ARCHIVUM MATHEMATICUM (BRNO)
Tomus 50 (2014), 237–253

ON CONFORMAL POWERS OF THE DIRAC OPERATOR ON SPIN MANIFOLDS

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Abstract. The well known conformal covariance of the Dirac operator acting on spinor fields does not extend to its powers in general. For odd powers of the Dirac operator we derive an algorithmic construction in terms of associated tractor bundles computing correction terms in order to achieve conformal covariance. These operators turn out to be formally (anti-) self-adjoint. Working out this algorithm we recover explicit formula for the conformal third and present a conformal fifth power of the Dirac operator. Finally, we will present polynomial structures for the first examples of conformal powers in terms of first order differential operators acting on the spinor bundle.

1. Introduction

In mathematics and physics differential operators are of central interest and a specific class is given by conformally covariant operators, i.e., they only depend on a given conformal class of metrics $[g]$ on a n-dimensional manifold. The most studied examples are the Dirac operator [22, 18, 9] and the Yamabe operator [27, 23, 2]. With the appearance of the ambient metric construction or equivalently Poincaré-Einstein metric, cf. [7, 8], a powerful tool emerged to deal with conformal structures. Using the ambient metric a sequence of conformally covariant differential operators $P_{2N}(g)$ acting on functions with leading part an N-th power of the Laplacian, for $N \in \mathbb{N}$ ($N \leq \frac{n}{2}$ for even n), was constructed in [15]. In even dimensions it was shown in [13] that in general no conformal N-th power of the Laplacian for $N > \frac{n}{2}$ does exist. These so-called GJMS operators are also linked with scattering matrices associated to Poincaré-Einstein metrics [16]. Another tool to deal with conformal structures are associated tractor bundles [26, 1]. Based on tractor bundles the GJMS operators were recovered by translation of the strongly invariant Yamabe operator by tractor D-operators in [14]. This technique is referred to as curved translation principle, cf. [5]. In case of even dimensions n their construction failed to produce the critical GJMS operator $P_n(g)$. Explicit formulas have rarely been

2010 *Mathematics Subject Classification*: primary 53A30; secondary 53C27.

Key words and phrases: conformal and spin geometry, conformal powers of the Dirac operator, conformal covariance, tractor bundle, tractor D-operator.

Received September 21, 2014. Editor J. Slovák.

Research partially supported by SFB 647 and grant P201/12/G028 of the Czech Science Foundation.

DOI: 10.5817/AM2014-4-237

produced, due to the complexity of the underlying algorithms. However, GJMS operators are in the case of flat manifolds just powers of the Laplacian, whereas in case of Einstein manifolds they are given by a product of shifted Laplacians [12]. Recent results [20, 21] describe the GJMS operators as polynomials in second order non-conformally covariant differential operators, which are of independent interest.

Concerning the spinor case not so much is known about conformal powers of the Dirac operator. It follows from [25] that no conformal even powers of the Dirac operator do exist. The existence of conformal odd powers was proven in [19] using the ambient metric. Again, in the even dimensional case their construction failed to give conformal powers when the order exceed the dimension. The first explicit formula for a conformal third power is due to Branson [3] using tractor techniques. Like in the scalar case, conformal powers are linked to scattering operators associated to Poincaré-Einstein metrics [17]. They also gave an explicit formula for the conformal third power, in agreement with the result of Branson. Explicit formulas for all orders are available for flat structures, for the standard round sphere [6] and for Einstein manifolds [11]. But in general, further examples were not known in the literature.

The main results of the article are the following: We construct conformal powers of the Dirac operator using associated tractor bundles, cf. Definition (4.1). Based on that we present for $n \neq 4$ an explicit formula for a conformal fifth power, cf. Theorem 4.3. Furthermore, we prove that the obtained operators are formally (anti-) self-adjoint, cf. Theorem 4.9. Finally, we identify conformal powers of the Dirac operator up to order five as polynomials in first order operators, cf. Theorem 4.10.

2. Preliminaries

This section fixes notations and conventions used throughout the article. As for literature we refer to standard books.

Riemannian geometry: Let (M,g) be a n-dimensional oriented semi-Riemannian manifold of signature (p,q). Such a structure is equivalent to a $SO(p,q) \subset Gl(n,\mathbb{R})$-reduction $\big(\mathcal{P}^g, \pi, M, SO(p,q)\big)$ of the frame bundle to the special orthogonal group. The associated Levi-Civita connection is denoted by ∇^g. The metric induces a point-wise isomorphism $\cdot^\natural : T^*M \to TM$. For vector fields X, Y, Z, $W \in \Gamma(TM)$ the curvature tensor is given by $R(X,Y)Z = \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z$, whereas the Riemannian curvature tensor is given by $\mathcal{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$. Furthermore, let us denote by Ric, τ and $2(n-1)J = \tau$ the Ricci-, scalar- and normalized scalar curvatures, respectively. The trace-adjusted Ricci- (Schouten tensor), Cotton-, Weyl- and Bach tensor are denoted by $P = (n-2)^{-1}(\mathrm{Ric} - Jg)$, $C(X,Y,Z) = (\nabla^g_X P)(Y,Z) - (\nabla^g_Y P)(X,Z)$, $W = \mathcal{R} + P \owedge g$, and $B(X,Y) = tr_g\big((\nabla^g C)(\cdot, X, Y)\big) + g(W(\cdot, X, Y, \cdot), P)$, respectively. The Kulkarni-Nomizu product $\owedge$ has the sign convention $P \owedge g(X,Y,Z,U) = P(X,Z)g(Y,U) + P(Y,U)g(X,Z) - P(X,U)g(Y,Z) - P(Y,Z)g(X,U)$.

Clifford algebra: Consider the vector space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{p,q})$ equipped with scalar product of signature (p,q), i.e., $\langle e_i, e_j\rangle_{p,q} = \varepsilon_i\delta_{ij}$, where $\{e_i\}$ is the standard basis, $\varepsilon_i = -1$ for $1 \le i \le p$; $\varepsilon_i = 1$, for $p+1 \le i \le n$. The Clifford algebra associated to $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{p,q})$ and denoted by $\mathcal{C}_{p,q}$ is $\mathbb{Z}_2$-graded, given by even and odd elements, i.e., $\mathcal{C}_{p,q} = \mathcal{C}^0_{p,q} \oplus \mathcal{C}^1_{p,q}$. The group of units of $\mathcal{C}_{p,q}$ contains two important subgroups, the Pin-group $\mathrm{Pin}(p,q)$, given by products of elements $x \in \mathbb{R}^n$ of length ± 1, and the Spin-group $\mathrm{Spin}(p,q) := \mathrm{Pin}(p,q) \cap \mathcal{C}^0_{p,q}$. Let us denote by $\lambda\colon \mathrm{Spin}(p,q) \to SO(p,q)$ the double covering map, $\lambda(g)y := gyg^{-1} \in \mathbb{R}^n$ for $g \in \mathrm{Spin}(p,q)$, $y \in \mathbb{R}^n$. We denote by κ_n the restriction to the Spin-group of the unique irreducible representation (for odd n one of the two unique irreducible representations) of the Clifford algebra on $\Delta_n := \mathbb{C}^{2^m}$, for $n = 2m$ or $n = 2m+1$. It is irreducible for odd n or decomposes into two non-equivalent irreducible representations on Δ_{n-1} for even n. A modification of the hermitian scalar product on Δ_n leads to a $\mathrm{Spin}_0(p,q)$-invariant scalar product $(\cdot,\cdot)_\Delta$, where $\cdot_0$ denotes the connected component containing the identity.

Spin geometry: Let (M,g) be a Spin-manifold (n-dimensional oriented semi-Riemannian Spin-manifold of signature (p,q)) and choose a spin structure $(\mathcal{Q}^g, f^g)$. The spinor bundle of (M,g) is the associated vector bundle $S(M,g) := \mathcal{Q}^g \times_{(\mathrm{Spin}_0(p,q),\kappa_n)} \Delta_n$. Due to the isomorphism $TM \simeq \mathcal{Q}^g \times_{(\mathrm{Spin}_0(p,q),\rho\circ\lambda)} \mathbb{R}^n$, where ρ is the standard representation of $SO(p,q)$, we denote by $X\cdot\psi = [q, x\cdot v]$ the Clifford multiplication between $X = [q,x] \in \Gamma(TM)$ and $\psi = [q,v] \in \Gamma\big(S(M,g)\big)$. It extends to $w \in \Lambda^k M$ by

$$w\cdot\psi = \sum_{1\le i_1<\ldots<i_k\le n} \varepsilon_{i_1}\cdots\varepsilon_{i_k} w(s_{i_1},\ldots,s_{i_k}) s_{i_1}\cdot\ldots\cdot s_{i_k}\cdot\psi\,,$$

which is independent of the chosen g-orthonormal frame $s = (s_1,\ldots,s_n)$. The scalar product $(\cdot,\cdot)_\Delta$ induces a metric structure on spinor bundle by $\langle\psi,\phi\rangle := (u,v)_\Delta$, for $\psi = [q,u], \phi = [q,v] \in S(M,g)$, which satisfies $\langle X\cdot\psi,\phi\rangle + (-1)^p\langle\psi, X\cdot\phi\rangle = 0$. The Levi-Civita connection lifts to a covariant derivative $\nabla^{g,S}$ on the spinor bundle, which satisfies the Leibniz rule with respect to Clifford multiplication and the $\langle\cdot,\cdot\rangle$-metricity. The Dirac operator on $S(M,g)$ is denoted by $\not{D} \overset{\text{loc.}}{=} \sum_i \varepsilon_i s_i\cdot\nabla^{g,S}_{s_i}$. It follows from Weitzenböck formula that $\not{D}^2\psi = -\Delta_g^{S(M,g)}\psi + \frac{\tau}{4}\psi$, where $\Delta^E_g := tr_g(\nabla^{T^*M\otimes E}\circ\nabla)$ is the Bochner Laplacian associated to a vector bundle with covariant derivative (E,∇) over (M,g). Concerning questions of (anti-) self-adjointness of certain operators on spinor fields we introduce a bracket notation. Let T be a symmetric $(0,2)$-tensor and ψ a spinor field. We define first a 1-form $T\cdot\psi$ with values in the spinor bundle by $T\cdot\psi(X) := T(X)^\natural\cdot\psi$. The following brackets are defined then:

$$(T,\eta) \overset{\text{loc.}}{:=} \sum_i \varepsilon_i T(s_i)^\natural\cdot\eta(s_i)\,, \tag{2.1}$$

$$(\nabla^{g,S}, T\cdot\psi) \overset{\text{loc.}}{:=} \sum_i \varepsilon_i(\nabla^{g,S}_{s_i} T\cdot\psi)(s_i)\,, \tag{2.2}$$

for $\eta \in \Omega^1\big(M, S(M,g)\big)$. Next, we define a symmetric $(0,2)$-tensor $T^2 := T\big(T(\cdot),\cdot\big)$, and introduce the notation

$$(C, P\cdot\psi) \overset{\text{loc.}}{:=} \sum_i \varepsilon_i C(s_i)\cdot P(s_i)\cdot\psi\,, \tag{2.3}$$

where the Cotton tensor is considered as $C(X) := C(\cdot,\cdot,X) \in \Omega^2(M)$. Analogously one defines $(P, C\cdot\psi)$. Two more product types, needed later on, are

$$W\cdot W\cdot\psi \overset{\text{loc.}}{:=} \sum_{i,j} \varepsilon_i\varepsilon_j W(s_i,s_j)\cdot W(s_i,s_j)\cdot\psi\,, \tag{2.4}$$

$$C\cdot W\cdot\psi \overset{\text{loc.}}{:=} \sum_{i,j} \varepsilon_i\varepsilon_j C(s_i,s_j,\cdot)\cdot W(s_i,s_j)\cdot\psi\,, \tag{2.5}$$

where Clifford multiplication of the 2-form $W(X,Y) := W(X,Y,\cdot,\cdot) \in \Omega^2(M)$ appears. Similary we define $W\cdot C\cdot\psi$.

Conformal geometry: Let $(M,[g])$ be a conformal manifold (n-dimensional oriented conformal manifold of signature (p,q)). This is equivalent to a $CO(p,q) \subset Gl(n,\mathbb{R})$-reduction $\big(\mathcal{P}^0,\pi,M,CO(p,q)\big)$ of the frame bundle to the conformal group $CO(p,q) = \mathbb{R}^+ \times SO(p,q)$. Levi-Civita connections corresponding to $g, \widehat{g} = e^{2\sigma}g \in [g]$ are related by $\nabla^{\widehat{g}}_X Y = \nabla^g_X Y + X(\sigma)Y + Y(\sigma)X - g(X,Y)\mathrm{grad}^g(\sigma)$, hence in general they are not invariant under conformal change of metrics. In contrast, the Weyl tensor is conformally invariant of weight 2, i.e., $W(\widehat{g}) = e^{2\sigma}W(g)$.

For metrics $g, \widehat{g} \in [g]$ there exists a vector bundle isomorphism $F_\sigma = \widehat{\cdot}\colon S(M,g) \to S(M,\widehat{g})$ induce by the isomorphism $L_\sigma\colon TM \to TM$, $L_\sigma(X) := e^{-\sigma}X$, which pulls back $\widehat{g}$ to g. Clifford multiplications for g and $\widehat{g}$ are related by $F_\sigma(X\cdot\psi) = L_\sigma(X)\cdot F_\sigma(\psi)$, for $X \in \Gamma(TM)$ and $\psi \in \Gamma\big(S(M,g)\big)$. Corresponding covariant derivatives $\nabla^{g,S}$ and $\nabla^{\widehat{g},S}$ are related by $\nabla^{\widehat{g},S}_X\widehat{\psi} = \widehat{\nabla^{g,S}_X\psi} - \frac{1}{2}(X\cdot\widehat{\mathrm{grad}(\sigma)}\cdot\psi + X(\sigma)\widehat{\psi})$, hence not conformally invariant in general. However, for corresponding Dirac operators we have $\widehat{D\!\!\!/}(e^{\frac{1-n}{2}\sigma}\widehat{\psi}) = e^{-\frac{n+1}{2}\sigma}\widehat{D\!\!\!/\psi}$, where $\widehat{D\!\!\!/}$ denotes the Dirac operator on $S(M,\widehat{g})$.

Let $(M,[g])$ be a conformal Spin-manifold. A conformal spin structure $(\mathcal{Q}^0, f^0)$ is by definition a λ^c-reduction of the conformal frame bundle, where $\lambda^c\colon C\,\mathrm{Spin}(p,q) \to CO(p,q)$ is given $\lambda^c(a,g) := a\lambda(g)$ and $C\,\mathrm{Spin}(p,q) := \mathbb{R}^+ \times \mathrm{Spin}(p,q)$ is the conformal Spin-group.

Let $E \to M$ and $F \to M$ be vector bundles over (M,g). We say that a linear differential operator $D(g)\colon \Gamma(E) \to \Gamma(F)$ is g-geometrical if it is locally a polynomial in g, g^{-1}, ∇^g and $\mathcal{R}$. A g-geometrical operator $D(g)$ is said to be conformally covariant of bi-degree (a,b) if there exists a, $b \in \mathbb{R}$ such that

$$D(e^{2\sigma}g)(e^{a\sigma}\psi) = e^{b\sigma}D(g)\psi\,,$$

for any metric $e^{2\sigma}g$, and $\psi \in \Gamma(E)$. Examples of conformally covariant operators are the GJMS operator $P_{2N}(g)$ with conformal bi-degree $(\frac{2N-n}{2}, -\frac{2N+n}{2})$ and the Dirac operator with bi-degree $(\frac{1-n}{2}, -\frac{1+n}{2})$.

Parabolic geometry for conformal spin structures: Let M be a manifold, G a Lie group and $H \subset G$ a closed subgroup. A Cartan geometry $(\mathcal{G}, \pi, M, H; w)$ of type (G, H) over M, consists of an H-principal bundle $\mathcal{G}$ over M with a Cartan connection $w \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e., (1) $w(\widetilde{X}) = X$ for every $X \in \mathfrak{h}$ (where $\widetilde{X}$ denotes the fundamental vector field of X), (2) $w\colon T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism, for every $u \in \mathcal{G}$, and (3) $(R_h)^*w = \mathrm{Ad}(h^{-1}) \circ w$, for every $h \in H$.

A Cartan geometry $(\mathcal{G}, w)$ of type (G, H), for which H is a parabolic subgroup inside a semisimple Lie group G, is referred to as a parabolic geometry.

An conformal manifold $(M, [g])$ can be described as a parabolic geometry as follows: Consider the special orthogonal group $G := SO(p+1, q+1)$. In terms of the standard orthonormal basis $\{e_\alpha\}_{\alpha=0}^{n+1}$ with respect to the standard semi-Riemannian metric $\langle \cdot, \cdot \rangle_{p+1,q+1}$ on $\mathbb{R}^{n+2}$, we consider the following basis $f_0 := \frac{1}{\sqrt{2}}(e_{n+1} - e_0), e_1, \dots, e_n, f_{n+1} := \frac{1}{\sqrt{2}}(e_{n+1} + e_0)$ on $\mathbb{R}^{n+2}$. The stabilizer $B := \mathrm{stab}_{\mathbb{R}f_0}(G)$ of the isotropic line $\mathbb{R}f_0$ defines a parabolic subgroup of G. Note that the Lie algebra $\mathfrak{g}$ of G is $|1|$-graded. In this setting it is shown in [4, Section 1.6] that there exists a parabolic geometry $(\mathcal{P}^1, w^{nc})$ of type (G, B) uniquely associated to the conformal structure. Geometrically speaking, the B-principal bundle $\mathcal{P}^1$, called first prolongation of the conformal frame bundle, is the collection of horizontal and torsion-free subspaces in $T\mathcal{P}^0$, and the normal conformal Cartan connection w^{nc} is a well-chosen extension of the soldering form of $\mathcal{P}^1$. Hence, to $(M, [g])$ is associated a distinguished Cartan geometry of type (G, B).

Let $(M, [g])$ be an conformal Spin-manifold. Denote by $\widetilde{\cdot}$ the pullback of the groups G, B by the covering map $\lambda\colon \mathrm{Spin}(p+1, q+1) \to SO(p+1, q+1)$. By fixing a conformal spin structure $(\mathcal{Q}^0, f^0)$ on $(M, [g])$, we obtain a first prolongation $(\mathcal{Q}^1, f^1)$ of the conformal spin structure as lift of the first prolongation $\mathcal{P}^1$ of the conformal structure. Furthermore, the normal conformal Cartan connection w^{nc} induces a Cartan connection $\widetilde{w}^{nc} := \lambda_* \circ w^{nc} \circ df^1 \in \Omega^1\big(\mathcal{Q}^1, \mathfrak{spin}(p+1, q+1)\big)$ on $\mathcal{Q}^1$. Hence, to a conformal Spin-manifold $(M, [g])$ is associated a distinguished Cartan geometry of type $(\widetilde{G}, \widetilde{B})$.

Tractor bundles: Consider the standard representation $\rho\colon G \to Gl(\mathbb{R}^{n+2})$ and spin representation $\widetilde{\rho} := \kappa_{n+2} : \widetilde{G} \to Gl(\Delta_{n+2})$. We may define the standard and spin tractor bundles by

$$\mathcal{T}(M) := \mathcal{P}^1 \times_{(B,\rho)} \mathbb{R}^{n+2},$$

$$\mathcal{S}(M) := \mathcal{Q}^1 \times_{(\widetilde{B}_0, \widetilde{\rho})} \Delta_{n+2}.$$

Both bundles can be equipped with a bundle metric: $g^{\mathcal{T}}(t_1, t_2) := \langle y_1, y_2 \rangle_{p+1,q+1}$, for $t_i = [H, y_i] \in \mathcal{T}(M)$, $i = 1, 2$; and $g^{\mathcal{S}}(s_1, s_2) := (v_1, v_2)_\Delta$, for $s_i = [\widetilde{H}, v_i] \in \mathcal{S}(M)$, $i = 1, 2$. They are well defined since $\langle \cdot, \cdot \rangle_{p+1,q+1}$ and $(\cdot, \cdot)_\Delta$ are invariant under B and $\widetilde{B}_0$, respectively. The Cartan connections w^{nc} and $\widetilde{w}^{nc}$ induce covariant derivatives $\nabla^{\mathcal{T}}$ and $\nabla^{\mathcal{S}}$, which are metric with respect to $g^{\mathcal{T}}$ and $g^{\mathcal{S}}$, respectively.

A metric g from the conformal class leads to the following isomorphisms:

$$
\begin{aligned}
\mathcal{T}(M) &\simeq \mathcal{P}^g \times_{(SO(p+1,q+1),\rho)} \mathbb{R}^{n+2}\,,\\
\mathcal{S}(M) &\simeq \mathcal{Q}^g \times_{(\mathrm{Spin}_0(p+1,q+1),\widetilde{\rho})} \Delta_{n+2}\,,\\
TM &\simeq \mathcal{P}^g \times_{(SO(p,q),\mathrm{Ad})} \mathfrak{b}_{-1} \simeq \mathcal{Q}^g \times_{(\mathrm{Spin}_0(p,q),\mathrm{Ad}\circ\lambda)} \mathfrak{b}_{-1}\,,\\
T^*M &\simeq \mathcal{P}^g \times_{(SO(p,q),\mathrm{Ad})} \mathfrak{b}_{1} \simeq \mathcal{Q}^g \times_{(\mathrm{Spin}_0(p,q),\mathrm{Ad}\circ\lambda)} \mathfrak{b}_{1}\,,\\
\mathfrak{so}(TM,g) &\simeq \mathcal{P}^g \times_{(SO(p,q),\mathrm{Ad})} \mathfrak{so}(p,q) \simeq \mathcal{Q}^g \times_{(\mathrm{Spin}_0(p,q),\mathrm{Ad}\circ\lambda)} \mathfrak{so}(p,q)\,.
\end{aligned}
$$

Hence we obtain a vector bundle isomorphisms

$$
\begin{aligned}
\Phi^g \colon \mathcal{T}(M) &\to \underline{M} \oplus TM \oplus \underline{M} =: \mathcal{T}(M)_g\,,\\
t = [e,y] &\mapsto (\alpha, X, \beta) =: t_g\,,
\end{aligned}
\tag{2.6}
$$

where $\underline{M} := M \times \mathbb{R}$ is the trivial bundle, $y \in \mathbb{R}^{n+2}$ has coordinates $(\alpha, x = (x_1,\dots,x_n), \beta)$ with respect to the basis $\{f_0, e_i, f_{n+1}\}$ of $\mathbb{R}^{n+2}$, and $X \in TM$ is the vector with coordinates $x = (x_1,\dots,x_n)$ with respect to $e \in \mathcal{P}^g$. Furthermore, we have the bundle isomorphism

$$
\begin{aligned}
\Psi^g \colon \mathcal{S}(M) &\to S(M,g) \oplus S(M,g) =: \mathcal{S}(M)_g\\
s = [q,v] &\mapsto (\psi, \phi) =: s_g,
\end{aligned}
\tag{2.7}
$$

where $\psi = [q, w_1]$ and $\phi = [q, w_2]$, with $w_1, w_2 \in \Delta_n$ being determined as follows: Consider the two $Spin(p,q)$-invariant subspaces $W^+ := \{v \in \Delta_{n+2} \mid f_{n+1}\cdot v = 0\}$ and $W^- := \{v \in \Delta_{n+2} \mid f_0 \cdot v = 0\}$ of Δ_{n+2}. Note that we naturally identify W^+ and Δ_n. Hence, $\widetilde{\rho}$ restricted to $\mathrm{Spin}(p,q)$ decomposes into two representations $\widetilde{\rho}^{\pm}\colon \mathrm{Spin}(p,q) \to Gl(W^{\pm})$, such that $\widetilde{\rho}|_{\mathrm{Spin}(p,q)} = \widetilde{\rho}^+ \oplus \widetilde{\rho}^-$. From the definition of $W^{\pm}$ it follows that $\widetilde{\rho}^{\pm}$ are equivalent with respect to the isomorphism $W^+ \ni w \mapsto f_0 \cdot w \in W^-$. Therefore, our element in question $v \in \Delta_{n+2}$ can be uniquely decomposed as $v = w_1 + f_0 \cdot w_2$ with $w_1, w_2 \in W^+$ due to the isomorphism $W^+ \times W^+ \ni (w_1, w_2) \mapsto w_1 + f_0 \cdot w_2 \in \Delta_{n+2}$.

The maps Φ^g and Ψ^g allow an interpretation of tractor objects in terms of g-geometrical data. For example, we have that

$$
\Phi^g \circ \nabla^{\mathcal{T}}_X \circ (\Phi^g)^{-1} = \begin{pmatrix} \nabla^g_X & -P(X,\cdot) & 0\\ X\cdot & \nabla^g_X & P(X)^{\natural}\cdot\\ 0 & -g(X,\cdot) & \nabla^g_X \end{pmatrix}
$$

and

$$
\Psi^g \circ \nabla^{\mathcal{S}}_X \circ (\Psi^g)^{-1} = \begin{pmatrix} \nabla^{g,S}_X & X\cdot\\ \frac{1}{2}P(X)^{\natural}\cdot & \nabla^{g,S}_X \end{pmatrix}.
$$

A further example is given by the bundle metrics, here we have that

$$
g^{\mathcal{T}}(t_1,t_2) = \alpha_1\beta_2 + g(X_1,X_2) + \alpha_1\beta_2,
\tag{2.8}
$$

for $t_i = [e, y_i]$, $i = 1,2$. Moreover, for $s_i = [q, v_i] \in \mathcal{S}(M)$, $i = 1,2$, we have that

$$
g^{\mathcal{S}}(s_1,s_2) = -2\sqrt{2}i^p\left(\langle\psi_2,\phi_1\rangle + (-1)^p\langle\psi_1,\phi_2\rangle\right).
\tag{2.9}
$$

Note that these relations are based on the isomorphisms (2.6) and (2.7). Furthermore, the relation among $\mathcal{T}(M)_g$ and $\mathcal{S}(M)_g$ for different representatives $g, \widehat{g} \in [g]$ are given by

$$T(g,\sigma) := \Phi^{\widehat{g}} \circ (\Phi^g)^{-1} = \begin{pmatrix} e^{-\sigma} & -e^{-\sigma} d\sigma & -\frac{1}{2} e^{-\sigma} |\mathrm{grad}^g(\sigma)|_g^2 \\ 0 & e^{-\sigma} & e^{-\sigma}\mathrm{grad}^g(\sigma) \\ 0 & 0 & e^{\sigma} \end{pmatrix},$$

and

$$T^{\mathcal{S}(M)}(g,\sigma) := \Psi^{\widehat{g}} \circ (\Psi^g)^{-1} = \begin{pmatrix} F_\sigma & 0 \\ 0 & F_\sigma \end{pmatrix} \circ \begin{pmatrix} e^{\frac{1}{2}\sigma} & 0 \\ \frac{1}{2} e^{-\frac{1}{2}\sigma}\mathrm{grad}^g(\sigma)\cdot & e^{-\frac{1}{2}\sigma} \end{pmatrix},$$

where $F_\sigma \colon S(M,g) \to S(M,\widehat{g})$ is the bundle isomorphism relating spinor bundles for two conformally related metrics g and $\widehat{g} = e^{2\sigma} g$.

3. Relevant differential operators

In this section we present some operators necessarily for the construction of conformal powers of the Dirac operator. Firstly we recall the tractor D-operator for functions and spinors, where the former extends to $\mathcal{S}^k(M) := \otimes^k \mathcal{T}(M) \otimes \mathcal{S}(M)$ for $k \in \mathbb{N}_0$ due to its strong invariance, cf. [1, 24]. Secondly we follow the curved translation principle to construct higher order conformally covariant differential operators acting on the spin tractor bundle.

The tractor D-operator $D(g,w) : \mathcal{C}^\infty(M) \to \Gamma(\mathcal{T}(M)_g)$, $w \in \mathbb{R}$, for functions is given by

$$D(g,w)\beta := \begin{pmatrix} -\Delta_g \beta - wJ\beta \\ (n+2w-2)(\nabla^g \beta)^\natural \\ w(n+2w-2)\beta \end{pmatrix}, \tag{3.1}$$

where $\Delta_g := tr_g(\nabla^{T^*M\otimes TM} \circ \nabla^g)$. For $w \in \mathbb{R}$, the tractor D-operator $D^{\mathrm{spin}}(g,w) \colon \Gamma\big(S(M,g)\big) \to \Gamma(\mathcal{S}(M)_g)$ for spinors is given by

$$D^{\mathrm{spin}}(g,w)\psi := \begin{pmatrix} (w + \frac{n-1}{2})\psi \\ \frac{1}{2} \not{D}\psi \end{pmatrix}. \tag{3.2}$$

They satisfy the following conformal transformation laws:

Proposition 3.1. *Let $\widehat{g} = e^{2\sigma} g$, $\beta \in \mathcal{C}^\infty(M)$ and $\psi \in \Gamma\big(S(M,g)\big)$. Then one has*

$$D(\widehat{g},w)(e^{w\sigma}\beta) = e^{(w-1)\sigma} T(g,\sigma) D(g,w)\beta\,,$$
$$D^{\mathrm{spin}}(\widehat{g},w)(e^{w\sigma}\widehat{\psi}) = e^{(w-\frac{1}{2})\sigma} T^{\mathcal{S}(M)}(g,\sigma) D^{\mathrm{spin}}(g,w)\psi\,,$$

for all $w \in \mathbb{R}$.

Proof. The proof is straightforward and based on equations (3.1), (3.2), formulas

$$\Delta_{\widehat{g}}\beta = e^{-2\sigma}\big[\Delta_g \beta + (n-2)\nabla^g_{\mathrm{grad}(\sigma)}\beta\big],$$
$$\Delta_g(e^{w\sigma}\beta) = e^{w\sigma}\big[\Delta_g\beta + 2w\nabla^g_{\mathrm{grad}(\sigma)}\beta + w^2|\mathrm{grad}(\sigma)|_g^2\beta + \Delta_g\sigma\beta\big],$$
$$\widehat{J} = e^{-2\sigma}\big[J - \Delta_g\sigma - \tfrac{n-2}{2}|\mathrm{grad}(\sigma)|_g^2\big],$$

as well as on $\not{D}(\beta \cdot \psi) = \mathrm{grad}(\beta) \cdot \psi + \beta \not{D}\psi$ and the conformal covariance of the Dirac operator. $\square$

Twisting the Levi-Civita connection with the covariant derivative $\nabla^{\mathcal{S}^k}$ on $\mathcal{S}^k(M)$, for $k \in \mathbb{N}_0$, yields an extension of the tractor D-operator for functions: $D^{\mathcal{S}^k}(g,w) : \Gamma\big(\mathcal{S}^k(M)\big) \to \Gamma\big(\mathcal{S}^{k+1}(M)\big)$, given by

$$D^{\mathcal{S}^k}(g,w)s = \begin{pmatrix} -\square^k_w s \\ (n+2w-2)(\nabla^{\mathcal{S}^k} s)^\natural \\ w(n+2w-2)s \end{pmatrix} \tag{3.3}$$

for $s \in \Gamma\big(\mathcal{S}^k(M)\big)$ and $\square^k_w s := \Delta_g^{\mathcal{S}^k(M)} s + wJs$. It satisfies the same conformal transformation law as the tractor D-operator $D(g,w)$. The conformal transformation laws for $D^{\mathcal{S}^k}(g,w)$ and $D^{\mathrm{spin}}(g,w)$ for $w = \frac{n-2}{2}$ and $w = \frac{n-1}{2}$, respectively, gives the following:

Corollary 3.2. *The operator $\square^k_{\frac{2-n}{2}}$ on $\Gamma\big(\mathcal{S}^k(M)\big)$ is conformally covariant of bi-degree $(\frac{2-n}{2}, -\frac{2+n}{2})$.*

The Dirac operator $\not{D}$ on $\Gamma\big(S(M,g)\big)$ is conformally covariant of bi-degree $(\frac{1-n}{2}, -\frac{1+n}{2})$.

For later purposes, let us define

$$\begin{aligned} C^{\mathrm{spin}}(g,w)\colon \Gamma(\mathcal{S}(M)_g) &\to \Gamma\big(S(M,g)\big) \\ s_g = (\psi,\phi) &\mapsto \tfrac{1}{2}\not{D}\psi - \big(w + \tfrac{n}{2}\big)\phi\,, \end{aligned}$$

and

$$\begin{aligned} C^{\mathcal{S}^k}(g,w)\colon \Gamma\big(\mathcal{S}^{k+1}(M)\big) &\to \Gamma\big(\mathcal{S}^k(M)\big) \\ (s_1,\eta,s_2) &\mapsto \big(n(n+1+w) + (n+w)(2w-2)\big)s_1 \\ &\qquad + (n+2w)\,\mathrm{div}(\eta) - \square^k_{(1-n-w)} s_2\,, \end{aligned} \tag{3.4}$$

where $s_1,\ s_2 \in \Gamma\big(\mathcal{S}^k(M)\big)$ and $\eta \in \Gamma\big(TM \otimes \mathcal{S}^k(M)\big)$. Note that for $Y \otimes s \in \Gamma\big(TM \otimes \mathcal{S}^k(M)\big)$ we used $\mathrm{div}(Y \otimes s) := \mathrm{div}(Y)s + \nabla_Y^{\mathcal{S}^k(M)} s \in \Gamma\big(\mathcal{S}^k(M)\big)$, and the divergence of a vector field is defined by

$$\mathrm{div}(Y) \overset{\text{loc.}}{:=} \sum_i \varepsilon_i g(\nabla^g_{s_i} Y, s_i)\,.$$

By the proposition below, they are the formal adjoints of corresponding tractor D-operators.

Proposition 3.3. *Let $k \in \mathbb{N}_0$. As formal adjoints with respect to the corresponding L^2-scalar product we have that*

$$\begin{aligned}
\left(D^{\mathcal{S}^k}(g,w)\right)^* &= C^{\mathcal{S}^k}(g, 1-n-w)\,,\\
\left(C^{\mathcal{S}^k}(g,w)\right)^* &= D^{\mathcal{S}^k}(g, 1-n-w)\,,\\
\left(D^{\mathrm{spin}}(g,w)\right)^* &= -2\sqrt{2}i^p C^{\mathrm{spin}}\left(g, \tfrac{1}{2}-n-w\right),\\
\left(C^{\mathrm{spin}}(g,w)\right)^* &= -\frac{1}{2\sqrt{2}} i^p D^{\mathrm{spin}}\left(g, \tfrac{1}{2}-n-w\right).
\end{aligned}$$

Proof. Let us set $w_1 := (n+2w-2)$ and $w_2 := (w + \frac{n-1}{2})$. Using the formulas (2.8) and (2.9) for the scalar products $g^{\mathcal{T}}$ and $g^{\mathcal{S}}$ we compute, for $k=0$, that

$$\begin{aligned}
&g^{\mathcal{T}\otimes\mathcal{S}}\left(D^{\mathcal{S}^0}(g,w)s, \begin{pmatrix} s_1\\ \eta \\ s_2\end{pmatrix}\right)_{L^2}\\
&= -g^{\mathcal{S}}(\Box_w^k s, s_2)_{L^2} + g^{TM\otimes\mathcal{S}}\left(w_1(\nabla^{\mathcal{S}}s)^\natural, \eta\right)_{L^2} + g^{\mathcal{S}}(ww_1 s, s_1)_{L^2}\\
&= -g^{\mathcal{S}}(s, \Box_w^k s_2)_{L^2} - g^{\mathcal{S}}\left(s, w_1 \operatorname{div}(\eta)\right)_{L^2} + g^{\mathcal{S}}(s, ww_1 s_1)_{L^2}\\
&= g^{\mathcal{S}}\left(s, C^{\mathcal{S}^0}(g, 1-n-w)\begin{pmatrix} s_1\\ \eta \\ s_2\end{pmatrix}\right)_{L^2},
\end{aligned}$$

where we have used the self-adjointness of $\Delta_g^{\mathcal{S}^k(M)}$ and the $g^{\mathcal{S}}$-metricity of $\nabla^{\mathcal{S}}$. Note that the index $\cdot_{L^2}$ indicates the induced L^2-scalar product. The second assertion, for $k=0$, follows immediately. The case for $k>0$ runs along the same lines. Coming to the third one, we have that

$$\begin{aligned}
&g^{\mathcal{S}}\left(D^{\mathrm{spin}}(g,w)\psi, \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right)_{L^2}\\
&\quad= -2\sqrt{2}i^p\left(\langle \tfrac{1}{2}\not{D}\psi, \phi_1\rangle_{L^2} + (-1)^p\langle w_2\psi, \phi_2\rangle_{L^2}\right)\\
&\quad= -2\sqrt{2}i^p\left((-1)^p\langle \psi, \tfrac{1}{2}\not{D}\phi_1\rangle_{L^2} + (-1)^p\langle \psi, w_2\phi_2\rangle_{L^2}\right)\\
&\quad= -2\sqrt{2}i^p(-1)^p\left(\langle \psi, \tfrac{1}{2}\not{D}\phi_1 - \left(\tfrac{1}{2}-n-w+\tfrac{n}{2}\right)\phi_2\rangle_{L^2}\right)\\
&\quad= \left\langle \psi, (-2)\sqrt{2}i^p C^{\mathrm{spin}}\left(g, \tfrac{1}{2}-n-w\right)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right\rangle_{L^2},
\end{aligned}$$

where we have used the (anti-) self-adjointness of $\not{D}$. Also note the hermiticity of $\langle\cdot,\cdot\rangle_{L^2}$. The fourth equation follows immediately from the third, which completes the proof. $\square$

It follows from the last proposition and from the invariance of the corresponding scalar products with respect to g and $\widehat{g} = e^{2\sigma}g$, that:

Proposition 3.4. *For $\widehat{g} = e^{2\sigma} g$, $s_g \in \Gamma(\mathcal{S}(M)_g)$ and $(s_1, \eta, s_2) \in \Gamma\big(\mathcal{S}^{k+1}(M)\big)$, one has that*

$$\begin{aligned} C^{\mathcal{S}^k}(\widehat{g}, w)\big(e^{w\sigma} T(g,\sigma)(s_1,\eta,s_2)\big) =& e^{(w-1)\sigma} C^{\mathcal{S}^k}(g,w)(s_1,\eta,s_2), \\ C^{\mathrm{spin}}(\widehat{g}, w)(e^{w\sigma} s_{\widehat{g}}) =& e^{(w-\frac{1}{2})\sigma} \widehat{C^{\mathrm{spin}}(g,w)} s_g \, . \end{aligned}$$

As mentioned in Corollary 3.2, the operator $\Box^k_{\frac{2-n}{2}}$ on $\Gamma\big(\mathcal{S}^k(M)\big)$, for $k \in \mathbb{N}_0$, is conformally covariant. Hence we can use the curved translation principle to define $P^{\mathcal{S}(M)}_{2N}(g) : \Gamma(\mathcal{S}(M)_g) \to \Gamma(\mathcal{S}(M)_g)$, for $N \in \mathbb{N}$, by

$$\begin{aligned} P^{\mathcal{S}(M)}_2(g) :=& \Box^0_{\frac{2-n}{2}} = \Delta^{\mathcal{S}(M)}_g + \tfrac{2-n}{2} J \\ P^{\mathcal{S}(M)}_{2N}(g) :=& C^{\mathcal{S}^0}\left(g, -\tfrac{2(N-1)+n}{2}\right) \circ \cdots \circ C^{\mathcal{S}^{N-2}}\left(g, -\tfrac{2+n}{2}\right) \circ \Box^{N-1}_{\frac{2-n}{2}} \circ \\ & \circ D^{\mathcal{S}^{N-2}}\left(g, \tfrac{4-n}{2}\right) \circ \cdots \circ D^{\mathcal{S}^0}\left(g, \tfrac{2N-n}{2}\right), \quad N > 1 \, . \end{aligned} \tag{3.5}$$

These operators satisfy the following:

Proposition 3.5. *The operator $P^{\mathcal{S}(M)}_{2N}(g)$, for $N \in \mathbb{N}$, is conformally covariant of bi-degree $(\frac{2N-n}{2}, -\frac{2N+n}{2})$, i.e., for $\widehat{g} = e^{2\sigma} g$ we have*

$$P^{\mathcal{S}(M)}_{2N}(\widehat{g})(e^{\frac{2N-n}{2}\sigma} s_{\widehat{g}}) = e^{-\frac{2N+n}{2}\sigma} P^{\mathcal{S}(M)}_{2N}(g) s_g \, ,$$

for $s \in \Gamma\big(\mathcal{S}(M)\big)$. Its leading term is given by $c(n,N)(\Delta^{\mathcal{S}(M)}_g)^N$ for a constant $c(n,N) := (-1)^{N-1} \prod_{k=1}^{N-1} [k(2+2k-n)]$.

Proof. The conformal covariance follows from the well-chosen $w's$ in the composition. Furthermore it follows from the composition that $P^{\mathcal{S}(M)}_{2N}(g)$ has leading part $c(n,N)(\Delta^{\mathcal{S}(M)}_g)^N$, where the expression of $c(n,N)$ follows directly from (3.3) and (3.4). $\square$

Remark 3.6. In case of even n, the operator $P^{\mathcal{S}(M)}_{2N}(g)$, for $N \geq \frac{n}{2}$, is not identically zero as stated in [10, Proposition 5.26]. It is just of order less than $2N$, due to the fact that the constant $c(n,N)$ is zero.

Example 3.7. A straightforward computation yields explicit formulas for

$$P^{\mathcal{S}(M)}_2(g) = \Delta^{\mathcal{S}(M)}_g + \tfrac{2-n}{2} J = \begin{pmatrix} -\not{D}^2 & 2\not{D} \\ \frac{1}{2}(P, \nabla^{g,S}) + \frac{1}{2}(\nabla^{g,S}, P\cdot) & -\not{D}^2 \end{pmatrix},$$

and

$$\begin{aligned} P^{\mathcal{S}(M)}_4(g) &= C^{\mathcal{S}^0}\left(g, -\tfrac{2+n}{2}\right) \circ (\Delta^{\mathcal{S}^1(M)}_g + \tfrac{2-n}{2} J) \circ D^{\mathcal{S}^0}\left(g, \tfrac{4-n}{2}\right) \\ &= \begin{pmatrix} (n-4)A(g)\not{D} + W \cdot W \cdot & -4(n-4)A(g) \\ B(g) & (n-4)\not{D}A(g) + W \cdot W \cdot \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} A(g) :=& \not{D}^3 - (P, \nabla^{g,S}) - (\nabla^{g,S}, P\cdot)\,, \\ B(g) :=& (n-4)\big[A(g)\not{D}^2 + \not{D}^2 A(g) - 2\not{D}^2 + 2\big((P^2, \nabla^{g,S}) + (\nabla^{g,S}, P^2)\big)\big] \\ & - (n-4)(C,P) + 2(B, \nabla^{g,S}) + C\cdot W\cdot + W\cdot C\cdot\,. \end{aligned}$$

We can decompose $P_4^{\mathcal{S}(M)}(g) = P_4^{\mathrm{red}}(g) + R_4(g)$, where

$$R_4(g) := \begin{pmatrix} W\cdot W\cdot & 0 \\ C\cdot W\cdot + W\cdot C\cdot & W\cdot W\cdot \end{pmatrix}$$

is conformally covariant of bi-degree $(\frac{4-n}{2}, -\frac{4+n}{2})$. Actually, $R_4(g)$ is up to a multiple the square of the curvature associated to spin tractor covariant derivative $\nabla^{\mathcal{S}}$, hence conformally covariant. It then immediately follows that $P_4^{\mathrm{red}}(g)$ satisfies the same conformal transformation law as $P_4^{\mathcal{S}(M)}(g)$.

4. The construction of conformal powers of the Dirac operator and related structures

In this section we construct conformal odd powers of the Dirac operator by composition of the operators $P_{2N}^{\mathcal{S}(M)}(g)$, cf. Definition (3.5), with the tractor D-operator $D^{\mathrm{spin}}(g,w)$ and its formal adjoint $C^{\mathrm{spin}}(g,w)$. Furthermore, we present explicit formulas for lower order examples in general, and subsequently simplify in the Einstein case. We then go on to prove some formal (anti-) self-adjointness results. Based on explicit formulas we show that they are polynomials in first order differential operators.

For $N \in \mathbb{N}$ we define the differential operator

$$D_{2N+1}(g) := C^{\mathrm{spin}}\left(g, -\tfrac{2N+n}{2}\right) \circ P_{2N}^{\mathcal{S}(M)}(g) \circ D^{\mathrm{spin}}\left(g, \tfrac{2N+1-n}{2}\right) \tag{4.1}$$

acting on the spinor bundle.

Theorem 4.1. *Let $N \in \mathbb{N}$. The operator $D_{2N+1}(g)$ is conformally covariant of bi-degree $(\frac{2N+1-n}{2}, -\frac{2N+1+n}{2})$, i.e., for $\widehat{g} = e^{2\sigma} g$ and $\psi \in \Gamma\big(S(M,g)\big)$ we have*

$$D_{2N+1}(\widehat{g})(e^{\frac{2N+1-n}{2}\sigma}\widehat{\psi}) = e^{-\frac{2N+1+n}{2}\sigma}\widehat{D_{2N+1}(g)\psi}\,.$$

Its leading term is given by $(-1)^N \frac{N}{2} c(n,N)\not{D}^{2N+1}$.

Proof. The conformal covariance follows directly from the construction of $D_{2N+1}(g)$. $P_{2N}^{\mathcal{S}(M)}(g)$ has leading term $c(n,N)(\Delta_g^{\mathcal{S}(M)})^N$ and it holds that

$$(\Delta_g^{\mathcal{S}(M)})^N = \begin{pmatrix} (-1)^N \not{D}^{2N} + \mathrm{LOT} & (-1)^{N-1} 2N\not{D}^{2N-1} + \mathrm{LOT} \\ \mathrm{LOT} & (-1)^N \not{D}^{2N} + \mathrm{LOT} \end{pmatrix},$$

where LOT means lower order terms. Then it follows that

$$\begin{aligned} D_{2N+1}(g) &= c(n,N)\begin{pmatrix} \frac{1}{2}\not{D} & N \end{pmatrix} (\Delta_g^{\mathcal{S}(M)})^N \begin{pmatrix} N \\ \frac{1}{2}\not{D} \end{pmatrix} + \mathrm{LOT} \\ &= (-1)^N \tfrac{N}{2} c(n,N)\not{D}^{2N+1} + \mathrm{LOT}\,, \end{aligned}$$

which completes the proof. □

Remark 4.2. In case of even n, the operator $D_{2N+1}(g)$, for $N \geq \frac{n}{2}$, is not identically zero as stated in [10, Theorem 5.27]. It is just of order less than $2N+1$, due to the fact that the constant in front of $\not{D}^{2N+1}$ is zero. Thus the last theorem does not yield conformal powers of the Dirac operator.

Theorem 4.3. *Let (M, g) be a* Spin*-manifold. A conformal third power of the Dirac operator is given by*

$$D_3(g) = -\tfrac{1}{2}[\not{D}^3 - (P, \nabla^{g,S}) - (\nabla^{g,S}, P\cdot)],$$

whereas for $n \neq 4$ a fifth power is given by

$$\begin{aligned} D_5(g) = (n-4)\Big[&\not{D} D_3(g) \not{D} + 2\left(\not{D}^2 D_3(g) + D_3(g)\not{D}^2\right) - 4\not{D}^5 \\ &+ 4\left(2P^2 + \tfrac{1}{n-4}B, \nabla^{g,S}\right) + 4\left(\nabla^{g,S}, 2P^2 \cdot + \tfrac{1}{n-4}B\cdot\right) \\ &- 2(C, P\cdot) - 2(P, C\cdot)\Big] \\ &+ \not{D}(W\cdot W\cdot) + W\cdot W\cdot \not{D} + 4(C\cdot W\cdot + W\cdot C\cdot), \end{aligned}$$

where the bracket and product notations were introduced in Section 2.

Proof. It based on a straightforward computation using explicit formulas for $D^{\mathrm{spin}}(g, w)$, $C^{\mathrm{spin}}(g, w)$ and Example 3.7. □

Remark 4.4. We define a conformally covariant differential operator $L_k(g)\colon \Gamma(\mathcal{S}(M)_g) \to \Gamma(\mathcal{S}(M)_g)$, for $k \in \mathbb{N}$, of bi-degree $(\frac{k+1-n}{2}, -\frac{k+1+n}{2})$ by

$$L_k(g) := \tfrac{4}{k+1} D^{\mathrm{spin}}\left(g, -\tfrac{k+n}{2}\right) \circ D_k(g) \circ C^{\mathrm{spin}}\left(g, \tfrac{k+1-n}{2}\right), \tag{4.2}$$

where $D_k(g) : \Gamma\big(S(M,g)\big) \to \Gamma\big(S(M,g)\big)$ is some conformally covariant operator of bi-degree $(\frac{k-n}{2}, -\frac{k+n}{2})$. The case $D_1(g) = \not{D}$ was found in a joint work with Andreas Juhl, also cf. [24]. Assuming $D_1(g) = \not{D}$ and $D_3(g)$ is our conformal third powers of the Dirac operator, gives

$$L_1(g) = \begin{pmatrix} -\not{D}^2 & 2\not{D} \\ \frac{1}{2}\not{D}^3 & -\not{D}^2 \end{pmatrix}, \quad L_3(g) = \begin{pmatrix} -D_3(g)\not{D} & 4D_3(g) \\ \frac{1}{4}\not{D} D_3(g)\not{D} & -\not{D} D_3(g) \end{pmatrix}.$$

Note that the pairs $\big(P_2^{\mathcal{S}(M)}(g), L_1(g)\big)$ and $\big(P_4^{\mathcal{S}(M)}(g), L_3(g)\big)$ have same bi-degrees, cf. Theorem 3.5. However, more interesting is their weighted sum:

$$\begin{aligned} L_1(g) - P_2^{\mathcal{S}(M)}(g) &= \begin{pmatrix} 0 & 0 \\ D_3(g) & 0 \end{pmatrix}, \\ L_3(g) + \tfrac{1}{n-4} P_4^{\mathcal{S}(M)}(g) &= \begin{pmatrix} \frac{1}{n-4} W\cdot W\cdot & 0 \\ D_5(g) & \frac{1}{n-4} W\cdot W\cdot \end{pmatrix}. \end{aligned}$$

The decomposition $P_4^{\mathcal{S}(M)}(g) = P_4^{\mathrm{red}}(g) + R_4(g)$, cf. Example 3.7, induces a decomposition $D_5(g) = D_5^{\mathrm{red}}(g) + R_5^{\mathrm{spin}}(g)$ into conformally covariant operators, where

$$\begin{aligned} R_5^{\mathrm{spin}}(g) :&= C^{\mathrm{spin}}\left(g, -\tfrac{4+n}{2}\right) \circ R_4(g) \circ D^{\mathrm{spin}}\left(g, \tfrac{5-n}{2}\right) \\ &= \not{D}(W \cdot W \cdot) + W \cdot W \cdot \not{D} + 4(C \cdot W \cdot + W \cdot C \cdot)\,. \end{aligned}$$

Question: Is it possible to describe such a decomposition for $D_{2N+1}(g)$ or equivalently for $P_{2N}^{\mathcal{S}(M)}(g)$ by use of the operators $L_{2N-1}(g)$, cf. Remark 4.4, for appropriate $D_{2N-1}(g)$?

Let us denote the first three examples of conformal powers of the Dirac operator as follows:

$$\mathcal{D}_1 := \not{D}\,; \quad \mathcal{D}_3 := -2D_3(g)\,; \quad \mathcal{D}_5 := \tfrac{1}{n-4} D_5^{\mathrm{red}}(g)\,, \qquad (n \neq 4)\,. \tag{4.3}$$

These operators have an odd power of the Dirac operator as the leading term. Due to the explicit formulas, cf. Theorem 4.3, we can prove the following theorem.

Theorem 4.5. *Let (M, g) be a n-dimensional Einstein* Spin*-manifold, normalized by* $\mathrm{Ric} = \frac{2(n-1)J}{n}$*. Then one has*

$$\begin{aligned} \mathcal{D}_3 &= \left(\not{D} - \sqrt{\tfrac{2J}{n}}\right)\not{D}\left(\not{D} + \sqrt{\tfrac{2J}{n}}\right), \\ \mathcal{D}_5 &= \left(\not{D} - \sqrt{\tfrac{8J}{n}}\right)\left(\not{D} - \sqrt{\tfrac{2J}{n}}\right)\not{D}\left(\not{D} + \sqrt{\tfrac{2J}{n}}\right)\left(\not{D} + \sqrt{\tfrac{8J}{n}}\right). \end{aligned}$$

Proof. Since (M, g) is Einstein with normalization $\mathrm{Ric} = \frac{2(n-1)J}{n} g$ for $J \in \mathbb{R}$, it follows that $P = \frac{J}{n} g$. This shows, that

$$\begin{aligned} \mathcal{D}_3 &= \not{D}^3 - 2(P, \nabla^{g,S}) = \not{D}^3 - \tfrac{2J}{n}\not{D} \\ &= \left(\not{D} - \sqrt{\tfrac{2J}{n}}\right)\not{D}\left(\not{D} + \sqrt{\tfrac{2J}{n}}\right). \end{aligned}$$

Since Bach- and Cotton tensor vanish for Einstein metrics, we have

$$\begin{aligned} \mathcal{D}_5 &= \not{D}\mathcal{D}_3\not{D} + 2\big(\not{D}^2\mathcal{D}_3 + \mathcal{D}_3\not{D}^2\big) - 4\not{D}^5 + 16(P^2, \nabla^{g,S}) \\ &= \not{D}^5 - 5\tfrac{2J}{n}\not{D}^3 + 4\big(\tfrac{2J}{n}\big)^2\not{D} \\ &= \left(\not{D} - \sqrt{\tfrac{8J}{n}}\right)\left(\not{D} - \sqrt{\tfrac{2J}{n}}\right)\not{D}\left(\not{D} + \sqrt{\tfrac{2J}{n}}\right)\left(\not{D} + \sqrt{\tfrac{8J}{n}}\right), \end{aligned}$$

which completes the proof. □

Remark 4.6. The last theorem does not hold for the operator $D_5(g)$, since there exist examples of Einstein manifolds which are not conformally flat. Such an example is given by the Fubini-Study metric on $\mathbb{C}P^2$.

Consider the standard sphere with round metric. In this case we have $J = \frac{n}{2}$ and Theorem 4.5 agrees with the result obtained in [6], where it was proven that all conformal odd powers of the Dirac operator, constructed using the ambient metric, have such a product structure.

In order to prove some formal (anti-) self-adjointness results, we present the following proposition. It generalizes the formal (anti-) self-adjointness of the Dirac operator, which is given in terms of the bracket notation (2.1), (2.2) by

$$\not{D} = \tfrac{1}{2}\left((g, \nabla^{g,S}) + (\nabla^{g,S}, g\cdot)\right) ,$$

to arbitrary symmetric $(0,2)$-tensor fields T instead of g.

Proposition 4.7. *Let (M,g) be a* Spin*-manifold without boundary, and let T be a symmetric $(0,2)$-tensor field. The operator*

$$(T, \nabla^{g,S}) + (\nabla^{g,S}, T\cdot)\colon \Gamma\big(S(M,g)\big) \to \Gamma\big(S(M,g)\big)$$

is formally (anti-) self-adjoint with respect to the induced L^2-scalar product.

Proof. Let ψ, $\phi \in \Gamma_c\big(S(M,g)\big)$ be the compactly supported spinors, and define a 1-form $w(X) := \langle T(X)^\natural \cdot \psi, \phi\rangle$ with values in $\mathbb{C}$. Considering its dual Y_w, with respect to g, and taking its divergence we obtain

$$\begin{aligned}\operatorname{div}(Y_w) = \sum_i \varepsilon_i \big[&\langle T(s_i)^\natural \cdot \nabla^{g,S}_{s_i}\psi, \phi\rangle - (-1)^p\langle \psi, T(s_i)^\natural \cdot \nabla^{g,S}_{s_i}\phi\rangle\big] \\ &+ (-1)^p\langle \psi, (\delta^{\nabla^g} T)^\natural \cdot \phi\rangle ,\end{aligned}$$

where δ^{∇^g} is the co-differential of d^{∇^g} with respect to the L^2-scalar product induced by the metric g. Using Stokes' Theorem we get $\int_M \operatorname{div}(Y_w) Vol(g) = 0$, hence

$$\begin{aligned}&\langle (T, \nabla^{g,S}\psi) + (\nabla^{g,S}, T\cdot\psi), \phi\rangle_{L^2} \\ &\qquad= \int_M \langle (T, \nabla^{g,S}\psi) + (\nabla^{g,S}, T\cdot\psi), \phi\rangle \operatorname{Vol}(g) \\ &\qquad= (-1)^p \int_M \langle \psi, 2(T, \nabla^{g,S}\phi\rangle - (\delta^{\nabla^g} T)^\natural \cdot \phi\rangle \operatorname{Vol}(g) \\ &\qquad= (-1)^p \langle \psi, (T, \nabla^{g,S}\phi) + (\nabla^{g,S}, T\cdot\phi)\rangle_{L^2} ,\end{aligned}$$

which completes the proof. □

This leads us to the following result:

Corollary 4.8. *Let (M,g) be a* Spin*-manifold without boundary. The operators $\mathcal{D}_k$, $k = 1, 3, 5$, are formally (anti-) self-adjoint with respect to the induced L^2-scalar product, i.e.,*

$$\langle \mathcal{D}_k\psi, \phi\rangle_{L^2} = (-1)^p \langle \psi, \mathcal{D}_k\phi\rangle_{L^2}$$

for ψ, ϕ compactly supported sections of the spinor bundle.

Proof. This follows from Theorem 4.3, Proposition 4.7 and the fact that we have

$$\begin{aligned}\langle (C, P\cdot)\psi + (P, C\cdot)\psi, \phi\rangle &= \sum_i \varepsilon_i \langle C(s_i)\cdot P(s_i)\cdot\psi + P(s_i)\cdot C(s_i)\cdot\psi, \phi\rangle \\ &= (-1)^p \sum_i \varepsilon_i \langle \psi, P(s_i)\cdot C(s_i)\cdot\phi + C(s_i)\cdot P(s_i)\cdot\phi\rangle \\ &= (-1)^p \langle \psi, (C, P\cdot)\phi + (P, C\cdot)\phi\rangle ,\end{aligned}$$

for any ψ, $\phi \in \Gamma\big(S(M,g)\big)$, where $\{s_i\}$ is a g-orthonormal basis. □

This corollary (except the case $k = 1$, which is well known) is a special case of the following result:

Theorem 4.9. *Let (M, g) be a* Spin-*manifold without boundary. For $N \in \mathbb{N}$ the operator $D_{2N+1}(g)$ is formally (anti-) self-adjoint with respect to the induced L^2-scalar product, i.e.,*

$$\langle D_{2N+1}(g)\psi, \phi\rangle_{L^2} = (-1)^p \langle \psi, D_{2N+1}(g)\phi\rangle_{L^2}$$

for ψ, ϕ compactly supported sections of the spinor bundle.

Proof. First of all note that from Proposition 3.3 the operator $P_{2N}^{\mathcal{S}(M)}(g)$ is formally self-adjoint. Hence, by further use of Proposition 3.3, we get that

$$\begin{aligned}\langle D_{2N+1}(g)\psi, \phi\rangle_{L^2} &= \langle C^{\mathrm{spin}}\left(g, -\tfrac{2N+n}{2}\right) \circ P_{2N}^{\mathcal{S}(M)}(g) \circ D^{\mathrm{spin}}\left(g, \tfrac{2N+1-n}{2}\right)\psi, \phi\rangle_{L^2} \\ &= \langle \psi, i^p i^p C^{\mathrm{spin}}\left(g, -\tfrac{2N+n}{2}\right) \circ P_{2N}^{\mathcal{S}(M)}(g) \circ D^{\mathrm{spin}}\left(g, \tfrac{2N+1-n}{2}\right)\phi\rangle_{L^2} \\ &= \langle \psi, (-1)^p D_{2N+1}(g)\phi\rangle_{L^2}\,,\end{aligned}$$

which completes the proof. □

Now we are going to describe lower order conformal powers of the Dirac operator as polynomials in first order differential operators. From explicit formulas for $\mathcal{D}_k$, for $k = 1, 3, 5$, we can define differential operators M_k, for $k = 1, 3, 5$, by

$$\begin{aligned} M_1 :&= \mathcal{D}_1 - 0 \\ &= \tfrac{1}{2}\left((g, \nabla^{g,S}) + (\nabla^{g,S}, g\cdot)\right)\,, \\[1ex] M_3 :&= \mathcal{D}_3 - \mathcal{D}_1^3 \\ &= -\left((P, \nabla^{g,S}) + (\nabla^{g,S}, P\cdot)\right)\,, \\[1ex] M_5 :&= \mathcal{D}_5 - \mathcal{D}_1\mathcal{D}_3\mathcal{D}_1 - 2(\mathcal{D}_1^2\mathcal{D}_3 + \mathcal{D}_3\mathcal{D}_1^2) + 4\mathcal{D}_1^5 \\ &= 4\Big((2P^2 + \tfrac{1}{n-4}B, \nabla^{g,S}) + (\nabla^{g,S}, 2P^2\cdot + \tfrac{1}{n-4}B\cdot)\Big) \\ &\quad - 2\big((C, P\cdot) + (P, C\cdot)\big)\,. \end{aligned}$$

By definition they are first order operators. Just as for each $\mathcal{D}_k$, the M_k, for $k = 1, 3, 5$, are formally (anti-) self-adjoint with respect to the induced L^2-scalar product. More interesting, however, is the following result:

Theorem 4.10. *On a* Spin-*manifold (M, g) of dimension $n \neq 4$ we have*

$$\begin{aligned} \mathcal{D}_1 &= M_1\,, \\ \mathcal{D}_3 &= M_1^3 + M_3\,, \\ \mathcal{D}_5 &= M_1^5 + M_1 M_3 M_1 + 2(M_1^2 M_3 + M_3 M_1^2) + M_5\,. \end{aligned}$$

Proof. The proof based on explicit formulas, see Theorem 4.3, and the definition of M_k, $k = 1, 3, 5$, given above. □

Remark 4.11. Concerning the structure of GJMS operator Andreas Juhl found an inversion formula, see [21, Theorem 1.1], which states that all GJMS operators are polynomials in second order differential operator and vice versa. It is also remarkable, that these second order operators are the coefficients of a holographic deformation of the Yamabe operator (in terms of the Poincaré-Einstein metric). We believe, see Theorem 4.10, that there is a complete analogous picture for the conformal powers of the Dirac operator, i.e., for all $N \in \mathbb{N}_0$ ($N \leq n$ for even n) there exists a sequence $\{M_1, \dots, M_{2N+1}\}$ of formally (anti-) self-adjoint first order differential operator such that

$$\mathcal{D}_{2N+1} \in \mathbb{N}[M_1, \dots, M_{2N+1}]\,,$$

$$M_{2N+1} \in \mathbb{Z}[\mathcal{D}_1, \dots, \mathcal{D}_{2N+1}]\,.$$

Hence, it is natural to ask about the nature of M_{2N+1}. For example, is there a generating function for the series of M_{2N+1}, and how can one understand the coefficients arising in the polynomial description of $\mathcal{D}_{2N+1}$?

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