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NATURAL TRANSFORMATIONS OF CONNECTIONS ON THE FIRST PRINCIPAL PROLONGATION

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Abstract. We consider a vector bundle $E \to M$ and the principal bundle PE of frames of E. We determine all natural transformations of the connection bundle of the first order principal prolongation of principal bundle PE into itself.

Unless otherwise specified, we use the terminology and notation from the book [7]. All manifolds and maps are assumed to be infinitely differentiable.

In the monograph [7], the following assertion was deduced.

Proposition 1. *All natural operators transforming principal connection Λ on P^1M into principal connection Λ_1 on P^1M are of zero order and form a 3-parameter family*

$$\Lambda_1 = \Lambda + \Phi_1(\Lambda)$$

where $\Phi_1(\Lambda)$ is a natural $(1,2)$-tensor field of the form

$$\Phi_1 = a_1 T + a_2 \operatorname{Id}_{TM} \otimes c_1^1(T) + a_3 c_1^1(T) \otimes \operatorname{Id}_{TM} \qquad a_i \in \mathbb{R}\,,$$

where T denotes the torsion tensor of Λ and c_1^1 denotes the contraction to the first subscript.

We discuss the generalization of this problem, i.e. the case of the principal connections on the first order prolongation W^1P of a principal bundle P.

1. Principal connections

We consider a principal bundle $P = (P, M, \pi; G)$ with a structure group G. We denote by (x^λ, z^a) fibered coordinates on P, $\lambda = 1, \dots, \dim M$, $a = 1, \dots, \dim G$.

A principal connection on P is defined as lifting linear mapping

$$\Gamma\colon TM \to TP/G\,.$$

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In coordinates

$$\Gamma = \mathrm{d}^\lambda \otimes \big(\partial_\lambda + \Gamma^a{}_\lambda(x)\widetilde{b}_a\big), \tag{1}$$

where $\Gamma^a{}_\lambda(x)$ are functions on M and $(\widetilde{\mathfrak{b}}_a)$ is the base of vertical right invariant vector fields on P which are induced by the base $(\mathfrak{b}_a)$ of $\mathfrak{g}$.

If we identify Γ with the functions $\Gamma^a{}_\lambda(x)$ then Γ can be considered as a section of the bundle $QP \to M$ of principal connections on P.

Moreover, we have, [7],

Proposition 2. *Let $P \to M$ be a principal bundle and $QP \to M$ be the bundle of principal connections on P. Then QP is a 1-order G-gauge-natural affine bundle associated with the vector bundle $\mathrm{ad}(P) \otimes T^*M \to M$, which implies that the standard fiber of the functor Q is $\mathfrak{g} \otimes \mathbb{R}^{m*}$.*

2. Principal connections on W^1P

Definition 1. For every principal bundle $P = (P, M, \pi; G)$ we can define the principal bundle $W^1P = P^1M \times_M J^1P \equiv (W^1P, M, p; W^1_m G)$, where $P^1M = \mathrm{inv}\, J^1_0(\mathbb{R}^m, M)$. The principal bundle W^1P is called the principal first order gauge prolongation of P.

The group $W^1_m G$ is the group of 1-jets at $(0, e)$ of all automorphisms $\varphi\colon \mathbb{R}^m \times G \to \mathbb{R}^m \times G$ with $\underline{\varphi}(0) = 0$, where the multiplication μ is defined by the composition of jets,

$$\mu(j^1\varphi(0,e), j^1\psi(0,e)) = j^1(\psi \circ \varphi)(0,e)\,. \tag{2}$$

Let Γ be a principal connection on W^1P given in coordinates by

$$\Gamma_1 = d^\lambda \otimes \big(\partial_\lambda + \Lambda^\nu_{\mu\lambda}(x)\,\widetilde{\mathfrak{b}}^\mu_\nu + \Gamma^a_\lambda(x)\,\widetilde{\mathfrak{b}}_a + \Gamma^a{}_{\kappa\lambda}(x)\,\widetilde{\mathfrak{b}}^\kappa_a\big)\,.$$

We have two canonical group homomorphisms

$$\pi^1_0\colon W^1_m G \to G\,, \qquad p_1\colon W^1_m G \to G^1_m\,, \tag{3}$$

which induce homomorphisms of Lie algebras

$$\pi^1_0\colon \mathfrak{w}^1_{\mathfrak{m}}\mathfrak{g} \to \mathfrak{g}\,, \qquad p_1\colon \mathfrak{w}^1_{\mathfrak{m}}\mathfrak{g} \to \mathfrak{g}^1_{\mathfrak{m}}\,, \tag{4}$$

principal bundle morphisms, over the identity of M,

$$\pi^1_0\colon W^1P \to P\,, \qquad p_1\colon W^1P \to P^1M\,, \tag{5}$$

and homomorphisms of asociated vector bundles

$$\pi^1_0\colon \mathrm{ad}(W^1P) \to \mathrm{ad}(P)\,, \qquad p_1\colon \mathrm{ad}(W^1P) \to \mathrm{ad}(P^1M)\,. \tag{6}$$

The projections (5) of W^1P and the functor Q induce projections

$$\pi^1_0\colon QW^1P \to QP\,, \qquad p_1\colon QW^1P \to QP^1M\,, \tag{7}$$

so any principal connection Γ_1 on W^1P projects on a principal connection Λ_1 on P^1M and on a principal connection Γ_0 on P.

By Proposition 2, $QW^1P \to M$ is the affine bundle modeled over the vector bundle $\mathrm{ad}(W^1P) \otimes T^*M \to M$.

Proposition 3. *Let Γ_1 and $\bar{\Gamma}_1$ be two principal connections on W^1P such that they are over the same principal connections Λ_1 on P^1M and Γ_0 on P, then the difference $\Gamma_1 - \bar{\Gamma}_1$ is identified with a natural tensor field*

$$\Psi_1 \colon M \to \mathrm{ad}(P) \otimes T^*M \otimes T^*M\,.$$

Tensor field Ψ_1 has the maximal order 1 and is of the form

$$\Psi_1(j^1\Lambda, j^1\Gamma) = \bar{\Psi}_1(c, R[\widetilde{\Lambda}], R[K], T, \widetilde{\nabla}T)\,,$$

where $\bar{\Psi}_1$ is a zero order operator.

Proof. According to higher order Utiyama's invariant interaction, [3], any natural section Ψ_1 is given by tensorial operations from c, $R[\Gamma]$ and its covariant derivatives with respect to $\widetilde{\Lambda}$ and Γ, the curvature tensor $R[\widetilde{\Lambda}]$ of $\widetilde{\Lambda}$ and its covariant derivatives (with respect to $\widetilde{\Lambda}$) and the torsion tensor T and its covariant derivatives (with respect to $\widetilde{\Lambda}$). From the homogeneous function theorem it follows, that a natural tensor field $\Psi_1 \colon M \to \mathrm{ad}(P) \otimes T^*M \otimes T^*M$ can be constructed tensorially from $R[\Gamma]$, $R[\widetilde{\Lambda}]$ and from the covariant derivatives of T up to the order 1. So the maximal order of Ψ_1 is 1. □

Remark 1. Moreover, for every natural number r, $W^r \colon \mathcal{PB}_m(G) \to \mathcal{PB}_m(W^r_m G)$ is a functor. Let φ be a principal morphism in the category $\mathcal{PB}_m(G)$ over a base diffeomorphism $\underline{\varphi}$ then $W^r\varphi = (P^r\underline{\varphi}, J^r\varphi)$. W^r is then a gauge-natural bundle functor of order r and plays a fundamental role in the theory of gauge-natural bundles. Really, any r-th order G-gauge-natural bundle is a fiber bundle associated with W^rP.

Remark 2. If $G = \{e\}$ is the one-element group, then $M \times \{e\}$ is identified with M and $W^r(M \times \{e\}) = P^rM$. Hence many properties of W^rP can be viewed as a generalization of the case of P^rM.

3. The main result

Clearly, the solution of this problem depends essentially on the structure group of principal bundles. In this paper we give the answer for the linear gauge group $GL(n)$.

Let $E \to M$ be a vector bundle with m-dimensional base and n-dimensional fibers. Let us denote by (x^λ, y^i) local linear fiber coordinate charts on E. Let $PE \to M$ be the frame bundle of E, i.e. PE is the principal bundle with the structure group $GL(n)$ and the induced fiber coordinates (x^λ, x^i_j).

Principal connections on PE are in bijection with general linear connections on E. The coordinate expression of a linear connection K on E is of the type

$$K = \mathrm{d}^\lambda \otimes (\partial_\lambda + K_p{}^i{}_\lambda(x) y^p \partial_i)$$

and, if we consider K as a principal connection on PE,

$$K = \mathrm{d}^\lambda \otimes (\partial_\lambda + K_p{}^i{}_\lambda(x) x^p_j \partial^j_i) = \mathrm{d}^\lambda \otimes (\partial_\lambda + K_p{}^i{}_\lambda(x) \widetilde{b}^p_i)\,.$$

A principal connection Λ on P^1M has form

$$\Lambda = \mathrm{d}^\lambda \otimes \left(\partial_\lambda + \Lambda_\varrho{}^\mu{}_\lambda \widetilde{b}^\varrho_\mu\right).$$

We remark that principal connections on P^1M are in the bijection with classical connections on M (linear connections on TM).

A principal connection on W^1PE we denote by Γ_1 and we consider Γ_1 over connections Λ and K.

$$\Gamma_1 = \mathrm{d}^\lambda \otimes \left(\partial_\lambda + \Lambda_\varrho{}^\mu{}_\lambda \widetilde{b}^\varrho_\mu + K_p{}^i{}_\lambda \widetilde{b}^p_i + \Gamma_p{}^i{}_{\varrho\lambda} \widetilde{b}^{p\varrho}_i\right).$$

Lemma 1. *All natural operators transforming a classical connection Λ on M and a principal connection K on PE into principal connections Γ_0 on PE are of the maximal order zero and form a 1-parameter family*

$$\Gamma_0 = K + \Psi_0\,,$$

where Ψ_0 is a natural tensor field of the form

$$\Psi_0 = gId_E \otimes c^1_1(T),$$

where coefficient g is real number.

Proof. If we consider any natural principal connection Γ_0 on PE, then the difference $\Gamma_0 - K$ is a natural section $\Psi_0\colon M \to E \otimes E^* \otimes T^*M$. So to classify all natural Γ_0 it is sufficient to classify natural Ψ_0. Then from the reduction theorem and the homogeneous function theorem we obtain that Ψ_0 is of zero order and is obtained as a multiple of the tensor product of the identity of E and $(0,1)$-tensor field given by the contraction of the torsion of Λ. $\square$

Proposition 4. *All natural tensor fields $\Psi_1(\Lambda, K)\colon M \to E \otimes E^* \otimes \otimes^2 T^*M$ naturally given by a classical connection Λ on M and by a general linear connection K on E form a 10-parameter family of operators with expression given by*

$$\begin{aligned}\Psi_1 = a_1 R[K] + \mathrm{Id}_E \otimes \big[& a_2 c^1_1 R[K] + b_1 c^1_1 R[\Lambda] + b_2 c^1_2 R[\Lambda] \\ &+ c_1 c^{12}_{13}(T\otimes T) + c_2 c^{12}_{31}(T\otimes T) + c_3 c^{12}_{12}(T\otimes T) \\ &+ d_1 c^1_1(\nabla T) + d_2 c^1_1(\overline{\nabla T}) + d_3 c^1_3(\nabla T)\big]\end{aligned}$$

where all coefficients a_i, b_i, c_j, d_j are real numbers $i = 1, 2, \quad j = 1, 2, 3$.

Proof. Ψ_1 is the difference $\Gamma_1 - \bar{\Gamma}_1$, where Γ_1 and $\bar{\Gamma}_1$ be two principal connections on W^1PE such that they are over the same principal connections Λ_1 on P^1M and Γ_0 on PE. From the reduction theorems, [2], and the homogeneous function theorem, [7], follows, that Ψ_1 is given by the curvature tensor of K, by the tensor product of identity of E and by the contracted curvature tensor of Λ and of K, by contracted tensor product of T (two times) and by the contracted covariant differentials of T. $\square$

Remark 3. Let us remark that in [5] Proposition 4 was proved without using the reduction theorem, so the base of 10-parameter family Ψ_1 is different.

Theorem 1. *All natural operators transforming a pricipal connection Γ on W^1PE into principal connections Γ_1 on W^1PE are of the maximal order 1 and form a 14-parameter family.*

Proof. We have the 3-parameter family of connections Λ_1 given by Proposition 1, the 1-parameter family of connections Γ_0 given by Lemma 1 and the 10-parameter family of operators given by Proposition 4. □

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