *The actions $T^A l$ and $T^B l$ are μ-related, i.e. the following diagram commutes*

$$\begin{tikzcd} T^A G \arrow[d, "\mu_G"'] & \times & T^A S \arrow[r, "T^A l"] \arrow[d, "\mu_S"] & T^A S \arrow[d, "\mu_S"] \\ T^B G & \times & T^B S \arrow[r, "T^B l"] & T^B S \end{tikzcd} \tag{19}$$

Proof. Consider $j^A\gamma \in T^A G$, $j^A\sigma \in T^A S$. Clockwise we first obtain $T^A l(j^A\gamma, j^A\sigma)$ and then $T^B l(\gamma \circ \bar{\mu}, \sigma \circ \bar{\mu})$. Counterclockwise we first have $j^B(\gamma \circ \bar{\mu})$ and $j^B(\sigma \circ \bar{\mu})$ and then $T^B l(\gamma \circ \bar{\mu}, \sigma \circ \bar{\mu})$. $\square$

According to [4], if Γ is a principal connection on a principal bundle $P(M,G)$, then $\mathcal{T}^A\Gamma$ is a principal connection on the principal bundle $T^A P(T^A M, T^A G)$.

Every principal connection Γ on P induces a general connection Γ_S on every associated bundle $P[S,l]$, [5]. In [4], we deduced

$$\mathcal{T}^A\Gamma_{T^A S} = \mathcal{T}^A(\Gamma_S)\,. \tag{20}$$

By Propositions 9 and 10 we obtain easily

Proposition 11. *The connections $\mathcal{T}^A(\Gamma_S)$ and $\mathcal{T}^B(\Gamma_S)$ are μ-related.*

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Institute of Mathematics and Statistics,
Masaryk University, Kotlářská 2, 611 37 Brno,
Czech Republic
E-mail: kolar@math.muni.cz