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GENERALIZED SCHAUDER FRAMES

S.K. Kaushik and Shalu Sharma

Abstract. Schauder frames were introduced by Han and Larson [9] and further studied by Casazza, Dilworth, Odell, Schlumprecht and Zsak [2]. In this paper, we have introduced approximative Schauder frames as a generalization of Schauder frames and a characterization for approximative Schauder frames in Banach spaces in terms of sequence of non-zero endomorphism of finite rank has been given. Further, weak* and weak approximative Schauder frames in Banach spaces have been defined. Finally, it has been proved that E has a weak approximative Schauder frame if and only if E^* has a weak* approximative Schauder frame.

1. Introduction

Dennis Gabor [8] in 1946 introduced a fundamental approach to signal decomposition in terms of elementary signals. While addressing some deep problems in non-harmonic Fourier series, Duffin and Schaeffer [6] in 1952 abstracted Gabor's method to define frames for Hilbert spaces. Later, in 1986, Daubechies, Grossmann and Meyer [5] found new applications to wavelet and Gabor transforms in which frames played an important role.

Frames are generalizations of orthnormal bases in Hilbert spaces. The main property of frames which makes them useful is their redundancy. Representation of signals using frames is advantageous over basis expansions in a variety of practical applications. Many properties of frames make them useful in various applications in mathematics, science and engineering. In particular, frames are widely used in sampling theory, wavelet theory, wireless communication, signal processing, image processing, differential equations, filter banks, geophysics, quantum computing, wireless sensor network, multiple-antenna code design and many more. The reason for such wide applications is that frames provide both great liberties in design of vector space decompositions, as well as quantitative measure on computability and robustness of the corresponding reconstructions. For a nice and comprehensive survey on various types of frames, one may refer to [1, 4] and the references therein.

The notion of frames has been extended to Banach spaces by Feichtinger and Grochenig [7]. They introduced the notion of atomic decomposition for Banach spaces. Another notion called Banach frames for Banach spaces was introduced by

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Grochenig. Casazza, Han and Larson [3] carried out a study of atomic decompositions and Banach frames. Schauder frames for Banach spaces were introduced by Han and Larson [9] and were further studied in [10, 11, 12, 14].

Recently, sparsity has become a key concept in various areas of applied mathematics and engineering. Sparse signal processing methodologies explore the fundamental fact that many types of signals can be represented by only a few non-zero coefficients when choosing a suitable basis or, more generally, a frame. In this paper, we introduce a generalization of a Schauder frame called approximative Schauder frame which has sparsity in its nature in the sense that it can be characterized by a sequence of non-zero endomorphisms of finite rank (Theorem 3.6). A necessary and sufficient condition for approximative Schauder frames in Banach spaces is given. Commuting approximative Schauder frames in Banach spaces has been defined. A sufficient condition for shrinking commuting approximative Schauder frame has been proved. Weak* and weak approximative Schauder frames in Banach spaces have been defined. Finally it is shown that E has a weak approximative Schauder frames if and only if E^* has a weak* approximative Schauder frame.

2. Preliminaries

Throughout this paper E denotes an infinite dimensional Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and E^* denotes the conjugate space of E. For a sequence $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$, $[x_n]$ denotes the closure of linear span of $\{x_n\}$ in the norm toplogy of E and $[\tilde{f_n}]$ the closure of $\{f_n\}$ in $\sigma(E^*, E)$ topology.

Definition 2.1 ([9]). Let E be a Banach space and let $\{x_n\}$ be a sequence in E and $\{f_n\}$ be sequence in E^*. Then the pair $(\{x_n\}, \{f_n\})$ is called a Schauder frame for E if

$$x = \sum_{n=1}^{\infty} f_n(x) x_n \,, \quad \text{for all} \quad x \in E \,.$$

Definition 2.2 ([13]). A Banach space E is said to have bounded approximation property if there exists $\lambda \geq 1$ such that the identity operator $I_E \colon E \to E$ can be approximated, uniformly on every compact subset of E, by linear operators of finite rank, of norm $\leq \lambda$, that is, if there exists a constant $\lambda \geq 1$ with the property: for every compact subset $Q \subset E$ and for every $\epsilon > 0$ there exists an endomorphism $u = u_{Q,\epsilon} \in L(E, E)$ of finite rank, of norm $\|u\| \leq \lambda$, such that $\|u(x) - x\| < \epsilon$ $(x \in Q)$.

Definition 2.3 ([13]). A Banach space E is said to have a λ duality approximation property, if for every $\epsilon > 0$ and every pair of finite dimensional subspaces G of E and Γ of E^*, there exists an endomorphism $u \in L(E, E)$ of finite rank such that

$$\begin{aligned} \|u(y) - y\| &< \epsilon \|y\| \,, \quad (y \in G) \\ \|u^*(h) - h\| &< \epsilon \|h\| \,, \quad (h \in \Gamma) \\ \|u\| &< \lambda \,. \end{aligned}$$

3. Approximative Schauder frames

We begin with the following definition of approximative Schauder frames.

Definition 3.1. Let E be a Banach space, $\{x_n\} \subset E$ and $\{h_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}} \subset E^*$, where $\{m_n\}$ is an increasing sequence of positive integers. Then the pair $\big(\{x_n\}, \{h_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}\big)$ is called an approximative Schauder frame for E if

$$x = \lim_{n\to\infty} \sum_{i=1}^{m_n} h_{n,i}(x) x_i \quad \text{for all} \quad x \in E\,.$$

The following is an example of an approximative Schauder frame.

Example 3.2. Let $E = l^1$. Let $\{e_n\}$ be the sequence of unit vectors in E. Define $\{x_n\} \subset E$, and $\{f_n\} \subset E^*$ by

$$x_1 = \frac{e_1}{2}\,, \quad x_2 = \frac{e_1}{2}\,, \quad x_n = e_{n-1}\,, \quad n = 3, 4, \dots$$

$$f_1(x) = \xi_1\,, \quad f_2(x) = \xi_1\,, \quad f_n(x) = \xi_{n-1}\,, \quad n \geq 3\,, \quad x = \{\xi_n\} \in E\,.$$

Now, define $\{h_{n,i}\}_{i=1,2,\dots,n} \subset E^*$ by

$$h_{1,1} = f_1\,, \quad h_{2,1} = f_2\,, \quad h_{2,2} = f_2\,, \quad h_{n,i} = f_i\,, \quad i = 3, 4, \dots\,.$$

Note that

$$\lim_{n\to\infty} \sum_{i=1}^{n} h_{n,i}(x) e_i = \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x) e_i = x\,, \qquad x \in E\,.$$

Hence $(\{x_n\}, \{h_{n,i}\}_{i=1,2,\dots,n})$ is an approximative Schauder frame for E.

Remark 3.3.

(1) Every Schauder frame is an approximative Schauder frame. Indeed, let $(\{x_n\}, \{f_n\})$ be a Schauder frame for E. Put $h_{n,i} = f_i$, $i = 1, 2, \dots, n$; $n \in \mathbb{N}$. Then $(\{x_n\}, \{h_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}})$ is an approximative Schauder frame for E as

$$\lim_{n\to\infty} \sum_{i=1}^{n} h_{n,i}(x) x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x) x_i = x\,, x \in E\,.$$

(2) An approximative Schauder frame may not be a Schauder frame.

Next, we give an example of approximative Schauder frame which is not a Schauder frame.

Example 3.4. Let $E = c_0$, $\{e_n\}$ be the sequence of unit vectors in E and $\{f_n\}$ be the sequence of standard unit vectors in E^*. Then, $(\{e_n\}, \{f_n\})$ is a Schauder frame for E. Define $\{h_{n,i}\}_{i=1,2,\dots,n} \subset E^*$ by

$$h_{n,i} = f_i\,, \qquad i = 1, 2, \dots, n, \quad n \in \mathbb{N}\,.$$

Note that

$$\lim_{n\to\infty}\sum_{i=1}^{n} h_{n,i}(x)e_i = \lim_{n\to\infty}\sum_{i=1}^{n} f_i(x)e_i = x, \qquad x \in E\,.$$

Hence $(\{e_n\}, \{h_{n,i}\}_{i=1,2,\dots,n})$ is an approximative Schauder frame for E but not a Schauder frame for E. Indeed, if we let $x = \{\xi_n\} \in E$. Then

$$\begin{aligned} h_{1,1}(x)e_1 + h_{2,1}(x)e_2 + h_{2,2}(x)e_3 + h_{3,1}(x)e_4 + h_{3,2}(x)e_5 + h_{3,3}(x)e_6 + \dots \\ = \xi_1 e_1 + \xi_1 e_2 + \dots \\ = (\xi_1, \xi_1, \xi_2, \xi_1, \dots) \\ \neq x\,. \end{aligned}$$

In the following example, we construct an approximative Schauder frame for $\ell^2(\mathbb{N})$ from a sequence which is not a frame for $\ell^2(\mathbb{N})$.

Example 3.5. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and $\{e_n\}$ be the sequence of standard unit vectors in $\mathcal{H}$. The sequence $\{x_n\} \in \mathcal{H}$ defined by $x_n = \dfrac{e_n}{n}$, $n \in \mathbb{N}$, is not a frame for $\mathcal{H}$ but there exist a sequence $\{h_{n,i}\}_{\substack{i=1,2,\dots,n\\ n\in\mathbb{N}}} \in \mathcal{H}$ such that $(\{x_n\}, \{h_{n,i}\}_{\substack{i=1,2,\dots,n\\ n\in\mathbb{N}}})$ is an approximative Schauder frame for $\mathcal{H}$. In fact, if we take $h_{n,i} = ie_i$, $i = 1, 2, \dots, n$; $n \in \mathbb{N}$, then

$$\begin{aligned} \lim_{n\to\infty}\sum_{i=1}^{n} h_{n,i}(x)x_i &= \lim_{n\to\infty}\sum_{i=1}^{n} i\; e_i(x)\frac{e_i}{i} \\ &= x\,, \qquad x \in \mathcal{H}\,. \end{aligned}$$

Further, one may note that $(\{x_n\}, \{h_{n,i}\}_{\substack{i=1,2,\dots,n\\ n\in\mathbb{N}}})$ is not even a Schauder frame for $\mathcal{H}$.

Next, we give a characterization of an approximative Schauder frame in terms of a sequence of non zero endomorphisms of finite rank.

Theorem 3.6. *A Banach space E has an approximative Schauder frame if and only if there exists a sequence $\{v_n\} \subset B(E)$ of non zero endomorphisms of finite rank such that $x = \sum_{i=1}^{\infty} v_i(x)$, $x \in E$ and $\sup \|\sum_{i=1}^{n} v_i\| \leq \lambda$, for some $\lambda > 0$.*

Proof. Let $\{x_n\} \in E$ and $\{h_{n,i}\} \in E^*$ be the sequences such that $\big(\{x_n\}, \{h_{n,i}\}_{\substack{i=1,2,\dots,m_n\\ n\in\mathbb{N}}}\big)$ is an approximative Schauder frame for E where $\{m_n\}$ is an increasing sequence of positive integers. Define

$$u_n(x) = \sum_{i=1}^{m_n} h_{n,i}(x)x_i\,, \quad x \in E\,,\ n \in \mathbb{N}\,.$$

Then for each $n \in \mathbb{N}$, u_n is a well defined continuous linear mapping on E with $\dim u_n(E) < \infty$ such that $\lim\limits_{n\to\infty} u_n(x) = x$, $x \in E$. Also by using the principle of uniform boundedness, $\sup\limits_{1\leq n<\infty} \|u_n\| < \infty$. Without loss of generality we may assume that $u_1 \neq 0$ and $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Define $v_1 = u_1$, $v_{2n} = v_{2n+1} = \frac{1}{2}(u_{n+1} - u_n)$, for all $n \in \mathbb{N}$.

Then $\{v_n\}$ is a sequence of non zero endomorphism of finite rank in $B(E)$ such that

$$\begin{aligned}\sum_{i=1}^{n} v_i(x) &= u_1(x) + \frac{1}{2}\{(u_2(x) - u_1(x)) + (u_2(x) - u_1(x))\} \\ &\quad + \frac{1}{2}\{\,(u_3(x) - u_2(x)) + (u_3(x) - u_2(x))\,\} + \dots \\ &= u_n(x)\,, \qquad x \in E\,.\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} v_i(x) = \lim_{n\to\infty} u_n(x) = x\,, \qquad x \in E\,.$$

Also

$$\sup_{1\leq n<\infty} \Big\| \sum_{i=1}^{n} v_i \Big\| = \sup_{1\leq n<\infty} \|u_n\| < \infty\,.$$

Conversely, taking $u_n = \sum\limits_{i=1}^{n} v_i$, $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} u_n(x) = x$, $x \in E$. Since for each $n \in \mathbb{N}, u_n(E)$ is finite dimensional, there exists a sequence $\{y_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E and a total sequence $\{g_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E^* such that

$$u_n(x) = \sum_{i=m_{n-1}+1}^{m_n} g_{n,i}(x) y_{n,i}\,, \qquad x \in E\,,\ n \in \mathbb{N}\,,$$

where $\{m_n\}$ is an increasing sequence of positive integers with $m_0 = 0$. Define $\{x_n\} \in E$ and $\{h_{n,i}\} \in E^*$ by

$$x_i = y_{n,i}, \quad i = m_{n-1}+1, \dots, m_n\,, \quad n \in \mathbb{N}$$

and

$$h_{n,i} = \begin{cases} 0, & \text{if} \quad i = 1, 2, \dots, m_{n-1}; \\ g_{n,i} & \text{if} \quad i = m_{n-1}+1, \dots, m_n. \end{cases}$$

Then, for each $x \in E$ and $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} h_{n,i}(x) x_i = \lim_{n\to\infty} u_n(x) = x\,.$$

Hence $\Big(\{x_n\}, \{h_{n,i}\}_{\substack{i=1,2,\dots,m_n \\ n\in\mathbb{N}}}\Big)$ is an approximative Schauder frame for E. □

Remark 3.7. For the converse of the above result we do not require the assumption $\sup \Big\| \sum\limits_{i=1}^{n} v_i \Big\| < \infty$.

Now, in view of the Theorem 3.6 we give definition for approximative Schauder frame of operators and λ approximative frame of operators of a Banach space E.

Definition 3.8. A sequence of finite rank endomorphisms $\{u_n\} \subset L(E,E)$ is called an approximative Schauder frame of operators of Banach space E, if

$$x = \lim_{n\to\infty} u_n(x)\,, \qquad x \in E\,.$$

If $\sup\limits_{1\le n<\infty} \|u_n\| \le \lambda$, we say $\{u_n\}$ is a λ-approximative Schauder frame(of operators) of E.

The following result gives a relation between λ duality approximation property and λ- approximative Schauder frame.

Theorem 3.9. *Let E be a Banach space such that E^* is separabel. Let $\lambda \ge 1$. Then E has the λ-duality approximation property if and only if E has a λ-approximative Schauder frame $\{u_n\}_{n\in\mathbb{N}}$ satisfying*

$$f = \lim_{n\in\mathbb{N}} u_n^*(f)\,, \qquad f \in E^*\,.$$

Proof. Assume that E has λ-duality approximation property. Let $\{y_n\} \subset E$ and $\{h_n\} \subset E^*$ be dense sequences, $G_n = [y_1, y_2, \dots, y_n]$ and $\Gamma_n = [h_1, h_2, \dots, h_n]$. Then by λ-duality approximation property, for each $n \in \mathbb{N}$ there exists a finite rank endomorphism $\{u_n\} \in B(E)$ such that

$$\|u_n(x) - x\| \le \frac{1}{n}\|x\|\,, \qquad (x \in G_n, n = 1, 2, \dots)$$

$$\|u_n^*(f) - f\| \le \frac{1}{n}\|f\|\,, \qquad (f \in \Gamma_n, n = 1, 2, \dots)$$

$$\|u_n\| \le \lambda\,, \qquad (n = 1, 2, \dots)$$

since $G_n \subset G_{n+1}$, $\Gamma_n \subset \Gamma_{n+1}$ and $\|u_n\| \le \lambda$, $n \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} u_n(x) = x\,, \quad x \in E \quad \text{and} \quad f = \lim_{n\to\infty} u_n^*(f), f \in E^*$$

Conversely, let E has a λ-approximative Schauder frame satisfying

$$f = \lim_{n\to\infty} u_n^*(f)\,, \quad f \in E^*\,.$$

So, there exists a sequence of endomorphism $\{u_n\} \in L(E,E)$ of finite rank such that

$$\lim_{n\to\infty} u_n(x) = x\,, \quad x \in E$$

$$\|u_n\| < \lambda\,.$$

This implies that E has λ-duality approximation property. □

Definition 3.10. Let E be a Banach space. A sequence of non zero endomorphisms $\{u_n\} \subset L(E,E)$ is called a commuting λ- approximative Schauder frame for Banach space E if it is a λ- approximative Schauder frame satisfying

$$u_i u_j = u_j u_i = u_i, \qquad (i < j)$$

Next, we give a sufficient condition for shrinking commuting approximative Schauder frame.

Theorem 3.11. *Let E be a Banach space such that E^* is separable and let E has λ-duality approximation property for some $\lambda \geq 1$. Then E has a shrinking commuting approximating Schauder frame $\{v_n\}_{n\in\mathbb{N}}$ (i.e. such that $\{v_n^*\}_{n\in\mathbb{N}}$ is a commuting approximative Schauder frame of E^*).*

Proof. Let $0 < \epsilon_n < 1$, $\lim\limits_{n\to\infty} \epsilon_n = 0$ and let $\{y_n\}$ be a dense sequence in E. For any $f \in E^*$, the subspaces $G_1 = [y_1]$ of E, $\Gamma_1 = [f]$ of E^* and for ϵ_1 there exists an operator $v_1 \in L(E, E)$ of finite rank, such that

$$v_1|_{G_1} = I_{G_1}\,, \quad v_1^*|_{\Gamma_1} = I_{\Gamma_1}\,, \quad \|v_1\| \leq \lambda + \epsilon_1\,.$$

For $G_2 = [v_1(E) \bigcup \{y_2\}]$, $\Gamma_2 = v_1^*(E^*)$ and ϵ_2 there exists $v_2 \in L(E, E)$ of finite rank, such that

$$v_2|_{G_2} = I_{G_2}\,, \quad v_2^*|_{\Gamma_2} = I_{\Gamma_2}, \quad \|v_2\| \leq \lambda + \epsilon_2\,.$$

Taking $G_3 = [v_2(E) \bigcup \{y_3\}]$, $\Gamma_3 = v_2^*(E^*)$ and ϵ_3 and continuing in this way indefinitely we obtain two sequences of subspaces $\{G_n\}$, $\{\Gamma_n\}$ and a sequence of endomorphisms $\{v_n\} \subset L(E, E)$ of finite rank. Now for each $y \in G_n$ we have $y = v_n(y) \in v_n(E)$ and

$$G_1 \subset v_1(E) \subset G_2 \subset v_2(E) \subset G_3 \ldots$$

Therefore, $v_i(E) \subset G_j$ for all $i < j$. Hence, we have

$$v_j v_i(x) = v_i(x)\,, \qquad (i < j, x \in E)\ ,$$

$$\text{i.e.} \quad v_j v_i = v_i\,, \qquad (i < j)\ .$$

Similarly we have $\Gamma_n \subset v_n^*\,(E^*) = \Gamma_{n+1}$ $(n = 1, 2, \dots)$

$$v_j^* v_i^* = v_i^*\,, \qquad (i < j)\,.$$

Now

$$f\left(v_i v_j(x)\right) = v_j^* v_i^*(f)(x) = v_i^*(f)(x) = f\left(v_i(x)\right)\ , \quad x \in E\,,\ \ f \in E^*\,,\ \ i < j$$

$$v_i v_j(x) = v_i(x)\,, \quad (x \in E, i < j)\ .$$

Thus, we have

$$v_i v_j = v_j v_i = v_i\,, \qquad (i < j)\ .$$

Also

$$v_i^* v_j^*(f)(x) = f\left(v_j v_i\right)(x) = f\left(v_i(x)\right) = v_i^*(f)(x)\,,$$

$$v_i^* v_j^* = v_i^*\ .$$

Thus, we have

$$v_i^* v_j^* = v_i^* = v_j^* v_i^*\,, \qquad (i < j)\ .$$

Further, $\overline{\bigcup_n v_n(E)} = E$ and so for any arbitrary $y \in \bigcup_n v_n(E)$, $\lim_n v_n(y) = y$, for all $\in E$ which proves the result. ☐

In view of the above theorem we have the following corollary.

Corollary 3.12. *Let E be a Banach space. If E^* is separable and has bounded approximation property, then E has a shrinking commuting approximative Schauder frame.*

4. Weak* approximative Schauder frames

Definition 4.1. Let E be a Banach space. A sequence of finite rank endomorphisms $\{u_n^*\} \in L(E^*, E^*)$ is said to be a weak* approximative Schauder frame for E^* if it satisfies

$$f(x) = \lim_{n\to\infty} u_n^*(f)(x)\,, \qquad (x \in E,\ f \in E^*)\ .$$

If $\sup\limits_{1\le n<\infty} \|u_n^*\| \le \lambda$, we say $\{u_n^*\}$ is a weak* λ-approximative Schauder frame (of operators) of E.

Next, we give a characterization of weak* approximative Schauder frame.

Theorem 4.2. *A separable Banach space E has an approximative Schauder frame if and only if E^* has a weak* approximative Schauder frame.*

Proof. Let $\{v_n^*\} \in L(E^*, E^*)$ be a weak* approximative Schauder frame for E^*. Then $\sup\limits_{1\le n<\infty} \|v_n^*\| \le \lambda < \infty$ and for each finite dimensional subspace Γ of E^* and $n \in \mathbb{N}$ there exists a finite rank operator $t_{\Gamma,\frac{1}{n}}$ on E such that

$$t^*_{\Gamma,\frac{1}{n}} = v_n^*(h)\,, \quad (h \in \Gamma)$$

$$\left\|t_{\Gamma,\frac{1}{n}}\right\| \le \lambda + 1\,.$$

Now, let D be the directed set of all pairs $\left(\Gamma, \frac{1}{n}\right)$ where Γ is a finite dimensional subspace of E^* and $n \in \mathbb{N}$ and where $\left(\Gamma_1, \frac{1}{n_1}\right) \ge \left(\Gamma_2, \frac{1}{n_2}\right)$ if and only if $\Gamma_1 \supset \Gamma_2$ and $\frac{1}{n_1} \le \frac{1}{n_2}$. Furthermore, for each $d = \left(\Gamma, \frac{1}{n}\right) \in D$ let $t_d \in L(E, E)$ be a finite rank endomorphism such that

$$\|t_d^*(h) - h\| \le \epsilon\,\|h\|\,, \qquad (h \in \Gamma)$$

and

$$\|t_d\| \le \lambda\,.$$

If $f \in E^*$ and $\epsilon > 0$, then putting $d_0 = \left([f]\,, \frac{1}{n}\right)$ it follows that

$$\|t_d^*(f) - f\| \le \epsilon\,\|f\|\,, \qquad (d \ge d_0)\ .$$

Hence

$$\lim_{d\in D} t_d^*(f) = f\,, \quad (f \in E^*)$$

and

$$\|t_d\| \le \lambda\,.$$

Then, we have

$$\lim_{d\in D} f\left(t_d(x)\right) = \lim_{d\in D} t_d^*(f)(x) = f(x)\,, \qquad (x\in E, f\in E^*)$$

this implies

$$t_d(x) \xrightarrow{w} x\,, \qquad (x\in E)\ .$$

Let $\{y_n\}$ be a dense sequence in E. Then we have sequences $\{d_n\}\subset D$ and $\{m_n\}\subset \mathbb{N}$ with $m_1 < m_2 < \ldots$ and for each n, nonnegative numbers $\alpha_{m_{n-1}+1},\ldots\alpha_{m_n}$ with $\sum_{i=m_{n-1}+1}^{m_n}\alpha_i = 1$ such that

$$\Big\| \sum_{i=m_{n-1}+1}^{m_n} \alpha_i t_{d_i}(y_j) - y_j \Big\| < \frac{1}{n} \qquad (j=1,2,\ldots n; n=1,2,\ldots)\ .$$

Then for the finite rank operators,

$$u_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i t_{d_i}$$

we have

$$\lim_{n\to\infty} u_n(y_j) = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i t_{d_i}(y_j) = y_j\,, \qquad (j=1,2,\ldots)$$

and

$$\|u_n\| \leq \sum_{i=m_{n-1}+1}^{m_n} \alpha_i \left\|t_{d_i}\right\| \leq \lambda+1 \qquad (n=1,2,\ldots)\ .$$

This implies $\lim\limits_{n\to\infty} u_n(x) = x$, $x\in E$. So $\{u_n\}$ is an approximative Schauder frame for E.

Conversely, assume that $\{u_n\}\subset L(E,E)$ be an approximative Schauder frame for E. Then

$$\lim_{n\to\infty} u_n(x) = x\,, \qquad x\in E\,.$$

Now, we have

$$f(x) = \lim_{n\to\infty} f\left(u_n(x)\right) = \lim_{n\to\infty} u_n^*(f)(x)\,, \qquad (x\in E, f\in E^*)$$

this implies $\{u_n^*\}$ is a weak* approximative Schauder frame. □

5. Weak approximative Schauder frames

Definition 5.1. Let E be a Banach space. A sequence of finite rank endomorphisms $\{u_n\}\in L(E,E)$ is said to be a weak approximative Schauder frame for E if it satisfies

$$f(x) = \lim_{n\to\infty} f\left(u_n(x)\right)\ , \qquad (x\in E,\ f\in E^*)\ .$$

If here $\sup\limits_{1\leq n<\infty}\|u_n\|\leq\lambda$, we say $\{u_n\}$ is a weak λ-approximative Schauder frame (of operators) of E.

Next, we give a characterization of weak approximative Schauder frame.

Theorem 5.2. *A separable Banach space E has a weak approximative Schauder frame if and only if E^* has a weak* approximative Schauder frame.*

Proof. Assume that $\{u_n^*\}$ is a weak* λ- approximative Schauder frame for E^*. Then for each finite dimensional subspace Γ of E^* and $n \in \mathbb{N}$ there exists a finite rank operator $t_{\Gamma,\frac{1}{n}}$ on E such that

$$t^*_{\Gamma,\frac{1}{n}} = u_n^*(h) \qquad (h \in \Gamma)$$
$$\left\|t_{\Gamma,\frac{1}{n}}\right\| \leq \lambda + 1\,.$$

Now, let A be the directed set of all pairs $\left(\Gamma, \frac{1}{n}\right)$ where Γ is a finite dimensional subspace of E^* and $n \in \mathbb{N}$ and where $\left(\Gamma_1, \frac{1}{n_1}\right) \geq \left(\Gamma_2, \frac{1}{n_2}\right)$ if and only if $\Gamma_1 \supset \Gamma_2$ and $\frac{1}{n_1} \leq \frac{1}{n_2}$. Furthermore, for each $\alpha = \left(\Gamma, \frac{1}{n}\right) \in A$, $t_\alpha \in L(E, E)$ be a finite rank endomorphism. Let $x \in E, f \in E^*$ and $\epsilon > 0$ be arbitrary. Then by definition of weak* Schauder frame there exist $n_0 \in \mathbb{N}$ such that

$$|u_n^*(f)(x) - f(x)| < \epsilon \qquad (n \geq n_0)\ .$$

Putting $\alpha_0 = \left([f], \frac{1}{n_0}\right) \in A$ and using above inequality, we have

$$\begin{aligned}
|f\left(t_\alpha(x)\right) - f(x)| &= |t^*_\alpha(f)(x) - u_n^*(f)(x) + u_n^*(f)(x) - f(x)| \\
&\leq |t^*_\alpha(f)(x) - u_n^*(f)(x)| + |u_n^*(f)(x) - f(x)| \\
&= |u_n^*(f)(x) - f(x)| \\
&< \epsilon\,, \qquad (\alpha \geq \alpha_0)\ .
\end{aligned}$$

This implies $\lim\limits_{\alpha} f\left(t_\alpha(x)\right) = f(x)$, $x \in E$, $f \in E^*$. Hence E has a weak approximative schauder frame.

Conversely, assume that E has a weak approximative Schauder frame. Then we have

$$\lim_{n\to\infty} f\left(u_n(x)\right) = f(x)\,, \qquad x \in E,\ f \in E^*\,.$$

Now,

$$f(x) = \lim_{n\to\infty} f\left(u_n(x)\right) = \lim_{n\to\infty} u_n^*(f)(x)\,.$$

This implies $\{u_n^*\}$ is a weak* approximative Schauder frame. ☐

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S.K. Kaushik,
Department of Mathematics, Kirori Mal College,
University of Delhi,
Delhi 110 007, India
E-mail: `shikk2003@yahoo.co.in`

Shalu Sharma,
Department of Mathematics, University of Delhi,
Delhi 110 007, India
E-mail: `shalusharma_21@yahoo.co.in`