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ON LIFTS OF SOME PROJECTABLE VECTOR FIELDS ASSOCIATED TO A PRODUCT PRESERVING GAUGE BUNDLE FUNCTOR ON VECTOR BUNDLES

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Abstract. For a product preserving gauge bundle functor on vector bundles, we present some lifts of smooth functions that are constant or linear on fibers, and some lifts of projectable vector fields that are vector bundle morphisms.

1. Introduction

Weil functors (product preserving bundle functors on manifolds) were used in [2] to define some lifts of geometric objects namely, smooth functions, tensor fields and linear connections on manifolds.

Product preserving gauge bundle functor on vector bundles have been classified in [7]: The set of equivalence classes of such functors are in bijection with the set of equivalence classes of pairs (A, V), where A is a Weil algebra and V a A-module such that $\dim_{\mathbb{R}} (V) < \infty$.

In this paper, we adopt the approach of [2] to present some lifts of smooth functions that are constant or linear on fibers, and some lifts projectable vector fields that are vector bundle morphisms.

2. Algebraic description of Weil functors

Weil functors generalize through their covariant description ([3] and [6]) the classical bundle functors of velocities T^r_k. Their particular importance in differential geometry comes from the fact that there is a bijective correspondence between them and the set of product preserving bundle functors on the category of smooth manifolds.

2.1. **Weil algebra.**

A *Weil algebra* is a finite-dimensional quotient of the algebra of germs $\mathcal{E}_p = C^\infty_0(\mathbb{R}^p, \mathbb{R}) \quad (p \in \mathbb{N}\ )$.

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We denote by $\mathcal{M}_p$ the ideal of germs vanishing at 0; $\mathcal{M}_p$ is the maximal ideal of the local algebra $\mathcal{E}_p$.

Equivalently, a *Weil algebra* is a real commutative unital algebra such that $A = \mathbb{R}{\cdot}1_A \bigoplus N$, where N is a finite dimensional ideal of nilpotent elements.

Example 2.1. (1) $\mathbb{R} = \mathbb{R}{\cdot}1 \bigoplus \{0\} = \mathcal{E}_p/\mathcal{M}_p$ is a Weil algebra.

(2) $J_0^r(\mathbb{R}^p, \mathbb{R}) = \mathbb{R}{\cdot}1 \bigoplus J_0^r(\mathbb{R}^p, \mathbb{R})_0 = \mathcal{E}_p/\mathcal{M}_p^{r+1}$ is a Weil algebra.

2.2. **Weil functors.**

Let us recall the following lemma ([6, Lemma 35.8]): Let M be a smooth manifold and let $\varphi\colon C^\infty(M,\mathbb{R}) \to A$ be an algebra homomorphism into a Weil algebra A. Then there is a point $x \in M$ and some $k \geq 0$ such that $\ker\varphi$ contains the ideal of all functions which vanish at x up the order k i.e. $\ker\varphi \supset (I_x)^{k+1}$with $I_x = \{f \in C^\infty(M,\mathbb{R})/f(x) = 0\}$. More precisely, $\{x\} = \bigcap\limits_{f\in\ker\varphi} f^{-1}(\{0\})$.

One defines a functor $F_A\colon \mathcal{M}f \to \mathcal{E}ns$ by:

$$F_A M := \mathrm{Hom}(C^\infty(M,\mathbb{R}), A) \text{ and } F_A f(\varphi) := \varphi \circ f^*$$

for a manifold M and $f \in C^\infty(M,N)$, where $f^*\colon C^\infty(N,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ is the pull-back algebra homomorphism.

Equivalently, if for a manifold M and a point $x \in M$, $\mathrm{Hom}(C_x^\infty(M,\mathbb{R}), A)$ is the set of algebra homomorphisms from $C_x^\infty(M,\mathbb{R})$ into A, one may define a functor $G_A\colon \mathcal{M}f \to \mathcal{E}ns$ by:

$$G_A M := \bigcup_{x\in M} \mathrm{Hom}(C_x^\infty(M,\mathbb{R}), A) \text{ and } (G_A f)_x(\varphi_x) := \varphi_x \circ f_x^*$$

for a manifold M and $f \in C^\infty(M,N)$, where $f_x^*\colon C_{f(x)}^\infty(N,\mathbb{R}) \to C_x^\infty(M,\mathbb{R})$ is the pull-back algebra homomorphism induced by f^*.

F_A and G_A are equivalent functors: Indeed let $\varphi \in \mathrm{Hom}(C^\infty(M,\mathbb{R}), A)$ and $\{x\} = \bigcap\limits_{f\in\ker\varphi} f^{-1}(\{0\})$.

By the previous lemma, there is a unique $\varphi_x \in \mathrm{Hom}(C_x^\infty(M,\mathbb{R}), A)$ such that the diagram

$$\begin{tikzcd} C^\infty(M,\mathbb{R}) \arrow[r, "\varphi"] \arrow[d, "\pi_x"'] & A \\ C_x^\infty(M,\mathbb{R}) \arrow[ur, "\varphi_x"'] & \end{tikzcd}$$

commutes; the maps $\chi_M\colon F_A M \to G_A M, \varphi \mapsto \varphi_x$ are bijective and define a natural equivalence $\chi\colon F_A \to G_A$.

Now, let $T^A = G_A$ and $\pi_{A,M}\colon T^A M \to M, \varphi \ni (T^A M)_x \mapsto x$. $(T^A M, M, \pi_{A,M})$ is a well-defined fibered manifold. Indeed let $c = (U, u^i)$, $1 \leq i \leq m$ be a chart of M; then the map

$$\begin{aligned} \phi_c\colon (\pi_{A,M})^{-1}(U) &\longrightarrow U \times N^m \\ \varphi_x &\longmapsto \left(x, \varphi_x\left(\mathrm{germ}_x(u^i - u^i(x)\right)\right) ; \end{aligned}$$

is a local trivialization of T^AM. Given another manifold M' and a smooth map $f\colon M \to M'$, let

$$\begin{aligned} T^Af\colon T^AM &\longrightarrow T^AM' \\ \varphi_x &\longmapsto \varphi_x \circ f_x^* \,. \end{aligned}$$

Then T^Af is a fibered map and $T^A\colon \mathcal{M}f \to \mathcal{FM}$ is a product preserving bundle functor called the *Weil functor* associated to A.

Let $c = (U, u^i,)$, $1 \le i \le m$ be a chart of M; a fibered chart $(\pi_{A,M}^{-1}(U), u^{i,\alpha})$, $1 \le i \le m$, $1 \le \alpha \le K$ $(= \dim A)$ of T^AM is defined by $u^{i,\alpha} = \mathrm{pr}_\alpha \circ B \circ T^A(u^i)$, where $B\colon A \to \mathbb{R}^K$ is a linear isomorphism and $\mathrm{pr}_\alpha\colon \mathbb{R}^K \to \mathbb{R}$ the α-th projection.

3. Product preserving gauge bundle functor on vector bundles

3.1. **Product preserving gauge bundle functor on $\mathcal{VB}$.**

Let $F\colon \mathcal{VB} \to \mathcal{FM}$ be a covariant functor from the category $\mathcal{VB}$ of all vector bundles and their vector bundle homomorphisms into the category $\mathcal{FM}$ of fibered manifolds and their fibered maps. Let $B_{\mathcal{VB}}\colon \mathcal{VB} \to \mathcal{M}f$ and $B_{\mathcal{FM}}\colon \mathcal{FM} \to \mathcal{M}f$ be the respective base functors.

Definition 3.1. F is a *gauge bundle functor on* $\mathcal{VB}$ when the following conditions are satisfied:

• (**Prolongation**) $B_{\mathcal{FM}} \circ F = B_{\mathcal{VB}}$ i.e. F transforms a vector bundle $E \xrightarrow{q} M$ in a fibered manifold $FE \xrightarrow{p_E} M$ and a vector bundle morphism $E \xrightarrow{f} G$ over $M \xrightarrow{\overline{f}} N$ in a fibered map $FE \xrightarrow{Ff} FG$ over $\overline{f}$.

• (**Localization**) For any vector bundle $E \xrightarrow{q} M$ and any inclusion of an open vector subbundle $i\colon \pi^{-1}(U) \hookrightarrow E$, the fibered map $F\pi^{-1}(U) \to p_E^{-1}(U)$ over id_U induced by Fi is an isomorphism; then the map Fi can be identified with the inclusion $p_E^{-1}(U) \hookrightarrow FE$.

Given two gauge bundle functors F_1, F_2 on $\mathcal{VB}$, by *a natural transformation* $\tau\colon F_1 \to F_2$ we shall mean a system of base preserving fibered maps $\tau_E\colon F_1E \to F_2E$ for every vector bundle E satisfying $F_2f \circ \tau_E = \tau_G \circ F_1f$ for every vector bundle morphism $f\colon E \to G$.

A gauge bundle functor F on $\mathcal{VB}$ is *product preserving* if for any product projections $E_1 \xleftarrow{\mathrm{pr}_1} E_1 \times E_2 \xrightarrow{\mathrm{pr}_2} E_2$ in the category $\mathcal{VB}$, $FE_1 \xleftarrow{F\,\mathrm{pr}_1} F(E_1 \times E_2) \xrightarrow{F\,\mathrm{pr}_2} FE_2$ are product projections in the category $\mathcal{FM}$. In other words, the map $(F\,\mathrm{pr}_1, F\,\mathrm{pr}_2\colon F(E_1 \times E_2) \to F(E_1) \times F(E_2)$ is a fibered isomorphism over $\mathrm{id}_{M_1 \times M_2}$.

Example 3.1. (a) Each Weil functor T^A induces a product preserving gauge bundle functor $T^A\colon \mathcal{VB} \to \mathcal{FM}$ in a natural way.

(b) Let $A = \mathbb{R}\cdot 1_A \bigoplus N$ be a Weil algebra and V a A-module such that $\dim_{\mathbb{R}}(V) < \infty$. For a vector bundle (E, M, q) and $x \in M$, let

$$T_x^{A,V}E = \\ \left\{(\varphi_x, \psi_x)/\varphi_x \in \mathrm{Hom}(C_x^\infty(M,\mathbb{R}), A) \text{ and } \psi_x \in \mathrm{Hom}_{\varphi_x}(C_x^{\infty, f\cdot l}(E), V)\right\}$$

where $\mathrm{Hom}(C_x^\infty(M,\mathbb{R}), A)$ is the set of algebra homomorphisms φ_x from the algebra $C_x^\infty(M,\mathbb{R}) = \{\mathrm{germ}_x(g) \ / \ g \in C^\infty(M,\mathbb{R})\}$ into A and $\mathrm{Hom}_{\varphi_x}(C_x^{\infty,\mathrm{fl}}(E), V)$ is the set of module homomorphisms ψ_x over φ_x from the $C_x^\infty(M,\mathbb{R})$-module $C_x^{\infty,\mathrm{fl}}(E,\mathbb{R}) = \{\mathrm{germ}_x(h) \ / \ h\colon E \to \mathbb{R} \text{ is fibre linear}\}$ into $\mathbb{R}$.
Let $T^{A,V}E = \bigcup_{x\in M} T_x^{A,V}E$ and $p_E^{A,V} : T^{A,V}E \to M$, $(\varphi,\psi) \ni T_x^{A,V}E \mapsto x$.
$(T^{A,V}E, M, p_E^{A,V})$ is a well-defined fibered manifold. Indeed, let $c = (q^{-1}(U), x^i = u^i \circ q, y^j)$, $1 \le i \le m$, $1 \le j \le n$ be a fibered chart of E; then the map

$$\begin{aligned} \phi_c\colon (p_E^{A,V})^{-1}(U) &\longrightarrow U \times N^m \times V^n \\ (\varphi_x, \psi_x) &\longmapsto \left(x, \varphi_x(\mathrm{germ}_x(u^i - u^i(x))), \psi_x(\mathrm{germ}_x(y^j))\right); \end{aligned}$$

is a local trivialization of $T^{A,V}E$. Given another vector bundle (G, N, q') and a vector bundle homomorphism $f\colon E \to G$ over $\overline{f}\colon M \to N$, let

$$\begin{aligned} T^{A,V}f\colon T^{A,V}E &\longrightarrow T^{A,V}G \\ (\varphi_x, \psi_x) &\longmapsto (\varphi_x \circ \overline{f}_x^*, \psi_x \circ f_x^*), \end{aligned}$$

where $\overline{f}_x^*\colon C_{\overline{f}(x)}^\infty(N) \to C_x^\infty(M)$ and $f_x^*\colon C_{\overline{f}(x)}^{\infty,\mathrm{fl}}(G) \to C_x^{\infty,\mathrm{fl}}(E)$ are given by the pull-back with respect to $\overline{f}$ and f. Then $T^{A,V}f$ is a fibered map over $\overline{f}$. $T^{A,V} : \mathcal{VB} \to \mathcal{FM}$ is a product preserving gauge bundle functor (see [7]).

Remark 3.1. Let $F\colon \mathcal{VB} \to \mathcal{FM}$ be a product preserving gauge bundle functor.

(a) F associates the pair $\left(A^F, V^F\right)$ where $A^F = F\left(\mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}\right)$ is a Weil algebra and $V^F = F\left(\mathbb{R} \to pt\right)$ is a A^F-module such that $\dim_{\mathbb{R}}(V^F) < \infty$. Moreover there is a natural isomorphism $\Theta\colon F \to T^{A^F,V^F}$ and equivalence classes of functors F are in bijection with equivalence classes of pairs (A^F, V^F). In particular, the product preserving gauge bundle functor associated to the Weil functor T^A is equivalent to $T^{A,A}$.

(b) Let $K = \dim A^F$ and $L = \dim V^F$; if $c = (\pi^{-1}(U), x^i, y^j)$, $1 \le i \le m$, $1 \le j \le n$ is a fibered chart of E, a fibered chart $(p_E^{-1}(U), x^{i,\alpha}, y^{j,\beta})$, $1 \le \alpha \le K$, $1 \le \beta \le L$ of FE is defined by

$$\begin{cases} x^{i,\alpha} = \lambda_B^\alpha \circ F\left(x^i\right) \\ y^{j,\beta} = \mu_C^\beta \circ F\left(y^j\right) \end{cases} \tag{3.1}$$

where $\lambda_B^\alpha = \mathrm{pr}_\alpha \circ B$, $\mu_C^\beta = \mathrm{pr}_\beta \circ C$ and $B\colon A^F \to \mathbb{R}^K$, $C\colon V^F \to \mathbb{R}^L$ are linear isomorphisms. In the particular case $F = T$, $A^F = V^F = \mathbb{D} = \mathbb{R}\left[X\right]/\left(X^2\right)$ is the algebra of dual numbers and the previous coordinate system is denoted

$\big(x^i, \dot{x}^i, y^j, \dot{y}^j\big)$ where

$$x^i = \lambda^1_B \circ F(x^i),\ \dot{x}^i = \lambda^2_B \circ F(x^i), y^j = \mu^1_B \circ F(y^j), \dot{y}^j = \mu^2_B \circ F(y^j)$$

and $B\colon \mathbb{D} \to \mathbb{R}^2$ the canonical isomorphism.

3.2. **The natural isomorphism $\kappa\colon F \circ T \to T \circ F$.**

For a product preserving gauge bundle functor $F\colon \mathcal{VB} \to \mathcal{FM}$, let $\kappa\colon F \circ T \to T \circ F$ be the canonical natural isomorphism associated to the exchange isomorphism $(\mathbb{D} \otimes_{\mathbb{R}} A^F, \mathbb{D} \otimes_{\mathbb{R}} V^F) \cong (A^F \otimes_{\mathbb{R}} \mathbb{D}, V^F \otimes_{\mathbb{R}} \mathbb{D})$ ([7, Corollary 3]). Using the definition of composed functors $F \circ T$ and $T \circ F$, one can check that locally

$$\kappa_E\colon \big(x^{i,\alpha}, \dot{x}^{i,\alpha}, y^{j,\beta}, \dot{y}^{j,\beta}\big) \mapsto \big(x^{i,\alpha}, x^{\dot{i,\alpha}}, y^{j,\beta}, y^{\dot{j,\beta}}\big)$$

with $x^{\dot{i,\alpha}} = \dot{x}^{i,\alpha}$ and $y^{\dot{j,\beta}} = \dot{y}^{j,\beta}$. The following assertion is clear.

Proposition 3.1. (a) *The diagram*

$$\begin{tikzcd} F(TE) \arrow[r, "\kappa_E"] \arrow[dr, "F(\pi_E)"'] & T(FE) \arrow[d, "\pi_{FE}"] \\ & FE \end{tikzcd}$$

commutes for any vector bundle $E \xrightarrow{q} M$.

(b) *If (F, π') is an excellent pair (i.e. $\pi'\colon F \to \mathrm{id}_{\mathcal{VB}}$ is a natural epimorphism), the diagram*

$$\begin{tikzcd} F(TE) \arrow[r, "\kappa_E"] \arrow[dr, "\pi'_{FE}"'] & T(FE) \arrow[d, "T(\pi'_E)"] \\ & TE \end{tikzcd}$$

commutes for any vector bundle $E \xrightarrow{q} M$.

Let $X \in \mathfrak{X}_{\mathrm{proj}}(E)$ be a *projectable vector field* on E i.e. a fibered map over a vector field $\overline{X}$ on M. We assume that X is a vector bundle morphism. The commutative diagram

$$\begin{tikzcd} FE \arrow[r, "(\mathcal{F}_E)X"] \arrow[d, "FX"'] & TFE \\ FTE \arrow[ur, "\kappa_E"'] & \end{tikzcd}$$

defines a vector field on FE called the *complete lift* of X and denoted X^c. The natural operator $\mathcal{F}\colon T \rightsquigarrow TF$ is called the *flow operator* of F.

When we restrict ourself to vector bundles of the form (M, M, id_M), κ and $\mathcal{F}$ are exactly the canonical flow natural equivalence and the flow operator associated to the Weil functor T^{A^F}, [6].

4. Prolongation of functions

We generalize for a product preserving gauge bundle $F\colon \mathcal{VB} \to \mathcal{FM}$ some prolongations of [2].

Let $f\colon E \to \mathbb{R}$ be a smooth function defined on a vector bundle $q\colon E \to M$.

Definition 4.1.

(a) The *λ-lift* of f constant on fibers is $f^{(\lambda)} := \lambda \circ Ff$, for $\lambda \in C^\infty(A^F, \mathbb{R})$.

(b) The *μ-lift* of f linear on fibers (i.e. $f \in C^\infty_\ell(E, \mathbb{R})$) is $f^{(\mu)} := \mu \circ Ff$, for $\mu \in C^\infty(V^F, \mathbb{R})$.

In the particular case $(E, M, q) = (M, M, \mathrm{id}_M)$, lifts of functions constant on fibers correspond to lifts of functions associated to the Weil functor T^{A^F} [2]. Lifts of functions linear on fibers correspond to lifts of smooth sections of the dual bundle $q^*\colon E^* \to M$.

It is easy to check that $(f \circ h)^{(\lambda)} = f^{(\lambda)} \circ Fh$, for $h\colon G \to E$ a vector bundle morphism and $(f_1 + f_2)^{(\lambda)} = f_1^{(\lambda)} + f_2^{(\lambda)}$ when λ is a linear map. Replacing λ with μ, the previous identities hold.

According to (3.1),

$$x^{i,\alpha} = (x^i)^{(\lambda^\alpha_B)} \quad \text{and} \quad y^{j,\beta} = (y^j)^{(\mu^\beta_C)}\,,$$

hence the following result holds:

Proposition 4.1. *If two vector fields $\widehat{X}$, $\widehat{Y}$ on FE satisfy*

$$\widehat{X}(f^{(\lambda)}) = \widehat{Y}(f^{(\lambda)}) \quad \textit{and} \quad \widehat{X}(g^{(\mu)}) = \widehat{Y}(g^{(\mu)})\,,$$

for any f constant on fibers, g linear on fibers, $\lambda \in C^\infty(A^F, \mathbb{R})$ and $\mu \in C^\infty(V^F, \mathbb{R})$, then $\widehat{X} = \widehat{Y}$.

5. Prolongation of projectable vector fields

5.1. Natural transformations $Q(a)\colon T \circ F \to T \circ F$.

For a vector bundle $q\colon E \to M$, let us denote $\mu_E\colon \mathbb{R} \times TE \to TE$, $(\alpha, k_u) \ni \mathbb{R} \times T_uE \mapsto \alpha \cdot k_u \in T_uE$, the fibered multiplication. This is a vector bundle morphism over the projection $\mathbb{R} \times E \to E$, hence for any $a \in A^F$, we have a natural transformation $FT \to FT$ given by the partial maps $F\mu_E(a, \cdot)\colon FTE \to FTE$. If $\kappa\colon T \to TF$ is the canonical natural isomorphism (subsection 3.2), one can deduce a natural transformation $Q(a)\colon TF \to TF$ by

$$Q(a) = \kappa \circ F\mu(a, \cdot) \circ \kappa^{-1}\,. \tag{5.1}$$

The restriction of $Q(a)$ to vector bundles of the form (M, M, id_M) is just the natural affinor associated to a [5].

Let us compute the algebra homomorphism $\mu_a\colon A^{FT} \to A^{FT}$ and the module homomorphisms $\upsilon_a\colon V^{FT} \to V^{FT}$ over μ_a associated to the natural transformation $F\mu\,(a, \cdot)$: Let $p_1\colon \mathbb{D} \to \mathbb{R}1$, $p_2\colon \mathbb{D} \to \mathbb{R}\delta$ the canonical projections associated to the algebra of dual numbers and $i_u\colon \mathbb{R} \to \mathbb{D}$, $t \mapsto tu$, $u \in \mathbb{D}$; since $\mu_{\mathbb{R}\to\mathbb{R}}\,(s, u) =$

$p_1(u) + p_2(su) = T(ad_{\mathbb{R}})(p_1(u), p_2 \circ m \circ (\mathrm{id}_{\mathbb{R}} \times \mathrm{id}_{\mathbb{D}})(s,u))$, $m(s,u) = su$ then for $b \in A^F$

$$\begin{aligned}
\mu_a(Fi_u(b)) &= F(Tad_{\mathbb{R}})\big(Fi_{p_1(u)}(b), F(p_2 \circ m)(a, Fi_u(b))\big)\\
&= F(Tad_{\mathbb{R}})\big(Fi_{p_1(u)}(b), F(p_2 \circ m)(F\,\mathrm{id}_{\mathbb{R}}(a), Fi_u(b))\big)\\
&= F(Tad_{\mathbb{R}})\big(Fi_{p_1(u)}(b), F(p_2 \circ m \circ \mathrm{id}_{\mathbb{R}} \times i_u)(a,b)\big)\\
&= F(Tad_{\mathbb{R}})\big(Fi_{p_1(u)}(b), F\big(i_{p_2(u)} \circ m_0\big)(a,b)\big),\ m_0(s,t) = st\\
&= Fi_{p_1(u)}(b) + Fi_{p_2(u)}(ab)\\
&= \mu_{FT}(p_1(u) \otimes b + p_2(u) \otimes ab)\ ,
\end{aligned}$$

where $\mu_{FT} \colon \mathbb{D} \otimes_{\mathbb{R}} A^F \to A^{FT}$ is the canonical algebra isomorphism. Hence

$$\begin{aligned}
\mu_{FT}^{-1} \circ \mu_a \circ \mu_{FT} \colon \mathbb{D} \otimes_{\mathbb{R}} A^F &\to \mathbb{D} \otimes_{\mathbb{R}} A^F\\
u \otimes b &\mapsto p_1(u) \otimes b + p_2(u) \otimes ab\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\upsilon_a(Fi_u(v)) &= F(Tad_{\mathbb{R}})\big(Fi_{p_1(u)}(v), F(p_2 \circ m)(a, Fi_u(v))\big)\\
&= F(Tad_{\mathbb{R}})\big(Fi_{p_1(u)}(v), F(p_2 \circ m)(F\,\mathrm{id}_{\mathbb{R}}(a), Fi_u(v))\big)\\
&= F(Tad_{\mathbb{R}})\big(Fi_{p_1(u)}(v), F(p_2 \circ m \circ \mathrm{id}_{\mathbb{R}} \times i_u)(a,v)\big)\\
&= F(Tad_{\mathbb{R}})\big(Fi_{p_1(u)}(v), F\big(i_{p_2(u)} \circ m_0\big)(a,v)\big),\ m_0(s,t) = st\\
&= Fi_{p_1(u)}(v) + Fi_{p_2(u)}(a \cdot v)\\
&= \upsilon_{FT}(p_1(u) \otimes v + p_2(u) \otimes a \cdot v)\ ,
\end{aligned}$$

where $\upsilon_{FT} \colon \mathbb{D} \otimes_{\mathbb{R}} V^F \to V^{FT}$ is the natural module isomorphism over μ_{FT}. Hence

$$\begin{aligned}
\upsilon_{FT}^{-1} \circ \upsilon_a \circ \upsilon_{FT} \colon \mathbb{D} \otimes_{\mathbb{R}} V^F &\to \mathbb{D} \otimes_{\mathbb{R}} V^F\\
u \otimes v &\mapsto p_1(u) \otimes v + p_2(u) \otimes a \cdot v\,.
\end{aligned}$$

5.2. **Prolongation of projectable vector fields.**

In this subsection, all projectable vector fields are vector bundle morphisms.

Definition 5.1. For a *projectable vector field* $X \in \mathfrak{X}_{proj}(E)$, its a-lift is given by $X^{(a)} = Q(a)_E \circ (\mathcal{F}_E)X$, where $\mathcal{F} \colon T \rightsquigarrow TF$ is the flow operator of F.

Let $\lambda \colon A^F \to \mathbb{R}$, $\mu \colon V^F \to \mathbb{R}$ linear maps and $\lambda_a \colon A^F \to \mathbb{R}$, $\mu_a \colon V^F \to \mathbb{R}$ given by $\lambda_a(x) = \lambda(ax)$, $\mu_a(v) = \mu(a \cdot v)$, for $a \in A^F$.

Theorem 5.1. *We have*

$$X^{(a)}(f^{(\lambda)}) = (X(f))^{(\lambda_a)} \quad \text{and} \quad X^{(a)}(g^{(\mu)}) = (X(g))^{(\mu_a)}\ ,$$

for any f constant on fibers and g linear on fibers.

Proof. Let $\tau \in C^\infty_\ell(T\mathbb{R}, \mathbb{R})$ such that $\tau_x \colon T_x\mathbb{R} \to \mathbb{R}$, $x \in \mathbb{R}$ the canonical isomorphism. Identifying the module of 1-forms on a manifold M with $C^\infty_\ell(TM, \mathbb{R})$, the differential of $f \in C^\infty(M, \mathbb{R})$ is given by $df = \tau \circ Tf$ and for a vector field $\overline{X}$ on M, $\overline{X}(f) = \langle \overline{X}, df \rangle = \tau \circ Tf \circ \overline{X}$.

We have

$$\begin{aligned} X^{(a)}(f^{(\lambda)}) &= \tau \circ Tf^{(\lambda)} \circ \kappa_E \circ F\mu_E\,(a,\cdot) \circ FX \\ &= \tau \circ T\lambda \circ T\,(Ff) \circ \kappa_E \circ F\mu_E\,(a,\cdot) \circ FX \\ &= \tau \circ T\lambda \circ \kappa_{\mathbb{R}\to\mathbb{R}} \circ F\,(\mu_{\mathbb{R}\to\mathbb{R}}\,(a,\cdot)) \circ F\,(Tf \circ X) \\ &= \lambda_a \circ F\,(\tau) \circ F\,(Tf \circ X) \\ &= (X\,(f))^{(\lambda_a)}\,. \end{aligned}$$

The second equality is proved in the same way. □

Corollary 5.1. *For any vector fields $X, Y \in \mathfrak{X}_{\mathrm{proj}}\,(E)$, we have*

$$\left[X^{(a)}, Y^{(b)}\right] = [X,Y]^{(ab)}\ .$$

Proof. Direct consequence of the previous result and Proposition 4.1. □

This generalizes (for product preserving gauge bundle functors on vector bundles) some results of [2].

References

[1] Cordero, L.A., Dobson, C.T.J., De León, M., *Differential geometry of frame bundles*, Kluwer Academic Publishers, 1989.

[2] Gancarzewicz, J., Mikulski, W.M., Pogoda, Z., *Lifts of some tensor fields and connections to product preserving functors*, Nagoya Math. J. **135** (1994), 1–14.

[3] Kolář, I., *Covariant approach to natural transformations of Weil bundles*, CMUC **27** (1986), 723–729.

[4] Kolář, I., *On the natural operators on vector fields*, Ann. Global Anal. Geom. **6** (1988), 119–117.

[5] Kolář, I., Modugno, M., *Torsions of connections on some natural bundles*, Differential Geom. Appl. **2** (1992), 1–16.

[6] Kolář, I., Slovák, J., Michor, P.W., *Natural operations in differential geometry*, Springer-Verlag Berlin–Heidelberg, 1993.

[7] Mikulski, W.M., *Product preserving gauge bundle functors on vector bundles*, Colloq. Math. **90** (2001), no. 2, 277–285.

[8] Ntyam, A., Mba, A., *On natural vector bundle morphisms $T^A \circ \otimes_s^q \to \otimes_s^q \circ T^A$ over* id_{TA}, Ann. Polon. Math. **96** (2009), no. 3, 295–301.

[9] Ntyam, A., Wouafo, K.J., *New versions of curvature and torsion formulas for the complete lifting of a linear connection to Weil bundles*, Ann. Polon. Math. **82** (2003), no. 3, 133–140.

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