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STABILITY AND BOUNDEDNESS OF SOLUTIONS OF NONLINEAR VECTOR DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Abstract. The paper studies the equation

$$\dddot{X} + \Psi(\dot{X})\ddot{X} + \Phi(X)\dot{X} + cX = P(t)$$

in two cases:
(i) $P(t) \equiv 0$,
(ii) $P(t) \neq 0$.
In case (i), the global asymptotic stability of the solution $X = 0$ is studied; in case (ii), the boundedness of all solutions is proved.

1. Introduction

For over five decades, the study of the stability and boundedness of ordinary scalar and vector nonlinear differential equations of third order have received tremendous attention. For a comprehensive treatment of this subject we refer the reader to the book by Reissig et al [7], the papers by Chukwu [1], Ezeilo [2], Mehri and Shadman [4], Tejumola [8], Tunc ([10], [9]), Tunc and Ates [11], Omeike and Afuwape [5] and the references cited in this book and papers. Throughout the results presented in the book of Reissig et al [7] and the papers mentioned above, Liapunov's second method ([3]) has been used as a basic tool to verify the results established in these works.
The present work is concerned with the differential equation of the form

$$\dddot{X} + \Psi(\dot{X})\ddot{X} + \Phi(X)\dot{X} + cX = P(t) \tag{1}$$

or the equivalent system of the form

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -\Psi(Y)Z - \Phi(X)Y - cX + P(t) \end{aligned} \tag{2}$$

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which was obtained as usual by setting $\dot{X} = Y, \ddot{X} = Z$ in (1), where $t \in \mathbb{R}^+ = (0, \infty)$ and $X\colon \mathbb{R}^+ \to \mathbb{R}^n$, c is a positive constant, Ψ and Φ are $n \times n$ continuous symmetric positive definite matrix functions for the argument displayed explicitly and the dots indicate differentiation with respect to t and $P\colon \mathbb{R}^+ \to \mathbb{R}^n$. It is also assumed that P is continuous for the argument displayed explicitly. Moreover, the existence and the uniqueness of the solution of Eq. (1) will be assumed (see Picard-Lindelof theorem in Rao [6]). Eq. (1) represents a system of real third-order differential equations of the form

$$\dddot{x}_i + \sum_{k=1}^{n} \psi_{ik}(\dot{x}_1, \ldots, \dot{x}_n)\ddot{x}_k + \sum_{k=1}^{n} \phi_{ik}(x_1, \ldots, x_n)\dot{x}_k + cx_i = p_i(t)\,, \quad (i = 1, \ldots, n)\,.$$

We shall assume, as basic throughout what follows, that the derivative $\dfrac{\partial \psi_{ij}}{\partial \dot{x}_j}$ exist and are continuous for $(j = 1, \ldots, n)$. The motivation for the present work comes from the papers of Tunc [10], and Omeike and Afuwape [5], where they studied the stability and boundedness of solutions of Eq. (1) for which $\Phi(X) = B$ (an $n \times n$ symmetric positive definite matrix). With respect to our observations in the literature, no work based on Eq. (1) was found. Essentially, our subject is to establish some sufficient conditions for the stability and for the boundedness of solutions of (1) in the cases $P(t) \equiv 0$, $P(t) \neq 0$, respectively. Unlike in [5] and [10], $\Psi(Y)$ and $\Phi(X)$ do not necessarily commute. In addition, $\Phi(X)$ is not necessarily differentiable.

2. Main results

Before stating our main results, we give some well known algebraic results which will be required in the proofs.

Lemma 2.1. *Let A be a real symmetric positive definite $n \times n$ matrix. Then for any $X \in \mathbb{R}^n$,*

$$\delta_a \|X\|^2 \le \langle AX, X\rangle \le \Delta_a \|X\|^2\,,$$

where δ_a and Δ_a are respectively the least and greatest eigenvalues of the matrix A.

Proof of Lemma 2.1. See [10]. □

Lemma 2.2. *Subject to earlier conditions on Ψ, the following is true for all $t \in \mathbb{R}^+$ and Y, $Z \in \mathbb{R}^n$:*

$$\frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Y)Y, Y\rangle\, d\sigma = \langle \Psi(Y)Y, Z\rangle\,.$$

Proof of Lemma 2.2. See [10]. □

In the case $P \equiv 0$, the first main result of this paper is the following theorem.

Theorem 2.1. *Let all the basic assumptions imposed on Ψ, Φ and c hold. Further, suppose that there are positive constants a_0 and b_0 such that the following conditions are satisfied,*

$$a_0 b_0 - c > 0\,,\ b_0 \le \lambda_i(\Phi(X)) \le b_0 + \mu \quad \textit{and} \quad \lambda_i(\Psi(Y)) \ge a_0, \quad (i = 1, 2, \ldots, n)$$

for all $X, Y \in \mathbb{R}^n$, *where* $\mu = 4cb_0^{-2}(a_0b_0 - c) > 0$, *and* $\lambda_i(\Phi(X)), \lambda_i(\Psi(Y))$ *are "eigenvalues of the matrix indicated"* $\Phi(X)$ *and* $\Psi(Y)$, *respectively.*
Then the zero solution of system (2) *is globally asymptotic stable.*

Proof of Theorem 2.1. The proof of this theorem depends on a scalar differentiable Liapunov function $V = V(X, Y, Z)$. The idea of the Liapunov's method is to impose some conditions on the function V and its time derivative $\frac{d}{dt}V(X, Y, Z)$ which both imply the stability of the zero solution of Eq. (1). We define the Liapunov function V by

$$(1)\ \ 2V = \langle b_0Z, Z\rangle + 2\langle cY, Z\rangle + \langle cX + b_0Y, cX + b_0Y\rangle + 2\int_0^1 \sigma\langle c\Psi(\sigma Y)Y, Y\rangle d\sigma\,.$$

Now, it is clear from (1) that $V(0, 0, 0) = 0$. □

Next, in view of the assumptions on Theorem 2.1 and the above lemmas, respectively, it follows that

$$\langle b_0Z, Z\rangle = b_0\|Z\|^2\,,$$

$$2\langle cY, Z\rangle \geq -2c\|Y\|\|Z\|\,,$$

$$2\int_0^1 \sigma\langle c\Psi(\sigma Y)Y, Y\rangle\, d\sigma \geq a_0c\|Y\|^2\,.$$

Hence one can get from (1) that

$$\begin{aligned} V &\geq \frac{1}{2}b_0\|Z\|^2 - c\|Y\|\|Z\| + \frac{ca_0}{2}\|Y\|^2 + \frac{1}{2}\|cX + b_0Y\|^2 \\ &= \frac{1}{2}\|cX + b_0Y\|^2 + \frac{1}{2}b_0(\|Z\| - \frac{c}{b_0}\|Y\|)^2 + \frac{1}{2}\Big(\frac{a_0b_0 - c}{b_0}\Big)\|Y\|^2\,. \qquad (2)\end{aligned}$$

Thus, it is evident from the terms contained in (2) that there exists a sufficiently small positive constant D_1 such that

$$V \geq D_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2)\,. \tag{3}$$

Now, let $(X, Y, Z) = (X(t), Y(t), Z(t))$ be any solution of differential system (2). Differentiating the function $V(t) = V(X(t), Y(t), Z(t))$ with respect to t along system (2) and using Lemma 2.2, we obtain

$$\begin{aligned} \dot{V}(t) &= -\langle c(\Phi - b_0I)Y, Y\rangle + \langle b_0(\Phi - b_0I)Y, Z\rangle - \langle (b_0\Psi - cI)Z, Z\rangle \\ &= -\langle c(\Phi - b_0I)(Y - \frac{b_o}{2c}Z), (Y - \frac{b_o}{2c}Z)\rangle \\ &\quad - \langle [(b_0\Psi - cI) - \frac{b_0^2}{4c}(\Phi - b_0I)]Z, Z\rangle \\ &\leq -\langle [(b_0\Psi - cI) - \frac{b_0^2}{4c}(\Phi - b_0I)]Z, Z\rangle \\ &\leq -[(a_0b_0 - c) - \frac{b_0^2}{4c}\mu]\|Z\|^2 \leq 0\,. \qquad (4)\end{aligned}$$

In addition, one can easily see that

$$V(X,Y,Z) \to \infty \quad \text{as} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 \to \infty\,.$$

The whole discussion shows that the zero solution of (1) is globally asymptotic stable (see also Reissig et al. [[7], Theorem 1.5]).

Example 2.1. As a special case of system (2), let us take for $n = 2$ that

$$\Psi(Y) = \begin{pmatrix} 9+y^2 & 1 \\ 1 & 9+y^2 \end{pmatrix}, \quad \Phi(X) = \begin{pmatrix} 2 + \dfrac{1}{1+x^2} & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad c = 2\,.$$

By easy calculation, we obtain eigenvalues of the matrices $\Psi(Y)$ and $\Phi(X)$ as follows:

$$\lambda_1(\Psi) = 8 + y^2\,, \qquad\qquad \lambda_2(\Psi) = 10 + y^2$$

and

$$\lambda_1(\Phi) = 2 + \frac{1}{1+x^2}\,, \qquad \lambda_2(\Phi) = 2\,.$$

Next, it is clear that $\lambda(\Psi) \geq 8 = a_0$ and $b_0 = 2 \leq \lambda_i(\Phi) \leq 30$ since $b_0 + \mu = 30$ and $a_0 b_0 - c = 14 > 0$. Thus, all the conditions of Theorem 2.1 are satisfied. It should be noted that when $\Psi(Y)$ and $\Phi(X)$ reduce to the linear case our conclusion is also valid.

In the case $P \neq 0$, the second and last main result of this paper is the following theorem.

Theorem 2.2. *In addition to the conditions in Theorem 2.1, we suppose that there is a positive constant $K > 0$ and non-negative and continuous function $\theta = \theta(t)$ such that the following conditions are satisfied*
(i) $\|P(t)\| \leq \theta(t)$ *for all* $t \geq 0$, $\max \theta(t) < \infty$ *and* $\theta \in L^1(0,\infty)$,
where $L^1(0,\infty)$ is the space of integrable Lebesgue functions. Then there exists a constant $D > 0$ such that any solution $(X(t), Y(t), Z(t))$ of system (2) determined by

$$X(0) = X_0\,, \qquad Y(0) = Y_0\,, \qquad Z(0) = Z_0$$

satisfies,

$$\|X(t)\| \leq D\,, \qquad \|Y(t)\| \leq D\,, \qquad \|Z(t)\| \leq D$$

for all $t \in \mathbb{R}^+$.

Proof of Theorem 2.2. Our main tool for the proof of Theorem 2.2 is also the Liapunov function V defined in (1). Then under the assumptions of Theorem 2.2, we still obtain (3), and since $P(t) \neq 0$, it is also clear from (2), (1) and (4) that

$$\begin{aligned} \dot{V}(t) &\leq (\|b_0 Z\| + \|cY\|) \times \|P(t)\| \\ &= (b_0\|Z\| + c\|Y\|) \times \|P(t)\| \\ &\leq D_2(\|Z\| + \|Y\|) \times \theta(t)\,, \end{aligned} \tag{5}$$

where $D_2 = \max\{b_0, c\}$.
Now, in view of the inequalities

$$\|Y\| \le 1 + \|Y\|^2 , \quad \|Z\| \le 1 + \|Z\|^2$$

and (3), we have from (5) that

$$\begin{aligned} \dot{V}(t) &\le D_2(2 + \|Z\|^2 + \|Y\|^2) \times \theta(t) \\ &\le D_3\theta(t) + D_4\theta(t)V(t) \qquad (6) \end{aligned}$$

where $D_3 = 2D_2$ and $D_4 = D_1^{-1} D_2$.
Integrating both sides of (6) from 0 to t $(t \ge 0)$, one can easily obtain

$$V(t) - V(0) \le D_3 \int_0^t \theta(s)\, ds + D_4 \int_0^t V(s)\theta(s)\, ds\,.$$

Taking $D_5 = V(0) + D_3 K$, it follows that

$$V(t) \le D_5 + D_4 \int_0^t V(s)\theta(s)\, ds\,.$$

By using Gronwall-Bellman inequality (see Rao [6]), we conclude that

$$V(t) \le D_5 \exp\left(D_4 \int_0^t \theta(s)\, ds\right).$$

This result completes the proof of Theorem 2.2. □

Example 2.2. If in addition to Example 2.1, let

$$P(t) = \begin{pmatrix} \dfrac{1}{1+t^2} \\ \dfrac{1}{1+t^2} \end{pmatrix}.$$

Hence by elementary calculation, one can easily find
$\|P(t)\| = \dfrac{2}{1+t^2} \le \dfrac{3}{1+t^2} = \theta(t), \quad \max\theta(t) = 3 < \infty,$
and $\displaystyle\int_0^\infty \frac{3}{1+t^2}\, dt = \frac{3\pi}{2}$, that is $\theta \in L^1(0,\infty)$.
Thus, all the conditions of Theorem 2.2 are satisfied.

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