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LEFSCHETZ COINCIDENCE NUMBERS OF SOLVMANIFOLDS WITH MOSTOW CONDITIONS

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Abstract. For any two continuous maps f, g between two solvmanifolds of the same dimension satisfying the Mostow condition, we give a technique of computation of the Lefschetz coincidence number of f, g. This result is an extension of the result of Ha, Lee and Penninckx for completely solvable case.

1. Introduction

For two compact oriented manifolds M_1 and M_2 of the same dimension, for two continuous maps $f, g\colon M_1 \to M_2$, as generalizations of the Lefschetz number and the Nielsen number for topological fixed point theory, the Lefschetz coincidence number $L(f,g)$ and the Nielsen coincidence number $N(f,g)$ are defined. The Nielsen coincidence number $N(f,g)$ is a lower bound for the number of connected components of coincidences of f and g. But computing the Nielsen coincidence number is very difficult. For some classes of manifolds, we have relationships between the Lefschetz coincidence number $L(f,g)$ and the Nielsen coincidence number $N(f,g)$.

Let G be a simply connected solvable Lie group with a lattice (i.e. cocompact discrete subgroup of G) Γ. We call G/Γ a solvmanifold. If G is nilpotent, we call G/Γ a nilmanifold.

For two solvmanifolds G_1/Γ_1 and G_2/Γ_2 with two continuous maps f, $g\colon G_1/\Gamma_1 \to G_2/\Gamma_2$, in [18], Wang showed the inequality

$$|L(f,g)| \le N(f,g).$$

Hence by Lefschetz coincidence number $L(f,g)$ we can estimate the number of coincidences of f, g. Suppose that G_1 and G_2 are completely solvable i.e. for any element of G the all eigenvalues of the adjoint operator of g are real. Then the de Rham cohomologies of solvmanifolds G_1/Γ_1 and G_2/Γ_2 are isomorphic to the cohomologies of the Lie algebras of G_1 and G_2. Moreover for the induced maps $f_*, g_* : \pi_1(G_1/\Gamma_1) \cong \Gamma_1 \to \Gamma_2 \cong \pi_1(G_2/\Gamma_2)$, we can take homomorphisms Φ, $\Psi\colon G_1 \to G_2$ which are extensions of f_*, g_*. In [4], Ha, Lee and Penninckx

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computed the Lefschetz coincidence number $L(f,g)$ by using "linearizations" Φ, Ψ of f and g.

In this paper, for a solvmanifold G/Γ we consider the Mostow condition: "$\mathrm{Ad}(G)$ and $\mathrm{Ad}(\Gamma)$ have the same Zariski-closure in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$" where Ad is the adjoint representation of a Lie group G. The condition: "G is completely solvable" is a special case of the Mostow condition (see [17] and [3]). In [12], Mostow showed that for a solvmanifold G/Γ satisfying the Mostow condition, the de Rham cohomology of G/Γ is also isomorphic to the cohomology of the Lie algebra of G. However, for two solvmanifolds G_1/Γ_1 and G_2/Γ_2 satisfying the Mostow conditions, extendability of homomorphisms between lattices Γ_1 and Γ_2 is not valid. (For isomorphisms, "virtually" extendability is known ([17])). Thus in order to compute the Lefschetz coincidence number $L(f,g)$ of two continuous maps f, $g\colon G_1/\Gamma_1 \to G_2/\Gamma_2$ between solvmanifolds satisfying the Mostow condition, we should give new idea of "linearizations".

In this paper, we give a technique of linearizations of all maps between solvmanifolds satisfying the Mostow condition and we give a formula for the Lefschetz coincidence number which is similar to the result by Ha, Lee and Penninckx ([4]).

2. Lefschetz numbers and spectral sequences

Let V^* be a finite dimensional graded vector space and $f^*: V^* \to V^*$ a graded linear map. Then we denote

$$L(f) = \sum_i (-1)^i \mathrm{tr}\, f^i\,.$$

Lemma 2.1. *Let C^* be a bounded filtered cochain complex and $f^*\colon C^* \to C^*$ a morphism of filtered cochain complex with the induced map $H^*(f)\colon H^*(C^*) \to H^*(C^*)$. Consider the spectral sequences $E_r^{*,*}(C^*)$ of C^* and the map $E_r^{*,*}(f) : E_r^{*,*}(C^*) \to E_r^{*,*}(C^*)$ induced by f^*. Consider the graded linear map $\mathrm{Tot}^* E_r^{*,*}(f)\colon \mathrm{Tot}^* E_r^{*,*}(C^*) \to \mathrm{Tot}^* E_r^{*,*}(C^*)$ for the total complex. We suppose that for some integer s, for $r \geq s$, the E_r-term $E_r^{*,*}(C^*)$ is finite dimensional.*

Then for each $r \geq s$, we have

$$L(H^*(f)) = L(\mathrm{Tot}^* E_r^{*,*}(f))\,.$$

Proof. By the assumption, sufficiently large r, we have

$$E_r^{p+q}(C) \cong F^p H^{p+q}(C)/F^{p+1}H^{p+q}(C)\,.$$

Hence by using the property of trace (see [6, Proposition 2.3.11]) we have

$$\sum_{p+q=k} \mathrm{tr}\, E_r^{p,q}(f) = \mathrm{tr} H^k(f)\,.$$

By the Hopf lemma for trace (see [6, Lemma 2.3.23]), we have

$$\sum_{p,q} (-1)^{p+q} \mathrm{tr}\, E_r^{p,q}(f) = \sum_{p,q} (-1)^{p+q} \mathrm{tr}\, E_{r-1}^{p,q}(f)$$

and inductively for $s \le r$, we have

$$\sum_{p,q}(-1)^{p+q}\mathrm{tr}\, E_r^{p,q}(f)=\sum_{p,q}(-1)^{p+q}\mathrm{tr}\, E_s^{p,q}(f)\,.$$

Hence the lemma follows. □

Let A^* be a finite-dimensional graded commutative $\mathbb{C}$-algebra.

Definition 2.2. A^* is of degree n Poincaré duality type (n-PD-type) if the following conditions hold:

- $A^{*<0}=0$ and $A^0=\mathbb{R}1$ where 1 is the identity element of A^*.
- For some positive integer n, $A^{*>n}=0$ and $A^n=\mathbb{R}v$ for $v\neq 0$.
- For any $0<i<n$ the bi-linear map $A^i\times A^{n-i}\ni(\alpha,\beta)\mapsto\alpha\cdot\beta\in A^n$ is non-degenerate. Hence we have an isomorphism $D_i\colon A^{n-i}\cong(A^i)^*$ where $(A^i)^*$ is the dual space of A^i.

Let A_1^* and A_2^* be finite-dimensional graded commutative $\mathbb{R}$-algebras of n-PD-type and $f^*\colon A_2^*\to A_1^*$ and $g^*\colon A_2^*\to A_1^*$ graded linear maps. By isomorphisms $:\ A_1^i\cong(A_1^{n-i})^*$ and $:\ A_2^i\cong(A_2^{n-i})^*$, we have the map $D^i(g^*)\ :\ A_1^i\to A_2^i$ which corresponds to the dual map $(A_1^{n-i})^*\to(A_2^{n-i})^*$ of g^{n-i}. Define the map $\theta^i(f,g)=D^i(g^*)\circ f^i$. We denote

$$L(f,g)=L\left(\theta^i(f,g)\right)\,.$$

For two compact oriented manifolds M_1 and M_2 of the same dimension, for two continuous maps f, $g\colon M_1\to M_2$, we consider the induced maps $H^*(f)$, $H^*(g)\colon H^*(M_2)\to H^*(M_1)$. Then the Lefschetz coincidence number $L(f,g)$ is defined as $L(f,g)=L(H^*(f),H^*(g))$.

Definition 2.3. A differential graded algebra (DGA) is a graded commutative $\mathbb{R}$-algebra A^* with a differential d of degree $+1$ so that $d\circ d=0$ and $d(\alpha\cdot\beta)=d\alpha\cdot\beta+(-1)^p\alpha\cdot d\beta$ for $\alpha\in A^p$.

Definition 2.4. A finite-dimensional DGA (A^*,d) is of n-PD-type if the following conditions hold:

- A^* is a finite-dimensional graded $\mathbb{R}$-algebra of n-PD-type.
- $dA^{n-1}=0$ and $dA^0=0$.

As similar to the Poincaré duality of the cohomology of compact Riemannian manifold, we can prove the following lemma.

Lemma 2.5 ([7])**.** *Let (A^*,d) be a finite dimensional DGA of n-PD-type. Then the cohomology algebra $H^*(A)$ is a finite dimensional graded commutative $\mathbb{R}$-algebra of n-PD-type.*

Then the following lemma follows from Lemma 2.5 inductively.

Lemma 2.6. *Let A^* be a bounded filtered differential graded algebra. Suppose that:*

- *The cohomology $H^*(A^*)$ is a finite dimensional graded algebra of n-PD-type.*
- *For some integer s, the total complex $(\mathrm{Tot}^*\, E_s^{*,*}(A^*),d_s)$ of the E_s-term of the spectral sequence is a finite dimensional graded algebra of n-PD-type.*

Then for each $r \geq s$, the total complex $(\mathrm{Tot}^* E_r^{*,*}(\mathfrak{g}), d_r)$ of the E_r-term of the spectral sequence is also a graded algebra of n-PD-type.

Proof. Since we have $H^0(A^*) \cong \mathbb{R}$, $H^n(A^*) \cong \mathbb{R}$, $\mathrm{Tot}^0 E_s^{*,*}(A^*) \cong \mathbb{R}$ and $\mathrm{Tot}^n E_s^{*,*}(A^*) \cong \mathbb{R}$, we have $d_s(\mathrm{Tot}^0 E_s^{*,*}(A^*)) = 0$ and $d_s(\mathrm{Tot}^{n-1} E_s^{*,*}(A^*)) = 0$. Hence the total complex $(\mathrm{Tot}^* E_s^{*,*}(A^*), d_s)$ of the E_s-term is a DGA of n-PD-type and by Lemma 2.5, the total complex $\mathrm{Tot}^* E_{s+1}^{*,*}(A^*)$ is a graded algebra of n-PD-type. □

By Lemma 2.1, we have:

Lemma 2.7. *Let A_1^* and A_2^* be bounded filtered DGAs and f^*, $g^* \colon A_2^* \to A_1^*$ morphisms of filtered DGA with the induced maps $H^*(f)$, $H^*(g) \colon H^*(A_2^*) \to H^*(A_1^*)$. Consider the spectral sequences $E_r^{*,*}(A_1)$ and $E_r^{*,*}(A_2)$ of A_1^* and A_2^* and the maps $E_r^{*,*}(f)$, $E_r^{*,*}(g) \colon E_r^{*,*}(A_2) \to E_r^{*,*}(A_1)$ induced by f, g.*

We suppose that:

- *The cohomologies $H^*(A_1^*)$ and $H^*(A_2^*)$ are finite dimensional graded algebra of n-PD-type.*
- *For some integer s, the total complexes $\mathrm{Tot}^* E_r^{*,*}(A_1)$ and $\mathrm{Tot}^* E_r^{*,*}(A_2)$ of E_r-terms are finite dimensional graded algebras of n-PD-type. Hence inductively the lemma follows.*

Then for each $r \geq s$, we have

$$L(H^*(f), H^*(g)) = L\big(\mathrm{Tot}^* E_r^{*,*}(f), \mathrm{Tot}^* E_r^{*,*}(g)\big) .$$

3. The Ha-Lee-Penninckx formula

Let V be a n-dimensional vector space. Consider the exterior algebra $\bigwedge V$. Then $\bigwedge V$ is a finite-dimensional graded commutative $\mathbb{C}$-algebras of n-PD-type. In [4], Ha-Lee-Penninckx showed:

Theorem 3.1 ([4])**.** *Let V_1, V_2 be n-dimensional vector spaces and $\Phi, \Psi : V_2 \to V_1$ linear maps. Consider the exterior algebras $\bigwedge V_1$ and $\bigwedge V_2$ and the extended map $\wedge\Phi, \wedge\Psi \colon \bigwedge V_2 \to \bigwedge V_1$. Take representation matrices A, B of Φ and Ψ associated with basis of V_1 and V_2. Then we have*

$$L(\wedge\Phi, \wedge\Psi) = \det(A - B) .$$

4. Lie algebra cohomology

Let $\mathfrak{g}$ be a n-dimensional solvable Lie algebra. We consider the DGA $\bigwedge \mathfrak{g}^*$ with the differential d which is the dual to the Lie bracket of $\mathfrak{g}$. We suppose that $\mathfrak{g}$ is unimodular. Then $\bigwedge \mathfrak{g}^*$ is a DGA of n-PD-type. Take a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ and its dual basis $x_1, \dots, x_n$ of $\mathfrak{g}^*$.

Let $\mathfrak{n}$ be a ideal of $\mathfrak{g}$. We consider the spectral sequence $(E_r^{p,q}(\mathfrak{g}), d_r)$ given by the extension $0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{n} \to 0$. This spectral sequence is given by the filtration

$$F^p \bigwedge^{p+q} \mathfrak{g}^* = \{\omega \in \bigwedge^{p+q} \mathfrak{g}^* | \omega(Y_1, \dots, Y_{p+1}) = 0 \quad \text{for} \quad Y_1, \dots, Y_{p+1} \in \mathfrak{n}\} .$$

We have
$$E_0^{*,*}(\mathfrak{g}) = \bigwedge(\mathfrak{g}/\mathfrak{n})^* \otimes \bigwedge \mathfrak{n}^*$$
with the differential $d_0 = 1 \otimes d_{\bigwedge \mathfrak{n}^*}$,
$$E_1^{*,*}(\mathfrak{g}) = \bigwedge(\mathfrak{g}/\mathfrak{n})^* \otimes H^*(\mathfrak{n})$$
whose differential d_1 is the differential on $\bigwedge(\mathfrak{g}/\mathfrak{n})^* \otimes H^*(\mathfrak{n})$ twisted by the action of $\mathfrak{g}/\mathfrak{n}$ on $H^*(\mathfrak{n})$ and
$$E_2^{*,*}(\mathfrak{g}) = H^*\left(\mathfrak{g}/\mathfrak{n}, H^*(\mathfrak{n})\right).$$

Since we suppose that $\mathfrak{g}$ is unimodular, we have $d\left(\bigwedge^{n-1}\mathfrak{g}^*\right) = 0$ and so $\bigwedge \mathfrak{g}^*$ is a finite dimensional DGA of n-PD-type. By Lemma 2.6, the total complex $(\operatorname{Tot}^* E_r^{*,*}(\mathfrak{g}), d_r)$ of each E_r-term of the spectral sequence is also a graded algebra of n-PD-type.

5. de Rham cohomology of solvamanifolds with Mostow conditions

Let G be a simply connected solvable Lie group with a lattice Γ. We suppose the Mostow condition: $\mathrm{Ad}(G)$ and $\mathrm{Ad}(\Gamma)$ have the same Zariski-closure in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. Then we have:

Proposition 5.1 ([2])**.** *Discrete subgroups $[\Gamma,\Gamma]$ and $\Gamma \cap [G,G]$ are lattices in the Lie group $[G,G]$ and the subgroup $\Gamma[G,G]$ is closed in G.*

Set $[G,G] = N$, $G/N = A$ and $\mathfrak{n}$ the Lie algebra of N and $\mathfrak{a}$ the Lie algebra of A. By Proposition 5.1, we have the fiber bundle structure
$$N/\Gamma \cap N \to G/\Gamma \to G/\Gamma N$$
of the solvmanifold G/Γ with base space torus $G/\Gamma N = A/p(\Gamma)$ and fiber nilmanifold $N/\Gamma \cap N$ where $p\colon G \to G/N$ is the quotient map.

We consider the filtration
$$F^p \bigwedge^{p+q} \mathfrak{g}^* = \{\omega \in \bigwedge^{p+q} \mathfrak{g}^* | \omega(X_1, \dots, X_{p+1}) = 0 \text{ for } X_1, \dots, X_{p+1} \in \mathfrak{n}\}.$$
This filtration gives the filtration of the cochain complex $\bigwedge \mathfrak{g}^*$ and the filtration of the de Rham complex $A^*(G/\Gamma)$. We consider the spectral sequence $E_*^{*,*}(\mathfrak{g})$ of $\bigwedge \mathfrak{g}^*$ and the spectral sequence $E_*^{*,*}(G/\Gamma)$ of $A^*(G/\Gamma)$. Then we have the commutative diagram
$$\begin{tikzcd} E_2^{*,*}(\mathfrak{g}) \arrow[r] \arrow[d, "\cong"] & E_2^{*,*}(G/\Gamma) \arrow[d, "\cong"] \\ H^*\left(\mathfrak{a}, H^*(\mathfrak{n})\right) \arrow[r] & H^*\left(A/p(\Gamma), \mathbf{H}^*(N/\Gamma \cap N)\right) \end{tikzcd}$$
where $\mathbf{H}^*(N/\Gamma \cap N)$ is the local system on the cohomology of fiber induced by the fiber bundle (see [5], [15, Section 7]).

Theorem 5.2. *The induced map $E_2^{*,*}(\mathfrak{g}) \to E_2^{*,*}(G/\Gamma)$ is an isomorphism.*

Proof. We first show that for each r, the induced map $E_r^{*,*}(\mathfrak{g}) \to E_r^{*,*}(G/\Gamma)$ is injective. A simply connected solvable Lie group with a lattice is unimodular (see [15, Remark 1.9]). Let $d\mu$ be a bi-invariant volume form such that $\int_{G/\Gamma} d\mu = 1$. For $\alpha \in A^p(G/\Gamma)$, we have a left-invariant form $\alpha_{\mathrm{inv}} \in \bigwedge^p \mathfrak{g}^*$ defined by

$$\alpha_{inv}(X_1, \dots, X_p) = \int_{G/\Gamma} \alpha(\tilde{X}_1, \dots, \tilde{X}_p) d\mu$$

for $X_1, \dots, X_p \in \mathfrak{g}$ where $\tilde{X}_1, \dots, \tilde{X}_p$ are vector fields on G/Γ induced by $X_1, \dots X_p$. We define the map $I\colon A^*(M) \to \bigwedge \mathfrak{g}^*$ by $\alpha \mapsto \alpha_{\mathrm{inv}}$. Then this map is a cochain complex map (see [8]) such that $I \circ i = \mathrm{id}_{|_{\bigwedge \mathfrak{g}^*}}$. The map I is compatible with the filtration as above. Hence I induces a homomorphism $E_r^{*,*}(G/\Gamma) \to E_r^{*,*}(\mathfrak{g})$. This implies that the induced map $E_r^{*,*}(\mathfrak{g}) \to E_r^{*,*}(G/\Gamma)$ is injective.

Consider the A-action on $H^*(\mathfrak{n})$ which is the extension of the $\mathfrak{a}$-action on $H^*(\mathfrak{n})$ given by $0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{a} \to 0$. Since we have $H^*(\mathfrak{n}) \cong H^*(N/\Gamma \cap N)$. The local system $\mathbf{H}^*(N/\Gamma \cap N)$ is given by the Γ-action on $H^*(\mathfrak{n})$ which is the restriction of the A-action on $H^*(\mathfrak{n})$. Since $\mathrm{Ad}(G)$ and $\mathrm{Ad}(\Gamma)$ have the same Zariski-closure in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$, the images of actions $A \to \mathrm{Aut}(H^*(\mathfrak{n}))$ and $p(\Gamma) \to \mathrm{Aut}(H^*(\mathfrak{n}))$ have also the same Zariski-closure in $\mathrm{Aut}(H^*(\mathfrak{n}))$. Then by [15, Theorem 7.26] we have

$$H^*\left(\mathfrak{a}, H^*(\mathfrak{n})\right) \cong H^*\left(A/p(\Gamma), \mathbf{H}^*(N/\Gamma \cap N)\right)$$

Hence the theorem follows. $\square$

6. Linearizations of solvamanifolds with Mostow conditions

Consider two simply connected solvable Lie groups G_1 and G_2 with lattices Γ_1 and Γ_2. We assume that they satisfy the Mostow condition. Let $\phi\colon \Gamma_1 \to \Gamma_2$ be a homomorphism. Then we have

$$\phi([\Gamma_1, \Gamma_1]) \subset [\Gamma_2, \Gamma_2].$$

Hence ϕ induces the homomorphism $\bar{\phi}\colon \Gamma_1/[\Gamma_1, \Gamma_1] \to \Gamma_2/[\Gamma_2, \Gamma_2]$. We show

Lemma 6.1. $\phi(\Gamma_1 \cap [G_1, G_1]) \subset \Gamma_2 \cap [G_2, G_2]$.

Proof. Consider the surjection

$$\Gamma_1/[\Gamma_1, \Gamma_1] \ni (g \mod [\Gamma_1, \Gamma_1]) \mapsto (g \mod \Gamma_1 \cap [G_1, G_1]) \in \Gamma/\Gamma_1 \cap [G_1, G_1].$$

By Proposition 5.1, two nilpotent groups $[\Gamma_1, \Gamma_1]$ and $\Gamma_1 \cap [G_1, G_1]$ have same rank and hence the kernel of this surjection consists of torsions. This implies that for $g \in \Gamma_1 \cap [G_1, G_1]$, the element

$$\bar{\phi}(g \mod [\Gamma_1, \Gamma_1]) = \phi(g) \mod [\Gamma_2, \Gamma_2]$$

is a torsion. Since the group $\Gamma_2/\Gamma_2 \cap [G_2, G_2]$ is a lattice in $G_2/[G_2, G_2]$, $\Gamma_2/\Gamma_2 \cap [G_2, G_2]$ is torsion-free. Hence we have

$$(\phi(g) \mod \Gamma_2 \cap [G_2, G_2]) = (0 \mod \Gamma_2 \cap [G_2, G_2])$$

for $g \in \Gamma_1 \cap [G_1, G_1]$. Thus the lemma follows. $\square$

Set $N_1 = [G_1, G_1]$, $N_2 = [G_2, G_2]$, $A_1 = G_1/N_1$ and $A_2 = G_2/N_2$. Let $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be the Lie algebras of N_1, N_2, A_1 and A_2 respectively. Consider the quotient maps $p_1 \colon G_1 \to A_1$ and $p_2 : G_2 \to A_2$. By Lemma 6.1, we have the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \Gamma_1 \cap N_1 \arrow[r] \arrow[d, "\phi"] & \Gamma_1 \arrow[r] \arrow[d, "\phi"] & p_1(\Gamma_1) \arrow[r] \arrow[d, "\bar{\phi}"] & 1 \\ 1 \arrow[r] & \Gamma_2 \cap N_2 \arrow[r] & \Gamma_2 \arrow[r] & p_2(\Gamma_2) \arrow[r] & 1 \end{tikzcd}$$

Since $\Gamma_1 \cap N_1$, $\Gamma_2 \cap N_2$, $p_1(\Gamma_1)$ and $p_2(\Gamma_2)$ are lattices in N_1, N_2, A_1 and A_2 respectively, we can take unique Lie group homomorphisms $\Phi_1 \colon N_1 \to N_2$ and $\Phi_2 \colon A_1 \to A_2$ which are extensions of $\phi \colon \Gamma_1 \cap N_1 \to \Gamma_2 \cap N_2$ and $\bar{\phi} \colon p_1(\Gamma_1) \to p_2(\Gamma_2)$.

Lemma 6.2. *We consider the spectral sequences*

$$E_0^{*,*}(\mathfrak{g}_1) = \bigwedge \mathfrak{a}_1^* \otimes \bigwedge \mathfrak{n}_1^*,$$

$$E_0^{*,*}(\mathfrak{g}_2) = \bigwedge \mathfrak{a}_2^* \otimes \bigwedge \mathfrak{n}_2^*$$

and

$$E_1^{*,*}(\mathfrak{g}_1) = \bigwedge \mathfrak{a}_1^* \otimes H^*(\mathfrak{n}_1),$$

$$E_1^{*,*}(\mathfrak{g}_2) = \bigwedge \mathfrak{a}_2^* \otimes H^*(\mathfrak{n}_2)$$

Then the linear map

$$\wedge\Phi_2^* \otimes \wedge\Phi_1^* : E_0^{*,*}(\mathfrak{g}_2) = \bigwedge \mathfrak{a}_2^* \otimes \bigwedge \mathfrak{n}_2^* \to \bigwedge \mathfrak{a}_1^* \otimes \bigwedge \mathfrak{n}_1^* = E_0^{*,*}(\mathfrak{g}_1)$$

is a cochain complex map and induced map

$$\wedge\Phi_2^* \otimes H^*(\wedge\Phi_1^*) : E_1^{*,*}(\mathfrak{g}_2) = \bigwedge \mathfrak{a}_2^* \otimes H^*(\mathfrak{n}_2) \to \bigwedge \mathfrak{a}_1^* \otimes H^*(\mathfrak{n}_1) = E_1^{*,*}(\mathfrak{g}_1)$$

is a cochain complex map.

Proof. Since Φ_1 is a homomorphism of Lie group, the linear map

$$\wedge\Phi_2^* \otimes \wedge\Phi_1^* \colon E_0^{*,*}(\mathfrak{g}_2) = \bigwedge \mathfrak{a}_2^* \otimes \bigwedge \mathfrak{n}_2^* \to \bigwedge \mathfrak{a}_1^* \otimes \bigwedge \mathfrak{n}_1^* = E_0^{*,*}(\mathfrak{g}_1)$$

is cochain complex map. We consider the induced map

$$\wedge\Phi_2^* \otimes H^*(\wedge\Phi_1^*) \colon E_1^{*,*}(\mathfrak{g}_2) = \bigwedge \mathfrak{a}_2^* \otimes H^*(\mathfrak{n}_2) \to \bigwedge \mathfrak{a}_1^* \otimes H^*(\mathfrak{n}_1) = E_1^{*,*}(\mathfrak{g}_1).$$

We show that this map is a cochain complex homomophism.

We consider the group cohomologies $H^*(\Gamma_1 \cap N_1, \mathbb{R})$ and $H^*(\Gamma_2 \cap N_2, \mathbb{R})$ and the induced map $H^*(\phi) \colon H^*(\Gamma_2 \cap N_2, \mathbb{R}) \to H^*(\Gamma_1 \cap N_1, \mathbb{R})$ of $\phi : \Gamma_1 \cap N_1 \to \Gamma_2 \cap N_2$. By the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \Gamma_1 \cap N_1 \arrow[r] \arrow[d, "\phi"] & \Gamma_1 \arrow[r] \arrow[d, "\phi"] & p_1(\Gamma_1) \arrow[r] \arrow[d, "\bar{\phi}"] & 1 \\ 1 \arrow[r] & \Gamma_2 \cap N_2 \arrow[r] & \Gamma_2 \arrow[r] & p_2(\Gamma_2) \arrow[r] & 1\,, \end{tikzcd}$$

for the $p_1(\Gamma_1)$-action $\delta_1 : p_1(\Gamma_1) \to \mathrm{Aut}(H^*(\Gamma_1 \cap N_1, \mathbb{R}))$ and the $p_2(\Gamma_2)$-action $\delta_2 : p_2(\Gamma_2) \to \mathrm{Aut}(H^*(\Gamma_2 \cap N_2, \mathbb{R}))$, we have

$$H^*(\phi) \circ \delta_2(\bar{\phi}(g)) = \delta_1(g) \circ H^*(\phi)\,.$$

By the isomorphisms,

$$H^*(\Gamma_1 \cap N_1, \mathbb{R}) \cong H^*(N_1/\Gamma_1 \cap N_1, \mathbb{R}) \cong H^*(\mathfrak{n}_1)$$

and

$$H^*(\Gamma_2 \cap N_2, \mathbb{R}) \cong H^*(N_2/\Gamma_2 \cap N_2, \mathbb{R}) \cong H^*(\mathfrak{n}_2)\,,$$

we have $H^*(\phi) = H^*(\Phi_1)$. Consider the A_1-action $\Delta_1 \colon A \to \mathrm{Aut}(H^*(\mathfrak{n}_1))$ induced by the extension $1 \to N_1 \to G_1 \to A_1 \to 1$ and A_2-action $\Delta_2 : A \to \mathrm{Aut}(H^*(\mathfrak{n}_2))$ induced by the extension $1 \to N_2 \to G_2 \to A_2 \to 1$. By $H^*(\phi) = H^*(\Phi_1)$ and $H^*(\phi) \circ \delta_2(\bar{\phi}(g)) = \delta_1(g) \circ H^*(\phi)$, we have

$$H^*(\Phi_1) \circ \Delta_2(\Phi_2(v)) = \Delta_1(v) \circ H^*(\Phi_1)$$

for all $v \in p(\Gamma_1) \subset A_1$. By the Mostow condition, $\Delta_1(A_1) \times \Delta_2(\Phi_2(A_2))$ and $\Delta_1(p_1(\Gamma_1)) \times \Delta_2(\Phi_2(p_2(\Gamma_2)))$ have the same Zariski-closure in $\mathrm{Aut}(H^*(\mathfrak{n}_1)) \times \mathrm{Aut}(H^*(\mathfrak{n}_2))$. By this we have

$$H^*(\Phi_1) \circ \Delta_2(\Phi_2(v)) = \Delta_1(v) \circ H^*(\Phi_1)$$

for all $v \in A_1$.

Consider the Lie algebra homomorphism $\Phi_{2*} \colon \mathfrak{a}_1 \to \mathfrak{a}_2$ and the $\mathfrak{a}_1$-action $\Delta_{1*} \colon \mathfrak{a}_1 \to \mathrm{End}(H^*(\mathfrak{n}_1))$ and $\mathfrak{a}_2$-action $\Delta_{2*} \colon \mathfrak{a}_{2*} \to \mathrm{End}(H^*(\mathfrak{n}_2))$. Then we have

$$H^*(\Phi_1) \circ \Delta_{2*}(\Phi_{2*}(V)) = \Delta_{1*}(V) \circ H^*(\Phi_1)$$

for all $V \in \mathfrak{a}_1$. This implies that the map

$$\wedge\Phi_2^* \otimes H^*(\wedge\Phi_1^*) : E_1^{*,*}(\mathfrak{g}_2) = \bigwedge \mathfrak{a}_2^* \otimes H^*(\mathfrak{n}_2) \to \bigwedge \mathfrak{a}_1^* \otimes H^*(\mathfrak{n}_1) = E_1^{*,*}(\mathfrak{g}_1)\,.$$

is a cochain complex homomophism, since the differentials of the cochain complexes $E_1^{*,*}(\mathfrak{g}_1) = \bigwedge \mathfrak{a}_1^* \otimes H^*(\mathfrak{n}_1)$ and $E_1^{*,*}(\mathfrak{g}_2) = \bigwedge \mathfrak{a}_2^* \otimes H^*(\mathfrak{n}_2)$ are twisted by the $\mathfrak{a}_1$-action $\Delta_{1*} \colon \mathfrak{a}_1 \to \mathrm{End}(H^*(\mathfrak{n}_1))$ and the $\mathfrak{a}_2$-action $\Delta_{2*} \colon \mathfrak{a}_{2*} \to \mathrm{End}(H^*(\mathfrak{n}_2))$ respectively. □

Let $f \colon G_1/\Gamma_1 \to G_2/\Gamma_2$ be a continuous map. We consider the induced map $f_* \colon \pi_1(G_1/\Gamma_1) \cong \Gamma_1 \to \Gamma_2 \cong G_2/\Gamma_2$. We write $\phi = f_*$. In this case, the pair Φ_1, Φ_2 constructed as above is called the linearlization of f. Consider the spectral sequences $E_r^{*,*}(G_1/\Gamma_1)$ and $E_r^{*,*}(G_2/\Gamma_2)$ as Section 5. Then for $r \geq 2$, $E_r^{*,*}(G_1/\Gamma_1)$ and $E_r^{*,*}(G_2/\Gamma_2)$ are identified with the Leray-Serre spectral sequences. By commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_1 \cap N_1 \arrow[r] \arrow[d, "\phi"] & \Gamma_1 \arrow[r] \arrow[d, "\phi"] & p_1(\Gamma_1) \arrow[r] \arrow[d, "\bar{\phi}"] & 1 \\
1 \arrow[r] & \Gamma_2 \cap N_2 \arrow[r] & \Gamma_2 \arrow[r] & p_2(\Gamma_2) \arrow[r] & 1\,,
\end{tikzcd}$$

Any continous map from G_1/Γ_1 to G_2/Γ_2 is homotopic to a continous map $f\colon G_1/\Gamma_1 \to G_2/\Gamma_2$ which is a fiber-preserving map as

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_1/\Gamma_1 \cap N_1 & \longrightarrow & G_1/\Gamma_1 & \longrightarrow & A_1/p_1(\Gamma_1) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} & & \\ 1 & \longrightarrow & N_2/\Gamma_2 \cap N_2 & \longrightarrow & G_2/\Gamma_2 & \longrightarrow & A_2/p_2(\Gamma_2) & \longrightarrow & 1. \end{array}$$

Consider the induced map $E_r^{*,*}(f)\colon E_r^{*,*}(G_1/\Gamma_1) \to E_r^{*,*}(G_2/\Gamma_2)$. Then

$$E_2^{*,*}(f) : H^*\left(A_2/p(\Gamma_2), \mathbf{H}^*(N_2/\Gamma_2 \cap N_2)\right) \to H^*\left(A_1/p(\Gamma_1), \mathbf{H}^*(N_1/\Gamma_1 \cap N_1)\right)$$

is induced by the fiber map $f\colon N_1/\Gamma_1 \cap N_1 \to N_2/\Gamma_2 \cap N_2$ and the base space map $\bar{f}\colon A_1/p(\Gamma_1) \to A_2/p(\Gamma_2)$ (see [9]). Consider the linearlization Φ_1, Φ_2 of f and induced maps $\underline{\Phi_1}\colon N_1/\Gamma_1 \cap N_1 \to N_2/\Gamma_2 \cap N_2$ and $\underline{\Phi_2}\colon A_1/p(\Gamma_1) \to A_2/p(\Gamma_2)$. Then the fiber map $f\colon N_1/\Gamma_1 \cap N_1 \to N_2/\Gamma_2 \cap N_2$ and the base space map $\bar{f}\colon A_1/p(\Gamma_1) \to A_2/p(\Gamma_2)$ are homotopic to $\underline{\Phi_1}\colon N_1/\Gamma_1 \cap N_1 \to N_2/\Gamma_2 \cap N_2$ and $\underline{\Phi_2}\colon A_1/p(\Gamma_1) \to A_2/p(\Gamma_2)$ respectively. By Theorem 5.2, we have

$$H^*(\mathfrak{a}_1, H^*(\mathfrak{n}_1)) \cong H^*\left(A_1/p(\Gamma_1), \mathbf{H}^*(N_1/\Gamma_1 \cap N_1)\right)$$

and

$$H^*(\mathfrak{a}_2, H^*(\mathfrak{n}_2)) \cong H^*\left(A_2/p(\Gamma_2), \mathbf{H}^*(N_2/\Gamma_2 \cap N_2)\right).$$

By these isomorphisms, $E_2^{*,*}(f)$ is induced by $\wedge\Phi_1^*\colon \bigwedge \mathfrak{n}_2^* \to \bigwedge \mathfrak{n}_1^*$ and $\wedge\Phi_2^*\colon \bigwedge \mathfrak{a}_2^* \to \bigwedge \mathfrak{a}_1^*$. Hence by Lemma 6.2 we have:

Lemma 6.3. *The map*

$$E_2(f)\colon E_2^{*,*}(G_2/\Gamma_2) \to E_2^{*,*}(G_1/\Gamma_1)$$

is identified with the map

$$H^*(\wedge\Phi_2^*) \otimes H^*(\wedge\Phi_1^*)\colon E_2^{*,*}(\mathfrak{g}_2) = H^*\left(\mathfrak{a}_1, H^*(\mathfrak{n}_2)\right) \to H^*\left(\mathfrak{a}_1, H^*(\mathfrak{n}_1)\right) = E_2^{*,*}(\mathfrak{g}_1)$$

induced by the cochain complex map

$$\wedge\Phi_2^* \otimes H^*(\wedge\Phi_1^*)\colon E_1^{*,*}(\mathfrak{g}_2) = \bigwedge \mathfrak{a}_2^* \otimes H^*(\mathfrak{n}_2) \to \bigwedge \mathfrak{a}_1^* \otimes H^*(\mathfrak{n}_1) = E_1^{*,*}(\mathfrak{g}_1)$$

as in Lemma 6.2.

7. Lefschetz coincidence numbers of Mostow solvamanifolds

Theorem 7.1. *Let G_1 and G_2 be simply connected solvable Lie groups of the same dimension with lattices Γ_1 and Γ_2. We assume they satisfy the Mostow condition. Let f, $g\colon G_1/\Gamma_1 \to G_2/\Gamma_2$ be continuous maps. Take linearizations Φ_1, Φ_2 of f and Ψ_1, Ψ_2 of g as Section 6. Take representation matrices A_1, A_2, B_1 and B_2 of Φ_{1*}, Φ_{2*}, Ψ_{1*} and Ψ_{2*} associated with basis of Lie algebras. Let $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$. Then we have*

$$L(f, g) = \det(A - B).$$

Proof. By Lemma 2.7, we have

$$L(f,g) = L\big(\mathrm{Tot}^* E_2^{*,*}(f), \mathrm{Tot}^* E_2^{*,*}(g)\big)\,.$$

By Lemma 6.3 and the Hopf lemma, we have

$$L\big(\mathrm{Tot}^* E_2^{*,*}(f), \mathrm{Tot}^* E_2^{*,*}(g)\big) = L\big(\wedge \Phi_2^* \otimes \wedge\Phi_1^*, \wedge\Psi_2^* \otimes \wedge\Psi_1^*\big)\,.$$

Take bases $\{X_1^1, \dots, X_n^1\}$, $\{Y_1^1, \dots, Y_m^1\}$, $\{X_1^2, \dots, X_{n'}^2\}$ and $\{Y_1^2, \dots, Y_{m'}^2\}$ of $\mathfrak{n}_1$, $\mathfrak{a}_1$, $\mathfrak{n}_2$ and $\mathfrak{a}_2$ which give representation matrices A_1, A_2, B_1 and B_2 of Φ_{1*}, Φ_{2*}, Ψ_{1*} and Ψ_{2*} respectively. Consider the dual bases $\{x_1^1, \dots, x_n^1\}$, $\{y_1^1, \dots, y_m^1\}$, $\{x_1^2, \dots, x_{n'}^2\}$ and $\{y_1^2, \dots, y_{m'}^2\}$ of these bases respectively Then we have

$$\bigwedge \mathfrak{a}_1^* \otimes \bigwedge \mathfrak{n}_1^* = \bigwedge \langle x_1^1, \dots, x_n^1, y_1^1, \dots, y_m^1\rangle,$$

$$\bigwedge \mathfrak{a}_2^* \otimes \bigwedge \mathfrak{n}_2^* = \bigwedge \langle x_1^2, \dots, x_n^2, y_1^2, \dots, y_m^2\rangle$$

and the maps $\wedge\Phi_2^* \otimes \wedge\Phi_1^*$ and $\wedge\Psi_2^* \otimes \wedge\Psi_1^*$ are represented by $\wedge A^*$ and $\wedge B^*$ respectively. Hence we have

$$L(f,g) = L(\wedge\Phi_2^* \otimes \wedge\Phi_1^*, \wedge\Psi_2^* \otimes \wedge\Psi_1^*) = L(\wedge A^*, \wedge B^*).$$

By Theorem 3.1, we have

$$L(\wedge A^*, \wedge B^*) = \det(A^* - B^*) = \det(A - B)\,.$$

Hence the theorem follows. □

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