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HOW MANY ARE AFFINE CONNECTIONS WITH TORSION

Zdeněk Dušek and Oldřich Kowalski

Abstract. The question how many real analytic affine connections exist locally on a smooth manifold M of dimension n is studied. The families of general affine connections with torsion and with skew-symmetric Ricci tensor, or symmetric Ricci tensor, respectively, are described in terms of the number of arbitrary functions of n variables.

1. Introduction

When we consider an infinite family of well-determined geometric objects, it is natural to put the question about "how many" such objects there exist. In the real analytic case, the Cauchy-Kowalevski Theorem is the standard tool ([3], [7], [11]). Hence a natural way how to measure an infinite family of real analytic geometric objects is a finite family of arbitrary functions of k variables and (optionally) a family of arbitrary functions of $k-1$ variables, and, optionally, "modulo" another family of arbitrary functions of $k-1$ variables. The last (optional) family of functions corresponds to the family of automorphisms of any geometric object from the given family. A good example is the following question: How many there are real analytic Riemannian metrics in dimension 3? It is known (see [4], [8]) that every such metric can be put locally into a diagonal form and that all coordinate transformations preserving diagonal form of the given metric depend on 3 arbitrary functions of two variables. Hence all Riemannian metrics in dimension 3 can be locally described by 3 arbitrary functions of 3 variables modulo 3 arbitrary functions of 2 variables. An immediate question arise if we can "calculate the number" of more basic geometric objects, namely the affine connections, in an arbitrary dimension n. To the authors' knowledge, no attempts are known in this direction from the past. We shall be occupied with real analytic affine connections in arbitrary dimension n. Our first result is the description of this class of connections using $n(n^2-1)$ functions of n variables modulo $2n$ functions of $n-1$ variables. For this purpose, we use the existence of the system of pre-semigeodesic coordinates. (see [9] for the definition and various applications of this concept and [2] for the alternative proof of the existence of such a system of coordinates).

A well known fact from Riemannian geometry is that a Riemannian connection has symmetric Ricci form. Our next aim is to determine "how big" is the class

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of all real analytic affine connections with *skew-symmetric* Ricci form (again, in dimension n) and those with *symmetric* Ricci form. For this purpose, a direct approach using the Cauchy-Kowalevski Theorem can be used. Surprisingly, for the torsion-free connections with symmetric Ricci form, another method was necessary, see [2] for the details.

We prove that the class of real analytic connections with torsion and with skew-symmetric Ricci form depends on $n(2n^2-n-3)/2$ functions of n variables and $n(n+1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables. We prove further that the class of real analytic connections with symmetric Ricci form depends on $n(2n^2-n-1)/2$ functions of n variables and $n(n-1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.

2. The Cauchy-Kowalevski Theorem

For the aim of the next sections, and to remain self-contained, we shall formulate two important special cases of the Cauchy-Kowalevski Theorem, namely the case of order one and the case of "pure" order 2. We shall start with the complete and explicit version in order 1.

Theorem 1. *Consider a system of partial differential equations for unknown functions $U^1(x^1,\dots,x^n),\dots,U^N(x^1,\dots,x^n)$ on an open domain in $\mathbb{R}^n$ and of the form*

$$\begin{aligned}
\frac{\partial U^1}{\partial x^1} &= H^1\Big(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\Big),\\
\frac{\partial U^2}{\partial x^1} &= H^2\Big(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\Big),\\
&\vdots\\
\frac{\partial U^N}{\partial x^1} &= H^N\Big(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\Big),
\end{aligned}$$

where H^i, $i=1,\dots,N$, are real analytic functions of all variables in a neighbourhood of $(x_0^1,\dots,x_0^n,a^1,\dots,a^N,a_2^1,\dots,a_n^1,\dots,a_2^N,\dots,a_n^N)$, where x_0^j, a^i, a_j^i are arbitrary constants.

Further, let the functions $\varphi^1(x^2,\dots,x^n),\dots,\varphi^N(x^2,\dots,x^n)$ be real analytic in a neighbourhood of $(x_0^2,\dots,x_0^n)$ and satisfy $\varphi^i(x_0^2,\dots,x_0^n)=a^i$ for $i=1,\dots,N$ and

$$\Big(\frac{\partial\varphi^1}{\partial x^2},\dots,\frac{\partial\varphi^1}{\partial x^n},\dots,\frac{\partial\varphi^N}{\partial x^2},\dots,\frac{\partial\varphi^N}{\partial x^n}\Big)(x_0^2,\dots,x_0^n)=(a_2^1,\dots,a_n^1,\dots,a_2^N,\dots,a_n^N)\,.$$

Then the system has a unique solution $(U^1(x^1,\dots,x^n),\dots,U^N(x^1,\dots,x^n))$ which is real analytic around $(x_0^1,\dots,x_0^n)$, and satisfies

$$U^i(x_0^1,x^2,\dots,x^n)=\varphi^i(x^2,\dots,x^n)\,,\qquad i=1,\dots,N\,.$$

For the "pure" case of order 2, the basic assumptions about the system of PDEs are analogous: The left-hand sides are second derivatives

$$\frac{\partial^2 U^1}{(\partial x^1)^2}, \dots, \frac{\partial^2 U^N}{(\partial x^1)^2}$$

and the right-hand sides $H^1, \dots, H^N$ involve, as arguments, the original coordinates, the unknown functions $U^1, \dots, U^N$, their first derivatives and their second derivatives except the derivatives written on the left-hand sides. Thus, for each $i = 1, \dots, N$, on the right-hand side, we have the function

$$H^i\Big(x^j, U^p, \frac{\partial U^p}{\partial x^k}, \frac{\partial^2 U^p}{\partial x^k \partial x^l}\Big), \quad j,k = 1, \dots, n, \quad l = 2, \dots, n, \quad p = 1, \dots, N.$$

Then the statement of the theorem says that there exist locally a unique N-tuple $(U^1, \dots, U^N)$ of real analytic functions which is a solution of the new PDE system, and satisfies the initial conditions

$$U^i(x_0^1, x^2, \dots, x^n) = \varphi_0^i(x^2, \dots, x^n),$$
$$\frac{\partial U^i}{\partial x^1}(x_0^1, x^2, \dots, x^n) = \varphi_1^i(x^2, \dots, x^n).$$

The general solution then depends on $2N$ arbitrary functions φ_0^i, φ_1^i of $n-1$ variables. See [3], [7] and [11] for the general case and more details.

3. Transformation of the connection

We work locally with the spaces $\mathbb{R}[u^1, \dots, u^n]$, or $\mathbb{R}[x^1, \dots, x^n]$, respectively. We will use the notation $\mathbf{u} = (u^1, \dots, u^n)$ and $\mathbf{x} = (x^1, \dots, x^n)$. For a diffeomorphism $f\colon \mathbb{R}[\mathbf{u}] \to \mathbb{R}[\mathbf{x}]$, we write $x^k = f^k(u^l)$, or $\mathbf{x} = \mathbf{x}(\mathbf{u})$ for short. We start with the standard formula for the transformation of the connection, which is

$$\bar{\Gamma}^h_{ij}(\mathbf{u}) = \Big(\Gamma^\gamma_{\alpha\beta}\big(\mathbf{x}(\mathbf{u})\big)\frac{\partial f^\alpha}{\partial u^i}\frac{\partial f^\beta}{\partial u^j} + \frac{\partial^2 f^\gamma}{\partial u^i \partial u^j}\Big)\frac{\partial f^h}{\partial u^\gamma}. \tag{1}$$

Lemma 2 ([2]). *For any affine connection determined by $\Gamma^h_{ij}(\mathbf{x})$, there exist a local transformation of coordinates determined by $\mathbf{x} = f(\mathbf{u})$ such that the connection in new coordinates satisfies $\bar{\Gamma}^h_{11}(\mathbf{u}) = 0$, for $h = 1, \dots, n$. All such transformations depend on $2n$ arbitrary functions of $n-1$ variables.*

Proof. We substitute from (1) to the equations $\bar{\Gamma}^h_{11}(\mathbf{u}) = 0$ and we get

$$\Big(\Gamma^\gamma_{\alpha\beta}\big(\mathbf{x}(\mathbf{u})\big)\frac{\partial f^\alpha}{\partial u^1}\frac{\partial f^\beta}{\partial u^1} + \frac{\partial^2 f^\gamma}{(\partial u^1)^2}\Big)\frac{\partial f^h}{\partial u^\gamma} = 0, \qquad h = 1, \dots, n. \tag{2}$$

We suppose further that the determinant of the Jacobi matrix of the transformation $\mathbf{x} = f(\mathbf{u})$ is nonzero, which corresponds to the regularity condition for the transformation. We multiply these equations by the inverse of the Jacobi matrix and we obtain the equivalent equations

$$\frac{\partial^2 f^\gamma}{(\partial u^1)^2} = -\Gamma^\gamma_{\alpha\beta}\big(\mathbf{x}(\mathbf{u})\big)\frac{\partial f^\alpha}{\partial u^1}\frac{\partial f^\beta}{\partial u^1}, \qquad \gamma = 1, \dots, n. \tag{3}$$

On the right-hand sides, we have analytic functions depending on $f^1, \dots, f^n$ and their first derivatives. We choose arbitrary analytic functions $\varphi^i_\lambda(u^2, \dots, u^n)$, for $i = 1, \dots, n$ and $\lambda = 0, 1$. According to the Cauchy-Kowalevski Theorem (the case of pure order 2), there exist unique functions $f^i(u^1, \dots, u^n)$ such that

$$\begin{aligned} f^i(u^1_0, u^2, \dots, u^n) &= \varphi^i_0(u^2, \dots, u^n)\,, \\ \frac{\partial f^i}{\partial u^1}(u^1_0, u^2, \dots, u^n) &= \varphi^i_1(u^2, \dots, u^n)\,. \end{aligned} \tag{4}$$

Obviously, determinant of the Jacobi matrix for these functions will be nonzero for the generic choice of the functions $\varphi^i_\lambda(u^2, \dots, u^n)$. □

Remark 3. Hereby, the local existence of pre-semigeodesic coordinates is proved.

We finish this paragraph with the following existence therorem, which is a corollary of Lemma2.

Theorem 4. *All affine connections with torsion in dimension n depend locally on $n(n^2-1)$ arbitrary functions of n variables, modulo $2n$ arbitrary functions of $(n-1)$ variables.*

Proof. After the transformation into pre-semigeodesic coordinates, we obtain n Christoffel symbols equal to zero. We are left with $n^3 - n = n(n^2-1)$ functions. The transformations into pre-semigeodesic coordinates is uniquely determined up to the choice of $2n$ functions $\varphi^i_0(u^2, \dots, u^n), \varphi^i_1(u^2, \dots, u^n)$ of $n-1$ variables. □

4. The Ricci tensor

We consider the space $\mathbb{R}^n[u^i]$ with the coordinate vector fields $E_i = \frac{\partial}{\partial u^i}$. We will denote derivatives with respect to u^i by the bottom index i. Using the standard definition

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{5}$$

we calculate the curvature operators

$$R(E_i, E_j)E_k = (\Gamma^\alpha_{jk})_i E_\alpha - (\Gamma^\beta_{ik})_j E_\beta + \Gamma^\alpha_{jk}\Gamma^\gamma_{i\alpha} E_\gamma - \Gamma^\beta_{ik}\Gamma^\delta_{j\beta} E_\delta.$$

For the Ricci form

$$\mathrm{Ric}(X,Y) = \mathrm{trace}\big[W \mapsto R(W,X)Y\big]\,, \tag{6}$$

we obtain

$$\mathrm{Ric}(E_i, E_j) = \sum_{k,l=1}^n \left[(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{kj}\Gamma^k_{il} \right]. \tag{7}$$

5. Skew-symmetric Ricci tensor

We analyze the condition for the skew-symmetry of the Ricci form using formulas (7). We split the skew-symmetry conditions into four cases:

$$\begin{aligned} \operatorname{Ric}(E_1, E_1) &= 0\,, & & \\ \operatorname{Ric}(E_i, E_i) &= 0\,, & & i > 1, \\ \operatorname{Ric}(E_1, E_i) + \operatorname{Ric}(E_i, E_1) &= 0\,, & & i > 1, \\ \operatorname{Ric}(E_i, E_j) + \operatorname{Ric}(E_j, E_i) &= 0\,, & & 1 < i < j \le n\,. \end{aligned} \tag{8}$$

In each formula which follows, we denote by Λ'_{ij} the terms which involve first derivatives with respect to $u^2, \ldots, u^n$ and by Λ_{ij} the terms which do not involve any differentiation (and which form a homogeneous polynomial of degree 2 in Γ^k_{ij}). Corresponding to the four cases above, we obtain, using (7), the equations

$$\begin{aligned} \sum_{k=2}^{n} (\Gamma^k_{k1})_1 &= \Lambda'_{11} + \Lambda_{11}\,, & & \\ (\Gamma^1_{ii})_1 &= \Lambda'_{ii} + \Lambda_{ii}\,, & & i > 1\,, \\ (\Gamma^1_{i1})_1 - \sum_{k=2}^{n} (\Gamma^k_{ki})_1 &= \Lambda'_{1i} + \Lambda_{1i}\,, & & i > 1\,, \\ (\Gamma^1_{ij})_1 + (\Gamma^1_{ji})_1 &= \Lambda'_{ij} + \Lambda_{ij}\,, & & 1 < i < j \le n\,. \end{aligned} \tag{9}$$

Theorem 5. *The family of connections with torsion whose Ricci form is skew-symmetric depends locally on $\frac{n(2n^2-n-3)}{2}$ functions of n variables and $\frac{n(n+1)}{2}$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.*

Proof. After the transformation into pre-semigeodesic coordinates, the family of torsion-free connections depends on $q(n) = n(n^2-1)$ functions (Christoffel symbols). In the system of equations (9), there are $p(n) = n(n+1)/2$ conditions for the skew-symmetry of the Ricci form. These conditions involve first derivatives of the Christoffel symbols and they are written in a way that the derivatives with respect to the first coordinate are on the left-hand side and all the other terms are on the right-hand side. Any Christoffel symbol appears on the left-hand side of the mentioned equations at most once.

Now we select one Christoffel symbol in each of the equations, for example the following Christoffel symbols:

Γ^2_{21} for $i > 1$ (1 function),

Γ^1_{ii} for $i > 1$ (altogether $n-1$ functions),

Γ^1_{ij} for $i > j$ (altogether $n(n-1)$ functions).

We choose the other $q(n) - p(n) = n(2n^2-n-3)/2$ Christoffel symbols as arbitrary functions. We substitute the arbitrary functions chosen above into the system (9) and transport them to the right-hand side, if necessary. We obtain a new system of equations of the form

$$
\begin{aligned}
(\Gamma^2_{21})_1 &= -\sum_{k=3}^{n}(\Gamma^k_{1k})_1 + \Lambda'_{11} + \Lambda_{11}\,, \\
(\Gamma^1_{ii})_1 &= \Lambda'_{ii} + \Lambda_{ii}, \qquad i > 1\,, \\
(\Gamma^1_{i1})_1 &= -\sum_{k=2}^{n}(\Gamma^k_{ik})_1 + \Lambda'_{1i} + \Lambda_{1i}, \qquad i > 1\,, \\
(\Gamma^1_{ji})_1 &= \Lambda'_{ij} + \Lambda_{ij}, \qquad 1 < i < j \leq n\,,
\end{aligned} \tag{10}
$$

where the Christoffel symbols on the right-hand sides are already fixed. We have got a standard system of $p(n)$ equations for the last $p(n)$ functions for which the Cauchy-Kowalevski Theorem can be applied. The general solution depends on $p(n)$ arbitrary functions of $n-1$ variables and, because we used pre-semigeodesic coordinates, this number is to be reduced by $2n$ functions. □

6. Symmetric Ricci tensor

We recall the formula for the nondiagonal entries of the Ricci form

$$
\mathrm{Ric}(E_i, E_j) = \sum_{k,l=1}^{n}\left[(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{kj}\Gamma^k_{il}\right]. \tag{11}
$$

We analyze the symmetry conditions for the Ricci form, which is

$$
\mathrm{Ric}(E_i, E_j) - \mathrm{Ric}(E_j, E_i) = 0\,, \qquad 1 \leq i < j \leq n\,. \tag{12}
$$

We analyze these conditions using formulas (11). In each of the equations which follow, we denote by Λ_{ij} the terms which do not involve any differentiation and which form a homogeneous polynomial in Γ^k_{ij}.

$$
\sum_{k=1}^{n}\left[(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i - (\Gamma^k_{ji})_k + (\Gamma^k_{ki})_j\right] = \Lambda_{ij}\,, \qquad 1 \leq i < j \leq n\,. \tag{13}
$$

Now we denote by Λ'_{ij} the terms which involve first derivatives with respect to $u^2, \dots, u^n$ and we simplify the above sums. We will split the situation into the two cases, $i = 1$ and $i > 1$. We obtain

$$
\begin{aligned}
-\sum_{k=2}^{n}(\Gamma^k_{kj})_1 - (\Gamma^1_{j1})_1 &= \Lambda'_{1j} + \Lambda_{1j}\,, \qquad 1 < j \leq n\,, \\
(\Gamma^1_{ij})_1 - (\Gamma^1_{ji})_1 &= \Lambda'_{ij} + \Lambda_{ij}\,, \qquad 1 < i < j \leq n\,.
\end{aligned} \tag{14}
$$

Theorem 6. *The family of connections with torsion whose Ricci form is symmetric depends locally on $\frac{n(2n^2-n-1)}{2}$ functions of n variables and $\frac{n(n-1)}{2}$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.*

Proof. After the transformation into pre-semigeodesic coordinates, the family of connections depends on $q(n) = n^3 - n = n(n^2 - 1)$ functions (Christoffel symbols). In the system of equations (13) or (14), there are $p(n) = n(n-1)/2$ conditions for the symmetry of the Ricci form. We let the $p(n)$ Christoffel symbols Γ^1_{ji} to be determined later and we chose all the other $q(n) - p(n) = n(2n^2 - n - 1)/2$ Christoffel symbols as arbitrary functions. If we transport the chosen Christoffel symbols to the right-hand sides of the equations (14), we obtain a standard system of $p(n)$ equations for the last $p(n)$ functions for which the Cauchy-Kowalevski Theorem can be applied. The general solution depends on $p(n)$ arbitrary functions of $n-1$ variables and, because we used pre-semigeodesic coordinates, this number is to be reduced by $2n$ functions. $\square$

7. Final remarks

An interesting observation shows that, for $n \to \infty$, the number of functions of n variables describing the family of connections with skew-symmetric Ricci tensor behaves as n^3 and the number of functions describing the family of connections with symmetric Ricci tensor also behaves as n^3. On the other hand, exactly the same asymptotic estimate holds for the family of all connections. This is an interesting paradox related to the fact that the calculation of the Ricci tensor from a given connection is a nonlinear procedure.

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Zdeněk Dušek,
University of Hradec Králové, Faculty of Science,
Rokitanského 62, 500 03 Hradec Králové, Czech Republic
E-mail: zdenek.dusek@uhk.cz
and
University of Ostrava, Faculty of Science,
30. dubna 22, 701 03 Ostrava, Czech Republic

Oldřich Kowalski,
Charles University, Faculty of Mathematics and Physics,
Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail: kowalski@mff.cuni.cz