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DE LA VALLÉE POUSSIN TYPE INEQUALITY AND EIGENVALUE PROBLEM FOR GENERALIZED HALF-LINEAR DIFFERENTIAL EQUATION

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Abstract. We study the generalized half-linear second order differential equation via the associated Riccati type differential equation and Prüfer transformation. We establish a de la Vallée Poussin type inequality for the distance of consecutive zeros of a nontrivial solution and this result we apply to the "classical" half-linear differential equation regarded as a perturbation of the half-linear Euler differential equation with the so-called critical oscillation constant. In the second part of the paper we study a Dirichlet eigenvalue problem associated with the investigated half-linear equation.

1. Introduction

In this paper we deal with the so-called *generalized half-linear differential equation*

$$x'' + c(t)f(x, x') = 0\,, \tag{1}$$

where c is a continuous function and the function f satisfies the following assumptions introduced in [3, 4].

(i) The function f is continuous on $\Omega = \mathbb{R} \times \mathbb{R}_0$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$;
(ii) It holds $xf(x, y) > 0$ if $xy \neq 0$;
(iii) The function f is homogeneous, i.e., $f(\lambda x, \lambda y) = \lambda f(x, y)$ for $\lambda \in \mathbb{R}$ and $(x, y) \in \Omega$;
(iv) The function f is sufficiently smooth such that the solutions of (1) depend continuously and uniquely on the initial condition $x(t_1) = x_0$, $x'(t_1) = x_1$ for $(x_0, x_1) \in \Omega$;
(v) Let $F(t) := tf(t, 1)$, then

$$\int_{-\infty}^{\infty} \frac{dt}{1 + F(t)} < \infty \quad \text{and} \quad \lim_{|t|\to\infty} F(t) = \infty\,.$$

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A typical model of (1) is the "classical" half-linear differential equation

$$\big(\Phi(x')\big)' + c(t)\Phi(x) = 0\,, \quad \Phi(x) := |x|^{p-2}x\,, \quad p > 1\,, \tag{2}$$

which attracted considerable attention in the recent years, see [1, 11], and its investigation was initiated by the fundamental Elbert's paper [16] from 1979. Differentiating the first term in (2) we obtain $\big(\Phi(x')\big)' = (p-1)|x'|^{p-2}x''$ and hence (2) can be written as

$$x'' + \frac{c(t)}{p-1}\Phi(x)|x'|^{2-p} = 0$$

which is an equation of the form (1) (with $f(x,x') = \frac{1}{p-1}\Phi(x)|x'|^{2-p}$ and $F(t) = |t|^p$). Generalized half-linear equation and also equation (2) are sometimes considered in a more general form

$$\big(a(t)x'\big)' + c(t)f(x, a(t)x') = 0\,, \tag{3}$$

resp.

$$\big(a(t)\Phi(x')\big)' + c(t)\Phi(x) = 0 \tag{4}$$

with a positive continuous function a. However, the change of independent variable $s = \int^t a^{-1}(\tau)\,d\tau$ resp. $\int^t a^{1-q}(\tau)\,d\tau$, $\frac{1}{p} + \frac{1}{q} = 1$, converts (3) and (4) into an equation of the form (1) or (2), respectively. The terminology "half-linear" equation is justified by the fact that the solution space of (1) is homogeneous but not generally additive, i.e., it possesses just one half of the properties characterizing linearity.

A standard subject of the present investigation is what results of the deeply developed qualitative theory of the *linear* second order Sturm-Liouville differential equation

$$\big(a(t)x'\big)' + c(t)x = 0 \tag{5}$$

(which is the special case $p = 2$ in (4)) can be "half-linearized", i.e., extended to (2). From this point of view, it is also natural to look which results known for (classical) half-linear equation (2) can be extended to its more general form (1) or (3), see also [5, 12].

The "restored" interest in the investigation of generalized half-linear differential equations is motivated by the fact that the recently introduced so-called *modified Riccati differential equation* associated with (2), as appeared e.g. in [13, 14, 18, 23], is the special form of the Riccati type differential equation associated with (1), see the later given equation (10) and also equation (28) from Section 4. Modified Riccati equation turned out to be a very useful tool in the oscillation theory of (4) since it essentially substitutes the missing transformation theory for (4), see [9, 25] and also [11, Sec. 1.3].

2. Preliminaries

The classical de la Vallée Pousin inequality, as established in [7], concerns of the second order linear differential equation

$$x'' + b(t)x' + c(t)x = 0 \tag{6}$$

with continuous functions b, c and it claims that if $t_1 < t_2$ are consecutive zeros of a nontrivial solution of (6), $h := t_2 - t_1$, then $1 < Bh + \frac{1}{2}Ch^2$, where

$$B = \max_{t\in[t_1,t_2]} |b(t)|\,, \quad C = \max_{t\in[t_1,t_2]} |c(t)|\,.$$

This condition for the distance of consecutive zeros of (6) was improved in several subsequent papers [6, 10, 15, 21, 22, 24, 26], see also the survey paper [2]. Our presentation follows the line of [6], where it was proved that the distance of consecutive zeros satisfies

$$2\int_0^\infty \frac{dt}{1 + Bt + Ct^2} \le t_2 - t_1\,. \tag{7}$$

The proof of our extension of (7) to (1) is based on the relationship between (1) and the associated Riccati type differential equation which we present in the next part of this section which is taken from [17]. Let g be the differentiable function given by the formula

$$g(u) = \begin{cases} \int_{1/u}^\infty \dfrac{ds}{F(s)} & \text{if} \quad u > 0\,, \\ -\int_{-\infty}^{1/u} \dfrac{ds}{F(s)} & \text{if} \quad u < 0 \end{cases} \tag{8}$$

with the function F from the assumption (v) of Section 1, and $g(0) = 0$. Then g is increasing and

$$\lim_{u\to\pm\infty} g(u) = \pm\infty\,. \tag{9}$$

If x is a solution of (1) such that $x(t) \neq 0$, then the function $v = g(x'/x)$ solves the Riccati type differential equation

$$v' + c(t) + H(v) = 0\,, \tag{10}$$

where the function H is given by the formula

$$H(v) = [g^{-1}(v)]^2 g'\big(g^{-1}(v)\big) \tag{11}$$

with $H(0) = 0$ (g^{-1} being the inverse function of g). Moreover, the function H satisfies

$$\int_{-\infty}^{-\varepsilon} \frac{dv}{H(v)} < \infty\,, \quad \int_\varepsilon^\infty \frac{dv}{H(v)} < \infty \tag{12}$$

for some (and hence for every) $\varepsilon > 0$. Note that in case of the classical half-linear differential equation (2) we have $g(u) = \Phi(u)$ and $H(v) = (p-1)|v|^q$, where q is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

The existence of a Riccati type differential equation associated with (1) implies that the classical linear Sturmian theory can be extended to (1) and hence this equation can be classified as oscillatory or nonoscillatory similarly as in the linear case.

Next we present the generalized Prüfer transformation which we use in studying the Dirichlet eigenvalue problem associated with (1). Consider equation (1) with $c(t) = 1$, i.e., the equation $x'' + f(x, x') = 0$, together with the initial condition $x(0) = 0, x'(0) = 1$. The solution of this initial value problem we denote by $S = S(t)$. Following [17], we call this function the generalized sine function and its derivative $S'(t) =: C(t)$ the generalized cosine function. Using these functions we introduce the generalized Prüfer transformation as follows. Let x be a nontrivial solution of (1) and define continuous functions φ, ϱ as generalized polar coordinates

$$x(t) = \varrho(t)S\big(\varphi(t)\big)\,, \quad x'(t) = \varrho(t)C\big(\varphi(t)\big)\,. \tag{13}$$

We also define

$$\hat{\pi} = \int_{-\infty}^{\infty} \frac{ds}{1 + F(s)}\,, \tag{14}$$

where the function F is defined in (v) of Introduction. Then, the function φ is a solution of the equation

$$\varphi' = 1 + (c(t) - 1)G(\varphi)\,,$$

where

$$G(\varphi) = \begin{cases} \dfrac{F(T(\varphi))}{1 + F(T(\varphi))}\,, & \varphi \neq (2k+1)\frac{\hat{\pi}}{2}\,, \quad k \in \mathbb{Z} \\ 1\,, \quad \varphi = (2k+1)\dfrac{\hat{\pi}}{2}\,, & k \in \mathbb{Z}\,, \end{cases} \tag{15}$$

$T(\varphi) = \frac{S(\varphi)}{C(\varphi)}$ being the generalized tangent function. Together with (1) we consider another equation of the same form

$$y'' + C(t)f(y, y') = 0 \tag{16}$$

with a continuous function C and with the same functions f as in (1). We suppose that (16) is a majorant of (1), i.e., $C(t) \geq c(t)$ in an interval under consideration.

In the second part of the paper we use a comparison result for (1) and (16) that the boundary value problem associated with the generalized half-linear equation containing an eigenvalue parameter

$$x'' + \lambda c(t)f(x, x') = 0 \tag{17}$$

together with the Dirichlet boundary condition

$$x(0) = 0 = x(\hat{\pi}) \tag{18}$$

possesses a sequence of eigenvalues $\lambda_n \to \infty$ with the property that the eigenfunction x_n corresponding to λ_n has exactly $n - 1$ zeros on $(0, \hat{\pi})$. To prove this result we need the following fact. If y is a nontrivial solution of (16) and ψ is its Prüfer angle (defined analogically as in (13)), then by [17, Theorem 4.9] we have $\psi(t) \geq \varphi(t)$ if $\psi(t_0) \geq \varphi(t_0)$ at an initial condition. Moreover, we have $y(t) = 0$ just if $\psi(t) = k\hat{\pi}$, $k \in \mathbb{Z}$, and $\psi'(t) = 1$ at these points t since $S(\psi(t)) = 0$ if and only if $\psi\ (t) = 0$ (mod $\hat{\pi}$).

3. Vallée Poussin inequality

The main result of this section reads as follows.

Theorem 1. *Let $t_1 < t_2$ be consecutive zeros of a nontrivial solution of* (1)*. Then*

$$\int_{-\infty}^{\infty} \frac{du}{C + H(u)} \le t_2 - t_1 \,, \tag{19}$$

where

$$C = \max_{t \in [t_1, t_2]} |c(t)| \,,$$

and H is the function which appears in the generalized Riccati equation associated with (1)*, i.e., it is given by* (11) *with the function g given by* (8)*.*

Proof. Since equation (1) is homogeneous, without loss of generality we can suppose that $x(t) > 0$ for $t \in (t_1, t_2)$. Let c, d be the first and the last points of local maxima of x in (t_1, t_2), so that $c, d \in (t_1, t_2)$, $c \le d$, $x'(t) > 0$ for $t \in (t_1, c)$, $x'(c) = 0$, and $x'(d) = 0$, $x'(t) < 0$ for $t \in (d, t_2)$. Let $v = g(x'/x)$ be the solution of (10) generated by x. Then (9) implies that $v(t) > 0$ in (t_1, c), $v(t_1+) = \infty$, and $v(c) = 0$.

We have for $t \in (t_1, c)$

$$v' = -c(t) - H(v) \ge -C - H(v) \,, \tag{20}$$

hence

$$\int_{t_1}^{c} \frac{v'(t)dt}{C + H(v(t))} \ge -(c - t_1) \,.$$

Substituting $v(t) = u$, the integral takes form

$$\int_{\infty}^{0} \frac{du}{C + H(u)} \ge t_1 - c \,,$$

which means that

$$\int_{0}^{\infty} \frac{du}{C + H(u)} \le c - t_1 \,. \tag{21}$$

Now consider the interval $[d, t_2)$. In this interval $v(t) < 0$, $v(d) = 0$, and $v(t_2-) = -\infty$. Integrating (20) from d to t_2 we get

$$\int_{d}^{t_2} \frac{v'(t)dt}{C + H(v(t))} \ge -(t_2 - d) \,.$$

Using the previous substitution the integral now is

$$\int_{0}^{-\infty} \frac{du}{C + H(u)} \ge d - t_2,$$

i.e.,

$$\int_{-\infty}^{0} \frac{du}{C + H(u)} \le t_2 - d \,. \tag{22}$$

The summation of (21) and (22) gives

$$\int_{-\infty}^{\infty} \frac{du}{C+H(u)} \le t_2 - d + c - t_1$$

and since $c \le d$, we have $t_2 - d + c - t_1 \le t_2 - t_1$. This gives the required inequality (19). $\square$

Remark 1. (i) If t_2 is the right focal point of t_1, i.e., there exists a solution x of (1) such that $x'(t_1) = 0$, $x(t_2) = 0$, then using the same reasoning as in the proof of the previous theorem we obtain the inequality

$$\int_{-\infty}^{0} \frac{du}{C+V(u)} \le t_2 - t_1\,.$$

A similar inequality is obtained also for the distance $t_2 - t_1$ for a nontrivial solution satisfying $x(t_1) = 0$, $x'(t_2) = 0$.

(ii) In the linear case $f(x, x') = x$ we have $H(v) = v^2$, hence (19) reduces to (7) with $B = 0$.

(iii) Another important and frequently investigated inequality for the distance of consecutive zeros of various types of differential equations is the Lyapunov inequality, which for (2) reads

$$t_2 - t_1 \ge \frac{1}{2^p} \int_{t_1}^{t_2} \max\{0, c(s)\}\, ds\,.$$

The proof of inequalities of this form is mostly based on the relationship between an equation and its associated energy functional which in case of (2) is (see [16])

$$\mathcal{F}(y : t_1, t_2) = \int_{t_1}^{t_2} \left[|y'|^p - c(t)|y|^p\right] dt\,. \tag{23}$$

Concerning equation (1), we have not been able to find a functional which would play for (1) a similar role as (23) for (2) yet, so Lyapunov type inequality for (1) is missing till now. This problem is a subject of the present investigation.

4. Perturbed Euler equation

In this section we apply the method used in the previous section to the equation

$$\big(\Phi(x')\big)' + c(t)\Phi(x) = 0\,, \quad t \ge 0\,, \tag{24}$$

regarded as a perturbation of the "critical" half-linear Euler equation

$$\big(\Phi(x')\big)' + \frac{\gamma_p}{t^p}\Phi(x) = 0\,, \quad \gamma_p := \Big(\frac{p-1}{p}\Big)^p\,, \tag{25}$$

i.e., we rewrite (24) into the form

$$\big(\Phi(x')\big)' + \frac{\gamma_p}{t^p}\Phi(x) + \Big(c(t) - \frac{\gamma_p}{t^p}\Big)\Phi(x) = 0\,, \tag{26}$$

we refer to [18] concerning the adjective “critical” in the Euler equation. Let w be a solution of the Riccati equation associated with (24)

$$w' + c(t) + (p-1)\,|w|^q = 0$$

and put

$$v = t^{p-1}w - \Gamma_p\,, \quad \Gamma_p = \Big(\frac{p-1}{p}\Big)^{p-1}.$$

Then

$$\begin{aligned} v' &= (p-1)t^{p-2}w + t^{p-1}w' = (p-1)t^{p-2}\frac{v+\Gamma_p}{t^{p-1}} + t^{p-1}[-c-(p-1)\,|w|^q] \\ &= \frac{p-1}{t}v + \frac{(p-1)}{t}\Gamma_p - t^{p-1}c - (p-1)t^{p-1}\Big|\frac{v+\Gamma_p}{t^{p-1}}\Big|^q \\ &= -t^{p-1}c + \frac{\gamma_p}{t} - \frac{\gamma_p}{t} - \frac{p-1}{t}|v+\Gamma_p|^q + \frac{p-1}{t}v + \frac{p-1}{t}\Gamma_p \\ &= -t^{p-1}\Big(c - \frac{\gamma_p}{t^p}\Big) - \frac{p-1}{t}\Big(|v+\Gamma_p|^q - v + \frac{\gamma_p}{p-1} - \Gamma_p\Big)\,, \end{aligned}$$

where

$$\begin{aligned} \frac{\gamma_p}{p-1} - \Gamma_p &= \frac{1}{p-1}\Big(\frac{p-1}{p}\Big)^p - \Big(\frac{p-1}{p}\Big)^{p-1} = \Big(\frac{p-1}{p}\Big)^{p-1}\Big(\frac{1}{p}-1\Big) \\ &= -\Big(\frac{p-1}{p}\Big)^p = -\gamma_p\,. \end{aligned}$$

Thus we have obtained the differential equation for v

$$v' + t^{p-1}\Big(c(t) - \frac{\gamma_p}{t^p}\Big) + \frac{p-1}{t}(|v+\Gamma_p|^q - v - \gamma_p) = 0\,. \tag{27}$$

In this equation we denote

$$C(t) := t^{p-1}\Big(c(t) - \frac{\gamma_p}{t^p}\Big)\,, \quad H(v) := |v+\Gamma_p|^q - v - \gamma_p\,,$$

hence (27) can be written in the form

$$v' + \frac{C(t)}{t} + \frac{p-1}{t}H(v) = 0\,. \tag{28}$$

A direct computation verifies that $H(0) = 0 = H'(0)$, H is strictly convex and (12) holds, i.e., (28) is the Riccati type differential equation corresponding to a generalized half-linear equation. We change the independent variable $s = \lg t$ in (28) and denote by $\dot{} = \frac{d}{ds}$ the derivative with respect to s. Then, substituting $u(s) = v(e^s)$, we obtain the equation

$$\dot u + d(s) + (p-1)H(u) = 0\,, \quad d(s) := C(e^s)\,. \tag{29}$$

Now, let $t_1 < t_2$ be two consecutive zeros of a nontrivial solution x of (24). Then for $w = \Phi(x'/x)$ we have $w(t_1+) = \infty$, $w(t_2-) = -\infty$, from which $v(t_1+) = \infty$, $v(t_2-) = -\infty$ as well and from this $u(s_1+) = \infty$, $u(s_2) = -\infty$, where $s_1 = \lg t_1$ and $s_2 = \lg t_2$.

Now we can apply the idea of the proof of Theorem 1 to (29) and we conclude that

$$\int_{-\infty}^{\infty} \frac{du}{D + (p-1)H(u)} \le s_2 - s_1 = \lg t_2 - \lg t_1 = \lg\frac{t_2}{t_1}\,,$$

where

$$D = \max_{s\in[s_1,s_2]} |d(s)| = \max_{s\in[s_1,s_2]} |e^{ps}c(e^s) - \gamma_p| = \max_{t\in[t_1,t_2]} |t^p c(t) - \gamma_p| \,.$$

The previous considerations are summarized in the next statement which is the main result of this section.

Theorem 2. *Let $0 < t_1 < t_2$ be consecutive zeros of a nontrivial solution x of* (24)*. Then*

$$\int_{-\infty}^{\infty} \frac{du}{D + (p-1)H(u)} \le \lg \frac{t_2}{t_1} \,,$$

where

$$D = \max_{t\in[t_1,t_2]} |t^p c(t) - \gamma_p| \,, \quad \gamma_p = \Big(\frac{p-1}{p}\Big)^p$$

and

$$H(u) = |u + \Gamma_p|^q - u - \gamma_p \,, \quad \Gamma_p = \Big(\frac{p-1}{p}\Big)^{p-1} .$$

Remark 2. Following the idea introduced in [14], one can consider instead of (24) a more general equation

$$\big(t^{\alpha-1}\Phi(x')\big)' + c(t)\Phi(x) = 0 \,, \quad \alpha \in \mathbb{R} \,,\ \alpha \ne p \,.$$

However, the computations are similar to those given in the previous part of this section, so for the sake of simplicity we consider the case $\alpha = 1$ in Theorem 2. Actually, even a more general equation

$$\big(t^{\alpha-1}\Phi(x')\big)' + \frac{1}{t^{p-1-\alpha}} f(x) = 0 \tag{30}$$

is considered in [14] with f satisfying the sign condition $xf(x) > 0$, $x \ne 0$. It is a subject of the present investigation under which additional assumptions on the function f a Vallée-Poussin type inequality can be established also for (30).

5. Dirichlet eigenvalue problem

In this section we consider the eigenvalue problem

$$x'' + \lambda c(t) f(x, x') = 0 \,, \quad x(0) = 0 = x(\hat\pi) \,, \tag{31}$$

where λ is a real eigenvalue parameter, $c(t) > 0$ for $t \in [a, b]$ and $\hat\pi$ is defined in Section 2. We suppose that c is a continuous positive function for $t \in [0, \hat\pi]$. We will call λ an eigenvalue of (31) if this boundary value problem has a nontrivial solution, the corresponding nontrivial solution is called the *eigenfunction*.

Throughout this section we suppose that

$$\mathcal{F}(\mu) := \int_{-\infty}^{\infty} \frac{ds}{1 + \mu F(s)} \to 0 \quad \text{as} \quad s \to \infty \tag{32}$$

with the function F defined in (v) of Section 1. Obviously, in case of the classical half-linear equation when $F(t) = |t|^p$ this assumption is satisfied.

Theorem 3. *The eigenvalue problem* (31) *has infinitely many eigenvalues* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots$, $\lambda_n \to \infty$ *as* $n \to \infty$. *The* n*-th eigenfunction has exactly* $n-1$ *zeros on* $(0, \hat{\pi})$.

Proof. Let $x(t;\lambda)$ be a nontrivial solution of (17) satisfying the initial condition $x(0;\lambda) = 0$, $x'(0;\lambda) = 1$. Using the generalized Prüfer transformation we can express $x(t;\lambda)$ as

$$x(t;\lambda) = \varrho(t)S\big(\varphi(t)\big)\,, \quad x'(t;\lambda) = \varrho(t)C(\varphi(t))$$

where we take φ in such a way that $\varphi(0) = 0$. To emphasize the dependence of φ on λ we will write $\varphi = \varphi(t;\lambda)$. We have

$$\varphi'(t;\lambda) = 1 - G(\varphi) + \lambda c(t)G(\varphi)$$

where G is given by formula (15). Since $G(\varphi) = 0$ for $\varphi = k\hat{\pi}$, $k \in N$, we have $\varphi'(t;\lambda) = 1$ if $\varphi(t;\lambda) = k\hat{\pi}$. According to [17] the function $\varphi(\hat{\pi};\lambda)$ is monotonically increasing function function of λ. Denote $\bar{c} = \min_{t\in[0,\hat{\pi}]} c(t)$ and consider the auxiliary eigenvalue problem

$$x'' + \lambda\bar{c}f(x,x') = 0\,, \quad x(0) = 0 = x(\hat{\pi})\,.$$

The differential equation in this eigenvalue problem is a minorant of (17), hence for the Prüfer angle of the solution of this equation $\psi(t;\lambda)$ for which $\psi(0;\lambda) = 0$ we have $\psi(t;\lambda) \le \varphi(t;\lambda)$ for $t \in [0,\hat{\pi}]$. Denote $\mu := \lambda\bar{c}$. Then, of course, $\mu \to \infty$ as $\lambda \to \infty$. The number $\hat{\pi}$ is defined as $\mathcal{F}(1)$ (see (14)). Since the function $\mathcal{F}$ depends continuously on μ, there exists a sequence $\mu_n \to \infty$ such that $\mathcal{F}(\mu_n) = \frac{\hat{\pi}}{n}$. Then, repeating the construction of the generalized sine function $S_n(t)$ as the odd $2\frac{\hat{\pi}}{n}$ periodic solution of

$$x'' + \mu_n f(x,x') = 0\,, \quad x(0) = 0 = x\big(\hat{\pi}/n\big)\,, \tag{33}$$

we obtain that (33) has a solution with zeros at $\frac{k\hat{\pi}}{n}$, $k = 0,\dots,n$, i.e., for the Prüfer angle $\psi(\cdot,\mu_n)$ of the solution of (33) we have $\psi(t_k,\mu_n) = \frac{k\hat{\pi}}{n}$ for some $0 < t_1 < t_2 < \cdots < t_{n-1} < \hat{\pi}/n$.

Now, from the previous paragraph we see that the Prüfer angle φ of the solution of (1) satisfying $x(0) = 0$ grows faster than ψ, we have the sequence $\lambda_n \to \infty$ such that $\varphi(\hat{\pi},\lambda_n) = n\hat{\pi}$ and the corresponding eigenfunction x_n has exactly $n-1$ zeros on $(0,\hat{\pi})$. $\square$

Remark 3. We consider for simplicity the Dirichlet boundary condition in our treatment of generalized half-linear eigenvalue problem. However, since our method is similar to that used in the linear case, our results can be extended to general Sturm-Liouville boundary conditions.

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