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EIGHTY FIFTH ANNIVERSARY OF BIRTHDAY AND SCIENTIFIC LEGACY OF PROFESSOR MILOŠ RÁB

Ondřej Došlý, Eva Jansová, and Josef Kalas

Abstract. The paper is concentrated on Professor Miloš Ráb and his contribution to the theory of oscillatory properties of solutions of second and third order linear differential equations, the theory of differential equations with complex coefficients and dynamical systems, and the theory of nonlinear second order differential equations. At the beginning, we take a brief look at the most important moments in his life. Afterwards, we describe his scientific activities on mentioned theories.

Miloš Ráb – an outstanding mathematician

Professor Miloš Ráb was well remembered by the 20th century mathematicians in the Czech lands and worldwide. A number of papers, such as [44], were written on

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the occasion of his important anniversaries bringing details about his pedagogical activities and highlighting his scientific achievement.

This paper is written on the occasion of 85th anniversary of the birth of Professor Ráb to commemorate his contribution to the theory of differential equations investigated by him for more than half century. During this time, Professor Ráb was an outstanding teacher and a lecturer at the Faculty of Science of Masaryk University. He supervised nine doctoral students, two of them, namely Josef Kalas and Jaromír Šimša, currently work at the Faculty of Science of Masaryk University.

Professor Ráb influenced several fields of differential equations. At the beginning of his research, he focused his attention on the oscillatory and asymptotic properties of the solutions of second and third order linear differential equations, later on extending the domain investigated to two-dimensional systems of differential equations with real coefficients. In early seventies, his first papers on differential equations with complex coefficients appeared, and, concurrently, he was concerned with the qualitative theory of nonlinear second order differential equations. Hence, he devoted himself mainly to differential equations with complex coefficients and dynamical systems. However, he returned to the asymptotic properties of the solutions of second order linear differential equations responding to the current developments in this field.

Before describing the scientific activities of Professor Ráb, let us recall some important moments in his life.

Miloš Ráb was born in Újezd u Brna on September 30 in 1928. After his graduation from a gymnasium in Brno in 1947, he enrolled at the Faculty of Science of Masaryk University in Brno to study mathematics and descriptive geometry. He started his academic career as a teacher of mathematics at Brno University of Technology in 1950. In the same year, he became a member of the Union of Czechoslovak Mathematicians and Physicists, working in the Brno local branch from that time onwards. A year later, in 1951, he completed his studies with honours by passing the state examination in Mathematics and Descriptive Geometry. In 1952, he joined the Department of Mathematics of the Faculty of Science of Masaryk University in Brno. He earned the degree of Doctor of Natural Sciences (RNDr.) defending his thesis called "Contributions to the Theory of Second-Order Differential Linear Equations" in 1953. Led by Professor Borùvka, in 1955, he began external postgraduate studies finishing them after two years by defending his dissertation thesis named "Oscillatory and Asymptotic Properties of the Integrals of a Linear Differential Equation of the Third Order" to receive his Candidate of Science degree. In 1961 he was appointed Associate Professor of Mathematics. In 1967 he defended his doctoral thesis, "Asymptotic Formulas for the Solutions of Ordinary Differential Equations", to obtain the title of Doctor of Science (DrSc.). He was designated Professor of Mathematics in 1969. A year later, he became the head of the Department of Mathematical Analysis, a post he held for sixteen years. In addition to this important office, he was a long-time member of the Scientific Board of the Faculty of Science of Masaryk University (J. E. Purkyně University that time) and the first editor-in-chief of the international journal Archivum Mathematicum. In his life, he was honored many times, for example, in

1993 he was awarded the Gold Medal of Masaryk University in Brno and in 2000 appointed the first Professor emeritus of mathematics in the modern university history at the Faculty of Science of Masaryk University in Brno. Unexpectedly, he left the ranks of the Czech mathematicians on April 29 2007, passing away at the age of nearly 79 years.

With respect to the fact that in his long academic career Professor Ráb published about forty papers, two monographs and eight academic textbooks, we will mainly concentrate on his contribution to the theory of the oscillatory properties of solutions of second and third order linear differential equations, the theory of nonlinear second order differential equations and the theory of differential equations with complex coefficients and dynamical systems. We will look at his papers on these theories. Some other details on his contribution to the theory of oscillatory properties of solutions of second and third order linear differential equations can be found in [59] and some other details on the theory of nonlinear second order differential equations in [60]. Throughout the paper, we will try to keep the notation used by Professor Ráb in his papers.

1. Oscillation and asymptotic theory of linear differential equations

Let us now start with the beginnings of Professor Ráb's scientific work. His first papers [R1], [R5] and [R9] are concerned with third order linear differential equations, in particular, with describing their properties in terms of solutions of certain associated second-order equations. At the same time, he also concentrated on second-order linear differential equations. His papers [R4] and [R8] are devoted to this theory. His research in this field was mainly based on the results of transformation theory of Professor Borůvka [4].

1.1. Linear second order differential equations.

The paper [R8] by Professor Ráb devoted to the oscillatory properties of second order linear differential equations published in 1959 is commonly regarded as his principal contribution to the oscillation theory of these equations. This paper is cited e.g. in the monographs [14] by P. Hartman and [55] by W. T. Reid which are basic reference books in the oscillation theory of linear differential equations, so we comment some results of this paper in detail in this subsection.

In this paper, M. Ráb presented a unifying approach to almost all previously known oscillation criteria for the second order linear differential equation

$$[p(x)y']' + q(x)y = 0\,, \tag{1}$$

where p', q are continuous functions in the interval $J = [x_0, \infty)$ and $p(x) > 0$. The list of references to that paper consists of 43 items written by 32 authors (including Ráb's paper [R4]) concerned with this problem. Due to the length of the paper [R8], only its main and the most important parts will be highlighted here.

The following theorem can be seen as a fundamental result, it is cited e.g. in the very recent paper [12] and reads as follows.

Theorem 1 ([R8], Hauptsatz)**.** *Equation* (1) *is oscillatory if and only if there exists a positive differentiable function g in J such that*

$$\lim_{x\to\infty}\int_{x_0}^{x}\frac{1}{p(u)}\exp\Big\{2\int_{x_0}^{u}\Big[\frac{1}{p(s)g^2(s)}\int_{x_0}^{s}[q(t)g^2(t)-p(t)g'^2(t)]\,\mathrm{d}t-a\Big]\,\mathrm{d}s\Big\}\,\mathrm{d}u=\infty \tag{2}$$

for every constant a.

The main significance of [R8] lies in the fact that Theorem 1 covers in some cases many of the previously proved criteria for the oscillation of the solutions of equation (1). Moreover, many of these criteria and their corollaries had been proved before only in the case of equation (1) with $p(x)\equiv 1$.

In the first part of [R8], various sufficient (and/or necessary) conditions under which equation (1) is oscillatory are established. In these criteria, M. Ráb used the so-called Riccati technique, consisting in the relationship between disconjugacy of (1) and the solvability of the Riccati equation (related to (1) by the Riccati substitution $u=rx'/x$)

$$u'+\frac{u^2}{p(x)}+q(x)=0\,,$$

combined with the transformation method elaborated by O. Borůvka. To explain his idea more in details, let us recall that the transformation $y=f(x)z$ with a positive differentiable function f transforms equation (1) into an equation of the same form

$$[P(x)z']'+Q(x)z=0\,, \tag{3}$$

with

$$P(x)=p(x)f^2(x),\quad Q(x)=f(x)[p(x)f'(x)]'+q(x)f^2(x\,), \tag{4}$$

and that this transformation preserves oscillatory nature of transformed equations. In particular, when the transformation function is $f(x)=\sqrt{y_1^2(x)+y_2^2(x)}$, y_1,y_2 being solutions of (1) with the Wronskian $p(x)[y_1'(x)y_2(x)-y_1(x)y_2'(x)]\equiv\pm 1$, (such transformation is sometimes referred to as the Bohl transformation, [3]) then f satisfies the equation $f[(pf')'+qf]=p^{-1}f^{-2}$ and hence the resulting equation (3) takes the form

$$\Big(\frac{u'}{Q(x)}\Big)'+Q(x)u=0\,. \tag{5}$$

The last equation can be solved explicitly, its solutions are $u_1(x)=\sin\int^x Q(t)\,dt$, $u_2(x)=\cos\int^x Q(t)\,dt$ and this enabled the formulation of Theorem 1 in the above given form, since (5) is oscillatory if and only if $\int^\infty Q(x)\,\mathrm{d}x=\infty$. Note that this transformation was used earlier by W. Leighton in [37] and R. Moore in [39], but in a different context. Also, Ráb's idea of the proof of Theorem 1 was used later in various modifications to other linear equations. For example, in the paper [8] by O. Došlý, a statement similar to that of Theorem 1 is formulated for the second order difference equation

$$\Delta(p_k\Delta x_k)+q_kx_{k+1}=0\,,\quad \Delta x_k=x_{k+1}-x_k\,.$$

M. Ráb also dealt in [R8] with comparative oscillation criteria. The origin of this theory is connected with C. Sturm and his celebrated work [63], where (now called Sturmian) separation and comparison theorems were proved. M. Ráb proved (again using the transformation of (1) to (5)) that there exists no "ideal" comparative equation, in the following sense.

Theorem 2 ([R8], Satz 5)**.** *Suppose that equation* (1) *is oscillatory. There exists a continuous function* q_1*, such that* $q_1(x) < q(x)$*,* $x \in J$*, and the differential equation (the Sturmian minorant to* (1)*)*

$$[p(x)y']' + q_1(x)y = 0$$

is oscillatory as well.

The idea of the proof of Theorem 2 was consequently used to prove the nonexistence of an ideal comparison system for matrix second order differential equations and linear Hamiltonian systems in [9, 10] by O. Došlý.

As we have mentioned before, Theorem 1 covers many known oscillation criteria. Let us mention at least two of them. The next result was proved by P. Hartman in [13] in the particular case $p(x) \equiv 1$, $f(x) \equiv 1$. To prove this theorem, M. Ráb employed a mutual relationship between equations (1) and (3) emphasized beforehand, and the result reads as follows.

Theorem 3 ([R8], Satz 8)**.** *If there is a function* $f \in C^2$*,* $f(x) > 0$ *in* J*, such that the conditions*

$$-\infty < \int_{x_0}^{\infty} Q(x)\,\mathrm{d}x < \infty, \ \frac{1}{\ln x} \le P(x) \le \ln x\,,$$

$$\int_{x_0}^{\infty} \exp\Big(-4\int_{x_0}^{x} \frac{1}{P(t)} \int_{t}^{\infty} Q(s)\,\mathrm{d}s\,\mathrm{d}t\Big)\,\mathrm{d}x < \infty$$

hold for the functions (4)*, then equation* (1) *is oscillatory.*

Another well-known oscillation criterion (V.A. Kondratiev in [35]) for oscillation of (1) is a special case of the following theorem, where we put $g(x) = x^{\frac{1}{2}\alpha}$ with $\alpha < 1$ in the next statement.

Theorem 4 ([R8], Satz 9)**.** *Let* $\int_{x_0}^{\infty} \frac{\mathrm{d}x}{p(x)} = \infty$*. Suppose that there exists a function* $g \in C^1$*,* $g(x) > 0$ *for* $x \in J = [x_0, \infty)$*, such that*

$$\int_{x_0}^{x} [q(x)g^2(x) - p(x)g'^2(x)]\,\mathrm{d}x \ge M > -\infty\,,$$

$$\int_{x_0}^{\infty} \frac{G_+^2(x)}{p(x)g^2(x)}\,\mathrm{d}x = \infty\,, \quad \int_{x_0}^{\infty} \frac{G_-^2(x)}{p(x)g^2(x)}\,\mathrm{d}x = \infty\,,$$

where $G(x) = \int_{x_0}^{x} \{g'^2(t)p(t) - g^2(t)q(t)\}\,\mathrm{d}t + C - p(x)g'(x)g(x)$*,* $G_+ = \max(0, G)$*,* $G_- = \min(0, G)$*,* C *is a constant. Then equation* (1) *is oscillatory.*

Finally, in describing the results of the Ráb's fundamental paper [R8], note that the transformation of equation (1) to (3) and (5) is a "red fiber" in the whole paper [R8]. Roughly speaking, sufficient conditions for oscillation are formulated for equation (3) or (5) and then they are "transformed back" (in a sophisticated way) to the original equation (1).

At the end of this section, let us also briefly mention another M. Ráb's paper devoted to second order equation (1). In [R4] published in 1957, M. Ráb generalized the necessary condition for oscillation of the equation

$$y'' + A(x)y = 0, \quad A(x) \in C(J) \tag{6}$$

originally proved by L.D. Nikolenko in [42, 43], and a sufficient condition for the oscillation of (6) proved by M. Zlámal in [68].

1.2. **Linear third order differential equations.** M. Ráb's very first paper [R1] on this subject was published in 1955, the year in which he began his external postgraduate studies led by Professor Borůvka. In the first part of the paper, he investigated oscillatory properties of solutions of the self-adjoint third-order differential equation

$$y''' + 2A(x)y' + A'(x)y = 0\,, \tag{7}$$

in connection with solutions of the second-order differential equation

$$y'' + \frac{1}{2}A(x)\,y = 0\,, \tag{8}$$

assuming A to be a differentiable function in an interval (a, b) (this interval can be also unbounded). The principal idea used in this paper consists in the fact (see [58]) that a fundamental system of equation (7) can be chosen in the form y_1^2, y_1y_2, y_2^2, where y_1, y_2 are two linearly independent solutions of (8) and that any solution y of (7) satisfies the so-called Mammana's identity (see [38])

$$L[y(x)] = y(x)y''(x) - \frac{1}{2}y'^2(x) + A(x)y^2(x) = \text{const.}$$

In the second part of [R1], M. Ráb employed the previous results to study the solutions of the general linear third-order differential equation

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0\,, \tag{9}$$

with the functions $p_i \in C^{3-i}(J)$, $J = [x_0, \infty)$, $i = 1, 2, 3$. He used the transformation of (9) into the so-called canonical form (the transformation is described in [R5])

$$y''' + 2A(x)y' + (A'(x) + \omega(x))y = 0. \tag{10}$$

Professor Ráb derived a necessary and sufficient condition for the oscillation of a solution y of (10). He also derived a theorem about distribution of zeros of two linearly independent solutions of (10) in the interval (x_0, ∞), assuming that the perturbation term $\omega(x)$ in (10) is nonnegative. Note that the distribution of zeros of (10) along the line treated in [R1] was originally studied by G. D. Birkhoff in [2] (for $x \in [a, b]$) and by M. Zlámal in [69] (in an unbounded interval).

Equation (10) is as well the topic of Ráb's work [R5] published in 1958, but there it is assumed that the functions A and ω are continuous in the interval $[x_0, \infty)$. Professor Ráb examined the properties of solutions of (10) that are similar to those of solutions of this equation with constant coefficients. In the first part of [R5], he explored the properties of oscillatory and nonoscillatory solutions of this equation under the assumption that $\omega(x) \geq 0$. The second part is devoted to the case of $\omega(x) \leq 0$.

The paper [R9] is Ráb's last paper devoted to the oscillatory properties of third-order linear differential equations. In this paper, published in 1960, he generalized a sufficient condition for the existence of a non-oscillatory solution of (10), originally published by G. Sansone (see [57]). This paper on the relationship between oscillation of second and third order linear differential equations was a motivation for several subsequent works of Brno mathematicians on this subject. Let us mention here the monograph of F. Neuman [41] devoted to the transformation theory of higher order linear differential equation (there, the relationship between (7) and (8) and its extension to higher order equations plays the crucial role). The problem of oscillatory and asymptotic properties of solutions of third order linear and nonlinear differential equations has been deeply investigated by Z. Došlá in collaboration with M. Cecchi and M. Marini, see [7] and references therein. Recently, asymptotic properties of solutions of higher order nonlinear differential equations in connection with oscillatory properties of second order linear equation has been studied in [1].

1.3. **Asymptotic properties of solutions of linear differential equations.** In the sixties of the last century, Professor Ráb turned his attention to the asymptotic analysis of solutions of linear differential equations and systems. His results in this direction are also of unifying character and provide very precise estimates of the approximation of a solution using asymptotic formulas. Professor Ráb concentrated mainly on the second-order differential equations (1) and (6). He used, among others, the transformation theory of second order differential equations combined with a sophisticated application of fixed point theorems. The first paper in this area is [R6]. The main result of that paper states that if $h \in C^2[a, \infty)$ is a function satisfying

$$\int^{\infty} \left[h(x)(h''(x) - A(x)h(x)) - h^{-2}(x)\right] \, \mathrm{d}x < \infty \,,$$

then any solution of (6) and its derivative can be expressed by the formulas

$$\begin{aligned}
y(x) &= h(x)\Big[y_0 \sin\Big(\int_{x_0}^{x} \frac{1}{h^2(t)} + \phi_0\Big) + o(1)\Big]\,,\\
y'(x) &= h'(x)\Big[y_0 \sin\Big(\int_{x_0}^{x} \frac{1}{h^2(t)} + \phi_0\Big) + o(1)\Big]\\
&\quad + \frac{1}{h}\Big[y_0 \cos\Big(\int_{x_0}^{x} \frac{1}{h^2(t)} + \phi_0\Big) + o(1)\Big]\,, \quad y_0 \in \mathbb{R}\,.
\end{aligned}$$

In later papers, Professor Ráb extended the results of [R6] in various directions, in particular, to equation (1) both in oscillatory and nonoscillatory case. As a brief summary of Ráb's results in the asymptotic theory can be regarded the paper [R13]. Let us also mention that the results of [R13] are cited in the book [14] by P. Hartman, which is considered as one of the basic monographs on ordinary differential equations.

To complete survey of Ráb's contribution to the asymptotic theory, it is also worth mentioning his paper [R35], where he investigated certain linear integral inequalities containing multiple integrals. The results of this paper cover as particular cases the inequalities presented in several previous papers on the linear integral inequalities of Gronwall type. The results of [R35] were taken up, among others, by J. Šimša, Ráb's student at that time, in his paper [64]. Not only his students, but also other successful mathematicians were inspired by Ráb's sophisticated attitude to the research of integral inequalities. Let us mention at least B. G. Pachpatte's paper [46], in which some new discrete generalizations of one of Ráb's useful iterated integral inequality were established.

2. Qualitative properties of nonlinear differential equations

In the early 1970', first papers by M. Ráb [R26] and [R30] devoted to second order nonlinear differential equations appeared. At this time questions concerning the qualitative properties came to the fore. It was important to determine the required properties of solutions of a differential equation without solving it, that is, by using properties of coefficients of the equation.

2.1. **Periodic solutions of $x'' = f(x, x')$.** In 1972, M. Ráb presented his paper [R26] at the international conference Equadiff 3 in Brno. His results received a great deal of response thanks to their wide generalizing character. M. Ráb took inspiration in papers by G. Villari [67], S. Sedziwy [61] and J. W. Heidel [15]. These mathematicians determined conditions for the existence of periodic solutions of the equation $x'' + f(x)x'^{2n} + g(x) = 0$ with arbitrarily large periods. M. Ráb generalized both the equation and their results. In [R26], M. Ráb proved that, under suitable assumptions, there exist periodic solutions of $x'' = f(x, x')$ with arbitrarily large periods. In the proofs, he employed the equivalence of the equation $x'' = f(x, x')$ with the autonomous system $x' = y$, $y' = f(x, y)$ investigating the trajectories of this system. At the end of [R26], in the appendix *Periodic solutions of $x'' + f(x)x'^{2n} + g(x) = 0$ with arbitrarily large periods*, some applications of the preceding theorems to the Liénard's differential equation $x'' + f(x)x'^{2n} + g(x) = 0$ are given.

2.2. **Boundedness of solutions of second order nonlinear equations.**

In [R30], published in 1975, the equation

$$(p(t)x')' + q(t)x = h(t, x, x')\,, \tag{11}$$

is considered, where p and q are continuous functions on the interval $J = [a, \infty)$, $p(x) > 0$, and the function h is continuous on $J \times \mathbb{R}^2$. M. Ráb set the bounds for its solutions using the knowledge of the fundamental system of solutions of $(p(t)y')' + q(t)y = 0$. Another result presented here concerns the estimate of solutions of (11), where $q(t) \equiv 0$. In the paper [11] published in 1978, J. Futák, a Slovak mathematician, used similar methods as M. Ráb in [R30] to derive a statement for a nonlinear system with delay. Similar results, but for the third order nonlinear differential equation, were published in 1987 in [62] by S. Staněk.

2.3. **Asymptotic formulas for nonlinear equations with advanced argument.** In the paper [R39] published in 1987, the asymptotic formulas for solutions of the differential equation with advanced argument

$$\left(\frac{x'(t)}{r(t)}\right)' + q(t)f(x(g(t))) = 0 \tag{12}$$

are derived in terms of the so-called $(1, p)$-integral equivalence to the equation $(y'(t)/r(t))' = 0$. Two of three formulas presented here, but in a special case for the equation of the form $(x'(t))' + q(t)f(x(t)) = 0$ without deviated argument, were proved by M. Naito in [40]. At the end of [R39], M. Ráb applied the proved theorem to the generalized Emden-Fowler equation $(t^{-\mu}x')' + t^{\lambda}\sin t|x|^{\gamma}\operatorname{sgn} x = 0$, $t > 0$, $\gamma > 0$, $\mu \geq -1$, $\lambda < 0$.

3. Equations with complex-valued coefficients

Another large field of differential equations, to which Professor Ráb devoted himself and in which he achieved remarkable results, is the theory of differential equations with complex-valued coefficients. In total, M. Ráb published eleven papers concerning this topic. Papers on this theory can be divided in two groups. At first, Professor Ráb dealt with a linear second order differential equation with complex-valued coefficients and the Riccati differential equation with complex-valued coefficients. Later on, he investigated stability and asymptotic properties of dynamical systems in the plane.

3.1. **Global properties.** Let us focus on the results of global character. In these papers the asymptotic properties of solutions of autonomous differential equations with complex-valued coefficients are investigated in the whole $\mathbb{C}$, possibly in the widest suitable region $\Omega \subseteq \mathbb{C}$.

In [R21] published in 1970, the Riccati differential equation

$$z' = A(t) - z^2\,, \tag{13}$$

$A(t)$ being a continuous complex-valued function defined on $J = [t_0, \infty)$, is investigated. The investigation was motivated by the papers of C. Kulig [36] and Z. Butlewski [5], [6], where the asymptotic properties of real two-dimensional systems, which can be transformed into the real system of the form

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t} &= a(t) - X^2 + Y^2\,,\\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= b(t) - 2XY\,,\end{aligned} \tag{14}$$

were studied. Equation (13) is used to find conditions under which the trajectories of system (14), where $a(t)$, $b(t)$ are continuous real-valued functions on J, behave asymptotically like the trajectories of the autonomous system

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= \alpha - X^2 + Y^2\,, \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= \beta - 2XY\,, \end{aligned} \tag{15}$$

where α, β are real constants, $\alpha = \lim_{t\to\infty} a(t)$, $\beta = \lim_{t\to\infty} b(t)$. The idea of the proof consits in finding the detailed integral phase-portrait of the trajectories of (15) in the neighborhood of the stable singular point Λ, where Λ with $\operatorname{Re}\Lambda > 0$, is a square root of the number $\alpha + i\beta$, exploiting the Lyapunov function method and the theory of the Riccati differential equation (13) with complex-valued coefficients. The Riccati equation corresponding to system (15) is of the form $z' = A - z^2$, where $z = X + iY$, $A = \alpha + i\beta$. Some sufficient conditions are derived under which the trajectories of the equation (13) behave asymptotically like the trajectories of equation $z' = A - z^2$ with constant coefficients.

The paper [R23] is a continuation of [R21] and it was published a year later than [R21]. The case when the function $A(t)$ has a small modulus is studied here, using the same method as in [R21]. Some sufficient conditions are derived under which the trajectories of equation (13) behave asymptotically like the trajectories of the equation $z' = -z^2$.

In the paper [R31] published in 1975, the asymptotic behavior of the solutions of the differential equation

$$x'' + p(t)x' + q(t)x = 0 \tag{16}$$

with complex-valued continuous coefficients on $J = [t_0, \infty)$ is investigated. Some results concerning the real differential equation can be found in [R12]. It is obvious that the situation of the complex-valued coefficients is more complicated than the situation of the real ones. Professor Ráb employed the transformation of (16) to an associated Riccati differential equation $z' + z^2 + p(t)z + q(t) = 0$ and studied its solutions by means of the Lyapunov method and the Wazewski topological principle. Using this method, some results similar to those concerning the real case were proved. For example: let $p(t)$, $q(t) \in C^0(J)$ and let $\lim_{t\to\infty} p(t) = p_0$, $\lim_{t\to\infty} q(t) = q_0$. Further, let either $\int_{t_0}^{\infty} |\mathrm{d}p(t)| < \infty$ or $\int_{t_0}^{\infty} |\mathrm{d}q(t)| < \infty$. Define $\Delta^2 = p^2 - 4q$ and assume

$$\lim_{t\to\infty} \Delta(t) = \Lambda\,, \quad \operatorname{Re}\Lambda^{1/2} > 0\,.$$

Then there exists a fundamental system of solutions of (16) such that

$$u(t) \sim \exp\int^t \frac{1}{2}(-p + \Delta^{1/2})\,, \quad v(t) \sim \exp\int^t \frac{1}{2}(-p - \Delta^{1/2})\,.$$

In the paper [R33] published in 1977, Professor Ráb investigated the Riccati differential equation

$$z' = q(t) - p(t)z^2\,, \tag{17}$$

where p and q are complex-valued continuous functions defined on $J = [0, \infty)$. The asymptotic behavior of the solutions of (17) was established under the condition that there exists a complex number a such that $\mathrm{Re}[ap(t)] > 0$ and $\int_0^\infty \mathrm{Re}[ap(t)]\,\mathrm{d}t = \infty$. To prove this result Professor Ráb used the Lyapunov function method and the Wazewski topological principle. The idea is to consider equation (17) as a perturbation of the equation $w' = p(t)(a^2 - w^2)$, and then to compare the solutions of both equations. Professor Ráb employed the same method in another paper [R32] devoted to global properties of the Riccati differential equation of the form $g(t)w' + w^2 + p(t)w + q(t) = 0$ with continuous complex-valued coefficients.

Most of the results published in [R21], [R23], [R31] and [R33] dealing with the Riccati differential equation with complex-valued coefficients were generalized in a multitude of papers by J. Kalas and other authors. This is because the results on the asymptotic behavior of the solutions of certain general nonlinear differential equations studied in these papers can also be applied to the Riccati differential equation and other special types of equations.

The paper [R22], published in 1970, contains three theorems, proofs of which were omitted because of their simplicity. These theorems are important in the transformation theory of linear second order differential equations. Professor Ráb aimed to represent the solutions of the equation

$$[p(x)y']' + q(x)y = 0\,, \tag{18}$$

where $p(x) = p_1(x) + ip_2(x) \neq 0$, $q(x) = q_1(x) + iq_2(x)$ and p_1, p_2, q_1, q_2 are real continuous functions on the half-line $I = [a, \infty)$, with the aid of three specific systems.

In [R24] published in 1972, conditions are derived under which the solutions of the equation

$$[p(x)y']' - q(x)y = 0\,, \tag{19}$$

where p, q are continuous complex-valued functions on an interval $J = [a, \infty)$, $p(x) \neq 0$, can be aproximated by the solutions of the equation $[p(x)z']' = 0$. Professor Ráb was inspired by the paper [56] by U. Richard, where a detailed discussion of the asymptotic properties of (19) with real-valued coefficients were carried into effect. The problem of the asymptotic integration of (19) is based on finding of a differential equation of the same type with well known solutions approximating in a certain sense the solutions of (19). More generally, it is based on finding of a transformation of (19) into an equation which can be approximated in a suitable manner. In this paper, Professor Ráb concentrated on asymptotic expansion of solutions of (19), regarding this equation as a perturbation of the one-term equation $[p(x)z']' = 0$, which has the general solution

$$z(x) = \alpha + \beta \int_\zeta^x \frac{\mathrm{d}t}{p(t)}\,, \quad \zeta \in J\,.$$

Defining a suitable sequence $U^n z(x)$, it is proved that the series

$$y(x) = \sum_0^\infty U^n z(x)$$

is a formal solution of (19) with a formal quasi-derivative

$$p(x)y'(x) = \beta + \sum_1^\infty \int_\eta^x q(t)U^{n-1}z(t)\,\mathrm{d}t\,.$$

Under various assumptions on $p(x)$ and $q(x)$, the uniform convergence of these series on J is proved and the asymptotic properties of solutions of (19) and their derivatives are discussed. Finally, Professor Ráb noted that the proved theorems could have been applied to one certain equation the special form of which in the real case had appeared in the mentioned paper [56].

Let us turn our attention to the paper [R25] published in 1972. Here, oscillatory and asymptotic properties of solutions of the equation

$$[P(x)y']' + Q(x)y = 0 \tag{20}$$

are studied, where P and Q are complex-valued functions defined on an interval $J = (a, b)$, $a \geq -\infty$, $b \leq \infty$. The paper is divided into three parts. The first one aims to study the logarithmic derivative $y'(x)/y(x)$ of a solution $y(x)$ of (20), primarily at the study of its behavior in a neighbourhood of a zero of $y(x)$. In the second part, it is shown that every solution of (20) can be expressed in the form $y = r(x)\mathrm{e}^{i\theta(x)}$, where r, θ are real functions satisfying the system

$$[p(x)r']' + \left\{\frac{1}{4}p(x)\varphi'^2(x) + q(x)\cos[\psi(x) - \varphi(x)] - p(x)[\theta' + \frac{1}{2}\varphi'(x)]^2\right\} r = 0\,,$$

$$[p(x)r\theta']' + p(x)[\theta' + \varphi'(x)]r' + q(x)\sin[\psi(x) - \varphi(x)]r = 0\,,$$

where p, q, φ, ψ are real-valued functions, $p(x) > 0$, $p, q, \psi \in C^0(J)$, $\varphi \in C^1(J)$. Using this fact, it is possible to derive some disconjugacy criteria as well as estimates of the number of zeros of any solution in J and to deal with the growth properties of solutions of (20) for $t \to \infty$. Finally, the third part has the oscillatory and asymptotic properties of solutions of (20) as its subject. These properties are investigated using Sturm comparison theorems.

Some extensions of the results on the asymptotic behavior of the Riccati differential equation were accomplished by Z. Tesařová in [66] and by J. Kalas in [27]. The results of Professor Ráb on global properties of the equation (13) were generalized by J. Kalas to equations where the right-hand side is a perturbation of a holomorphic function in a simply connected region Ω containing zero. It is supposed that zero is a unique root of f and two cases are considered: (i) $z = 0$ is a simple root of f in Ω ([16]-[20]), (ii) $z = 0$ is a multiple root of f in Ω ([21]-[25]).

3.2. **Perturbed two-dimensional linear systems.** In the early nineties of the last century, M. Ráb and J. Kalas turned their attention to the investigation of the local behavior and estimations and stability properties of the solutions of the perturbed linear system

$$x' = A(t)x + h(t,x)\,, \tag{21}$$

where $A(t) = (a_{jk}(t))$, $j,k = 1,2$, is a real square matrix and a vector function h is defined by $h(t,x) = (h_1(t,x), h_2(t,x))$. By means of the method of complexification, they converted this system to one equation of the complex form

$$z' = a(t)z + b(t)\bar{z} + g(t,z)\,. \tag{22}$$

In the first paper [R43] devoted to dynamical systems published in 1990, Professor Ráb together with J. Kalas investigated the stability and asymptotic stability of the solutions of equation (22) in the stable case. Using the method of Lyapunov functions, various stability criteria for the zero solution of (22) and various of estimates for the solutions of (22) are given. The results complement or extend some previous results such as J. Osička [45] and K. Tatarkiewicz [65].

Using Wazewski topological principle, the same authors studied the asymptotic properties of system (21) in the semi-stable case in the paper [R45] published in 1992. The achieved results generalized those of J. Radzikowski [48]. The results were complemented by J. Kalas and J. Osička in [26], where the asymptotic behavior of solutions and the existence of bounded solutions of (21) in the unstable case were investigated.

The papers [R43] and [R45] motivated other authors for further investigations and the results for (21) were extended to more general equations. The asymptotic properties of the two-dimensional system with constant delay in the stable case were studied by J. Kalas and L. Baráová in [28], the asymptotic behavior of solutions under the conditions of instability was investigated by J. Kalas in [29], [30]. The equations with a finite system of constant delays were inspected by J. Rebenda in [49], [50]. The asymptotic properties of solutions of the two-dimensional system with nonconstant delays were investigated by J. Kalas and J. Rebenda in [31], [32], [34], [51]. The asymptotic behavior of the solutions of systems with a finite number of nonconstant delays was examined by J. Kalas and J. Rebenda in [33] and by J. Rebenda and Z. Šmarda in [52]-[54].

Epilogue

In his lifetime, Professor Ráb was known as really active person and anything mentioned above can hardly say enough about all his admirable results of both scientific and pedagogical work. In conclusion, we can say that the personality of Professor Miloš Ráb influenced not only the development of mathematics in Brno, but also the lives of many of his students and followers. Professor Ráb is remembered with respect and gratitude by all who had the opportunity to be with him at least for a short time.

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Department of Mathematics and Statistics,
Masaryk University Brno,
Kotlářská 2, 61137 Brno, Czech Republic
E-mail: dosly@math.muni.cz kalas@math.muni.cz jansova.e@fce.vutbr.cz