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COUNTABLY Z-COMPACT SPACES

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Abstract. In this work we study countably z-compact spaces and z-Lindelof spaces. Several new properties of them are given. It is proved that every countably z-compact space is pseudocompact (a space on which every real valued continuous function is bounded). Spaces which are countably z-compact but not countably compact are given. It is proved that a space is countably z-compact iff every countable z-closed set is compact. Characterizations of countably z-compact and z-Lindelof spaces by multifunctions are given.

1. Introduction

Countably z-compact spaces are introduced by Frolik [1]. As far as the author knows, no further study has been done about these spaces except one result (Theorem 4.6) appeared in [4]. In this paper we study some properties of countably z-compact spaces and z-Lindelof spaces. We relate countably z-compact spaces to pseudocompact spaces (Theorem 3.2). Then we give some characterizations of countably z-compact spaces. The collection of real valued continuous functions on a topological space X forms a ring denoted by $C(X)$ [2]. Characterizations of countably z-compact spaces in terms of z-filterbases and z-ideals are given, similar to countably compact case, where complete regularity is assumed. Here, no separation property is assumed unless otherwise is stated. For definitions and notations not stated here see [2].

2. Preliminaries

2.1. **Definitions.**

Recall that a *cozero set* in a space $X = (X, \tau)$ is an $f^{-1}[\mathbb{R} \setminus \{0\}]$ with a continuous function $f\colon X \to \mathbb{R}$. Cozero sets constitute a base of a topology $z\tau$ on X. (X, τ) is said to be z-compact, (countably z-compact resp. z-Lindelof) if $zX = (X, z\tau)$ is compact (countably compact, resp. Lindelof).

Filters and z-ideals here are modifications of their respective definitions [2] by taking z-closed set (closed in zX) instead of zero-set.

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Definition 2.1. A multifunction α of a space X into a space Y is a set valued function on X into Y such that $\alpha(x) \neq \Phi$ for every $x \in X$. The class of all multifunctions on X into Y is denoted by $m(X, Y)$.

Definition 2.2. A multifunction α on X into Y is called closed graph iff its graph $G(\alpha) = (x, y) \in X \times Y : y \in \alpha(x)$ is closed in $X \times Y$.

3. Some properties of countably z-compact spaces

The proof of the following theorem is straightforward.

Theorem 3.1. *The following statements about a space X are equivalent.*

(a) *X is countably z-compact.*

(b) *Every countable family of subsets of X each is an intersection of zero sets, with the finite intersection property has non empty intersection.*

(c) *Every countable z-filter on X is fixed.*

(d) *Every countable z-ideal in $C(X)$ is fixed.*

(e) *zX is countably compact.*

Theorem 3.2. *Every countably z-compact is pseudocompact.*

Proof. Let X be countably z-compact, f be a real valued function on X. The collection of all sets $An = \{x\colon \mid f(x) \mid < n\}$, n is a positive integer, is a countable open cover of zX, consequently it has a finite subcover $\{Ai\colon i = 1, 2, \ldots, m\}$. Then $X = A_m$, and $|f(x)| < m$ for each $x \in X$. Thus f is bounded. □

Example 3.3 ([3], [5])**.** Let N be the set of positive integers. Topologize N by taking a subbase the collection $\beta = \{U_p(b)\colon b + np \in N, p$ prime, b is not divisible by $p\}$.
This space is a T_2 countably z-compact Lindelof space which is not countably compact.

Theorem 3.4. *A space X is countably z-compact iff every countable closed set in zX is compact (z-compact).*

Proof. By Theorem 3.1(e), X is countably z-compact iff zX is countably compact, and zX is countably compact iff every countable closed set in zX is compact (z-compact) ([2], p.79). □

3.1. **The space Ψ.** The following well-known example Ψ [2] has many nice topological properties. Although it is not z-compact. It is Hausdorff, completely regular, first axiom pseudocompact and every subset of it is a G_δ (an intersection of a countable number of open sets). We describe this space for the sake of completeness. Let E be a maximal family of infinite subset of sets of natural numbers N such that the intersection of any two is finite. Let $\Psi = \{w_i\colon i \in E\}$ be a new set of distinct points. The topology on Ψ is defined as follows: Every point of N is isolated and the neighborhoods of w_i are sets containing w_i and all but finite numbers of E. This space is completely regular pseudocompact not countably z-compact.

4. A characterization of countable z-compactness in terms of multifunctions

We give here a characterization of countably z-compact space X in terms of multifunctions. Equivalently a characterization of countable compactness of zX. It is to be noted that a space is countably compact iff every countable family of closed sets with the finite intersection property has a non-empty intersection. We shall use this fact in the proof of the second part of the following theorem.

Theorem 4.1. *A space X is countably z-compact iff for every first countable space Y and closed graph multifunction $\alpha \in m(zX, Y)$, the image of every closed set in zX is closed in Y.*

Proof. Let zX be countably compact, Y be first countable space, $\alpha \in m(zX, Y)$ with closed graph.
Let K be closed in zX and $y \in Y \backslash \alpha(K)$.
Let $\{B_i : i = 1, 2, \cdots\}$ be a countable local base at y.
For each $x \in K$, there exist an open set V_x in zX and B_i in Y such that

$$(x, y) \in V_x \times B_i$$

and $(V_x \times B_i) \cap G(\alpha) = \Phi$. For each $i = 1, 2, \ldots$, let $W_i = \cup\{V_x : x \in K, (x, y) \in V_x \times B_i\}$. Then $\{W_i : i = 1, 2, \ldots\}$ is a countable open cover of K. So, it has a finite subcover $\{W_i : i = 1, 2, \ldots, n\}$. Now, let $W = \bigcap_{i=1}^{n} B_i$. Then W is open in Y with $y \in W$ and $W \cap \alpha(K) = \Phi$. So, $\alpha(K)$ is closed in Y. To prove the converse, let $\{Ki : i = 1, 2, \cdots\}$ be a countable family of closed sets in zX, with the finite intersection property, let $y_0 \notin zX$. Topologize $zX \cup \{y_0\}$ by taking open sets all subsets of zX and sets containing $y_0 \cup \alpha(K_i)$ for some $i = 1, 2, \ldots$.

Obviously, $zX \cup \{y_0\}$ is first countable. Let β be the closure of the identity function of zX. β has a closed graph and so, by hypothesis, it maps closed sets in zX onto closed subsets in Y. So, $\beta(K_i)$ is closed in $zX \cup \{y_0\}$, for every $i = 1, 2, \ldots$. So, $y_o \in \beta(Ki)$ for every $i = 1, 2, \ldots$. Hence $\{K_i, i = 1, 2, \ldots\}$ has a non-empty intersection. Therefore, zX is countably compact. □

5. z-Lindelof space

Hewitt's example [4] is z-Lindelof but not Lindelof.

Definition 5.1. A space X is a P-space [2] iff every G_σ set in X is open.
The following result about z-Lindelof space can be proved by the same technique of Theorems 4.1.

Theorem 5.2. *A space X is z-Lindelof iff for every P-space Y and z-closed graph multifunction $\alpha \in m(X, Y)$ the image of every z-closed set in X is closed in Y.*

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