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EFFECTIVE HOMOLOGY FOR HOMOTOPY COLIMIT AND COFIBRANT REPLACEMENT

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Abstract. We extend the notion of simplicial set with effective homology presented in [22] to diagrams of simplicial sets. Further, for a given finite diagram of simplicial sets $X\colon \mathcal{I} \to \mathsf{sSet}$ such that each simplicial set $X(i)$ has effective homology, we present an algorithm computing the homotopy colimit $\mathsf{hocolim}\, X$ as a simplicial set with effective homology. We also give an algorithm computing the cofibrant replacement X^{cof} of X as a diagram with effective homology. This is applied to computing of equivariant cohomology operations.

1. Introduction

In recent years a framework of *simplicial sets with effective homology* developed by Sergeraert et al. (see e.g. [22]) was used by a group of authors in papers [4, 5, 6, 13] to address a variety of computational problems in homotopy theory and algebraic topology. The main property of simplicial sets with effective homology is that we are able to perform homological computations with them. This is important when working with infinite simplicial sets such as the Eilenberg–MacLane spaces $K(\pi, n)$.

The framework also provides a collection of algorithms on simplicial sets with effective homology in such a way that if the inputs are simplicial sets with effective homology, the output are also simplicial sets with effective homology. These algorithms usually describe commonly used constructions from algebraic topology such as Cartesian product, loop space, bar construction, mapping cylinder, total space of fibration (see [4, 5, 22]) and homotopy pushout [6, 16].

The are however constructions such as colimits over finite diagrams of simplicial sets with effective homology that do not result in a simplicial set with effective homology. This is demonstrated later in Example 2.13.

Nevertheless, in this paper we enlarge the collection of constructions which behave well with respect to effective homology by presenting an algorithm that computes the Bousfield–Kan model of homotopy colimit of a finite diagram of

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simplicial sets with effective homology. We can formulate this in the following proposition.

Proposition 1.1. *Let $\mathcal{I}$ be a finite category and $X\colon \mathcal{I} \to \mathsf{sSet}$ a functor. Suppose that every $X(i)$ has effective homology for all $i \in \mathcal{I}$ and that all maps in the diagram X are computable. Then the space $\operatorname{\mathsf{hocolim}} X$ has effective homology.*

For the second main result, we define the notion of a diagram of simplicial sets with effective homology. This notion is stronger than the notion of pointwise effective homology of a diagram defined by the requirement that every simplicial set in the diagram has effective homology. We consider the projective model structure on the category of functors $\mathcal{I} \to \mathsf{sSet}$ [1, chapter 9] in which weak equivalences and fibrations are pointwise weak equivalences and fibrations, respectively. Our second result reads as

Proposition 1.2. *Let $\mathcal{I}$ be a finite category and $X\colon \mathcal{I} \to \mathsf{sSet}$ a functor. Suppose that every $X(i)$ has effective homology for all $i \in \mathcal{I}$ and that all maps in the diagram X are computable. Then there is an algorithm which provides a diagram X^{cof} which is a cofibrant replacement of X and has effective homology as a diagram.*

In [6], the authors describe algorithmic approach for computing homotopy classes between spaces X and Y with a free action of a finite group if the dimension of X is twice the connectivity of Y. The main motivation for Proposition 1.2 is the extension of this result to simplicial sets with nonfree action of a finite group using the fact the category sSet^G of simplicial sets with an action of a group G is Quillen equivalent to a category of functors $\mathcal{O}_G^{\mathsf{op}} \to \mathsf{sSet}$ where $\mathcal{O}_G$ is a category of orbits of the group G.

In Section 4, we present a special case of such a computation by computing the equivariant cohomology operations, i.e. sets of equivariant homotopy classes $[K(\pi, n), K(\rho, m)]_G$ where π and ρ are diagrams of abelian groups [2, 9, 20] equipped with actions of a finite group G. These can be computed as the cohomology groups of the cofibrant replacement of $K(\pi, n)$ on the level of diagrams.

2. Effective homology

We first repeat basic notions of effective homological algebra.

Locally effective simplicial sets. The basic building blocks of the theory are locally effective simplicial sets and computable maps between them.

Let X be a simplicial set. We say that X is *locally effective* if we are given a specified encoding of the simplices of X and a collection of algorithms computing the face and degeneracy operations on the simplices of X. The notation suggests that we do not have any global information about X. Similarly, a map of locally effective simplicial sets is *computable* if there is an algorithm that for any given simplex in the domain gives the encoding for its image.

Effective chain complexes, reductions and strong equivalences. Let C_* be a chain complex and let A be a set such that for every $\alpha \in A$ we are given $c_\alpha \in C_*$. We call C_* *free with basis* $\{c_\alpha\}$ if every chain $c \in C_*$ can be expressed uniquely as a combination

$$c = \sum k_\alpha c_\alpha \tag{1}$$

with integer coefficients k_α in $\mathbb{Z}$.

We call C_* *locally effective* if the elements of C_* have finite encoding and there are algorithms computing the multiplication, addition, zero, inverse and differential for the elements of C_*. Further there is a basis and an algorithm that for every $c \in C_*$ computes (1) and decides whether c lies in the basis.

The chain complex C_* is called *effective* if it is locally effective and there is an algorithm that for given $n \in \mathbb{N}$ generates a finite basis $c_\alpha \in C_n$.

Clearly, the effective chain complexes have nice computational properties e.g. one can compute the homology of such a chain complex.

Definition 2.1. Let C_*, C'_* be chain complexes. A *reduction* $C_* \Rightarrow C'_*$ is a triple (α, β, η) pictured as below, where α, β are chain maps and η is a morphism of graded groups.

$$(\alpha,\beta,\eta)\colon C_* \Rightarrow C'_* \quad \equiv \quad \begin{tikzcd} C_* \arrow[loop left, "\eta"] \arrow[r, bend left, "\alpha"] & C'_* \arrow[l, bend left, "\beta"] \end{tikzcd}$$

satisfying

$$\begin{aligned} &\eta\beta = 0 \qquad &&\alpha\eta = 0 \qquad &&\eta\eta = 0 \\ &\alpha\beta = \mathrm{id} \qquad &&\partial\eta + \eta\partial = \mathrm{id} - \beta\alpha\,. && \end{aligned} \tag{2}$$

A *strong equivalence* between chain complexes $C_* \Longleftrightarrow C'_*$ consists of a span of reductions

$$\begin{tikzcd} & \widehat{C}_* \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ C_* & & C'_* \end{tikzcd}$$

Strong equivalences can be composed [22], producing another strong equivalence. We say that a locally effective chain complex C_* is equipped with *effective homology* if there is a strong equivalence $C_* \Leftrightarrow C_*^{\mathrm{ef}}$ of C_* with some effective chain complex C_*^{ef}. A locally effective simplicial set X is said to have *effective homology* if its chain complex $C_*(X)$ does.

Objects with effective homology further have nice properties:

Lemma 2.2. *Let C_*, C'_* be chain complexes with effective homology and let X, Y be simplicial sets with effective homology. Then the following holds:*

(1) *$C_* \oplus C'_*$, $C_* \otimes C'_*$ have effective homology.*

(2) *The space $X \times Y$ has effective homology.*

Proof. The proof that $C_* \oplus C'_*$ have effective homology is easy, for the proof that $C_* \otimes C'_*$ has effective homology, we refer the reader to [22, Proposition 61].

For the second part, we use the Eilenberg–Zilber reduction $C_*(X \times Y) \Rightarrow C_*(X) \otimes C_*(Y)$ first introduced in [10, 11]. The proof is finished using the first part of the statement. □

Perturbation Lemmas. Consider a reduction $C_* \Rightarrow D_*$. If we change the differential of one of the complexes, the following perturbation lemmas provide us with a new reduction $C'_* \Rightarrow D'_*$ where the C'_*, D'_* are the original chain complexes with changed (perturbed) differential.

Definition 2.3. Let (C_*, ∂) be a chain complex. We call a collection of maps $\delta \colon C_* \to C_{*-1}$ *perturbation* if the sum $\partial + \delta$ is also a differential on C_*.

The following perturbation lemmas are well–known, they constitute one of the basic tools in homological perturbation theory. Their genesis can be traced back to [3, 10, 15, 23] and their full proofs can be found e.g. in [22].

Lemma 2.4 (Easy Perturbation Lemma)**.** *Let $(\alpha, \beta, \eta) \colon (C_*, \partial) \Rightarrow (C'_*, \partial')$ be a reduction. Suppose δ' is a perturbation of differential ∂'. Then there is a reduction $(\alpha, \beta, \eta) \colon (C, \partial + \beta\delta'\alpha) \Rightarrow (C', \partial' + \delta)$.*

Proof. Check the formulas for the new reduction given in the statement. □

Lemma 2.5 (Basic Perturbation Lemma)**.** *Let $(\alpha, \beta, \eta) \colon (C_*, \partial) \Rightarrow (C_*, \partial')$ be a reduction. Suppose δ is a perturbation of differential ∂ such that let for every $c \in C_*$ there is some $i \in \mathbb{N}$ satisfying $(\eta\delta)^i(c) = 0$. Then there is a reduction $(\alpha', \beta', \eta') \colon (C, \partial + \delta) \Rightarrow (C, \partial' + \delta')$.*

Proof. We describe the formulas for maps in the new reduction and the rest is left for the reader. We set

$$\varphi = \sum_{i=0}^{\infty} (-1)^i (\eta\delta)^i; \qquad \psi = \sum_{i=0}^{\infty} (-1)^i (\delta\eta)^i \,.$$

By the condition in the statement, both sums are finite. The maps in the reduction are given as follows:

$$\delta' = \alpha\psi\delta\beta = \alpha\delta\varphi\beta; \quad \alpha' = \alpha\psi; \quad \beta' = \varphi\beta; \quad \eta' = \varphi\eta = \eta\psi \,. \qquad \square$$

Effective homology for diagrams.

Definition 2.6. We call a diagram $C \colon \mathcal{I} \to \mathsf{Ch}_+$ of nonnegatively graded chain complexes *pointwise locally effective* if $C(i)$ is locally effective for every $i \in \mathcal{I}$. If further for every every morphism f in the category $\mathcal{I}$ is the morphism $C(f)$ computable, we call C *locally effective*.

We say that pointwise locally effective C has *pointwise effective homology* if for every $i \in \mathcal{I}$ there is an effective chain complex $C^{\mathrm{ef}}(i)$ and a strong equivalence of chain complexes $C(i) \Longleftrightarrow C^{\mathrm{ef}}(i)$.

Definition 2.7. Let $C \colon \mathcal{I} \to \mathsf{Ch}_+$. We say C is *cellular* if there is an indexing set A such that for every $\alpha \in A$ we are given

- $i_\alpha \in \mathcal{I}$,
- c_α is a chain in $C(i_\alpha)$,

such that the set
$$\{f_* c_\alpha \mid \alpha \in A, f \in \mathcal{I}(i_\alpha, i)\}$$
forms a basis for each $C(i)$.

We can formulate the cellularity also in a different way: given an element $c \in C(i)$ there is a unique description of c as

$$c = \sum_\alpha f_{\alpha *}(c_\alpha) \tag{3}$$

where $f_{\alpha *}$ is an element in $\mathbb{Z}\mathcal{I}(i_\alpha, i)$ the free abelian group on the set $\mathcal{I}(i_\alpha, i)$.

Definition 2.8. We call a locally effective diagram C *effective* if there is an algorithm that generates for given n finite list of all elements $c_\alpha \in C(i_\alpha)_n$ and an algorithm computing (3) for this basis.

We remark that the notion of a cellular functor C is somewhat related to the notion of a functor that is free or representable with models $\mathsf{ob}(\mathcal{C})$. One can find more details, e.g. in [19, p. 127].

Example 2.9. Let $\mathcal{I} = G$, where G is a finite group thought of as a category with one object and arrows labelled by the elements of G. Let $C\colon \mathcal{I} \to \mathsf{Ch}_+$. Then the cellularity means that every element $c \in C(i)$ can be described uniquely as
$$c = \sum_\alpha k_\alpha c_\alpha\,,$$
where $k_\alpha \in \mathbb{Z}G$. Hence, the chain complex is cellular only if G acts freely.

Example 2.10. Let $\mathcal{I}$ be a finite category. Then for any $i \in \mathcal{I}$ there is a functor $\mathbb{Z}\mathcal{I}(i, -)\colon \mathcal{I} \to \mathsf{Ab}$ the free abelian group on the set $\mathcal{I}(i, -)$. We think of this abelian group as a chain complex concentrated in degree 0 and thus obtain a functor $\mathbb{Z}\mathcal{I}(i, -)\colon \mathcal{I} \to \mathsf{Ch}_+$. This diagram of chain complexes is effective.

We first show it is cellular: The basis consists of one element only, namely id_i. Given some $j \in \mathcal{I}$, the elements $x \in \mathcal{I}(i, j)$ form the basis $\mathbb{Z}\mathcal{I}(i, -)$ and we can describe them as $x = x_*(\mathrm{id}_i)$. The finiteness of $\mathcal{I}$ now implies that $\mathbb{Z}\mathcal{I}(i, -)$ is effective.

To define diagrams with effective homology, we introduce reduction and strong equivalence of diagrams:

Definition 2.11. Let $C, C'\colon \mathcal{I} \to \mathsf{Ch}_+$ be diagrams of chain complexes. A *reduction* $C \Rightarrow C'$ is a triple of natural transformations (α, β, η).

$$(\alpha, \beta, \eta)\colon C \Rightarrow C' \quad \equiv \quad \begin{tikzcd} C \arrow[loop left, "\eta"] \arrow[r, bend left, "\alpha"] & C' \arrow[l, bend left, "\beta"] \end{tikzcd}$$

they again satisfy the conditions (2).

Similarly we define strong equivalence of diagrams of chain complexes and the strong equivalences can again be composed. We say that a locally effective diagram

$C\colon \mathcal{I} \to \mathsf{Ch}_+$ has *effective homology* if there is a strong equivalence of locally effective diagrams $C \Leftarrow \widehat{C} \Rightarrow C^{\mathrm{ef}}$ where $C^{\mathrm{ef}}\colon \mathcal{I} \to \mathsf{Ch}_+$ is an effective diagram of chain complexes. A diagram of simplicial sets $X\colon \mathcal{I} \to \mathsf{sSet}$ has *effective homology* if $C(X)$ has effective homology.

Proposition 1.1 states that homotopy colimits of finite diagrams of simplicial sets with pointwise effective homology have effective homology. The same does not hold for finite colimits. To demonstrate this, we present two examples. The first example shows that colimits of finite diagrams of chain complexes that are pointwise effective do not have effective homology in general. We will then use the idea of the first example to give a finite diagram of simplicial sets with pointwise effective homology with a colimit that does not have effective homology.

Example 2.12. Let $\mathcal{I} = \mathsf{C}_2$, where $\mathsf{C}_2 = \{1, t\}$ is the two element group. As a category, it has one object and two arrows. Any diagram of chain complexes $D\colon \mathcal{I} \to \mathsf{Ch}_+$ can be seen as a chain complex of $\mathbb{Z}\mathsf{C}_2$ modules. We take the following diagram of chain complexes $C\colon \mathcal{I} \to \mathsf{Ch}_+$, which is a free $\mathbb{Z}\mathsf{C}_2$ resolution of the module $\mathbb{Z}$ with trivial group action:

$$\cdots \xrightarrow{\partial_4} \mathbb{Z}\mathsf{C}_2 \xrightarrow{\partial_3} \mathbb{Z}\mathsf{C}_2 \xrightarrow{\partial_2} \mathbb{Z}\mathsf{C}_2 \xrightarrow{\partial_1} \mathbb{Z} \longrightarrow 0$$

The differentials $\partial_i\colon C_i \to C_{i-1}$ are $\partial_1(1) = \partial_1(t) = 1$ and for $i \geq 1$, we have

$$\partial_{2i-1}(c) = (1+t)\cdot c\,; \qquad \partial_{2i}(c) = (1-t)\cdot c\,.$$

This chain complex is contractible over $\mathbb{Z}$ as there is a reduction $(0,0,h)\colon X \Rightarrow 0$ to a trivial chain complex $0\colon \mathcal{I} \to \mathsf{Ch}_+$. The only nontrivial part of the reduction data is the homotopy $h_i\colon C_i \to C_{i+1}$ which is given by

$$h_{2i-1}(t) = t; h_{2i-1}(1) = 0 \qquad h_{2i}(t) = 0; h_{2i}(1) = 1 \qquad h_0(1) = 1$$

and it follows that C has effective homology.

The colimit $\mathsf{colim}_{\mathcal{I}} C$ is the following chain complex which we obtained by factoring out the group action:

$$\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \longrightarrow 0$$

where the differentials 0, 2 alternate. Notice that $\mathsf{colim}_{\mathcal{I}} C$ has nontrivial homology groups.

Let us now take $(\bigoplus_{\mathbb{N}} C) : \mathcal{I} \to \mathsf{Ch}_+$, the countable directed coproduct of copies of C. It is still contractible and has effective homology (the reduction is given by enlarging the reduction given above).

On the other hand, $\mathsf{colim}_{\mathcal{I}} (\bigoplus_{\mathbb{N}} C) = \bigoplus_{\mathbb{N}} \mathsf{colim}_{\mathcal{I}} C$ has infinitely generated homology groups and therefore cannot have effective homology.

Example 2.13. We describe the main steps of the construction and omit most of the details. The idea of this example is similar to the idea of Example 2.12. We take the space $E\mathsf{C}_2$, which is a contractible space with a free action of the group C_2. It can be seen as a diagram $E\mathsf{C}_2\colon \mathsf{C}_2 \to \mathsf{sSet}$. Using a specified simplicial model of $E\mathsf{C}_2$ described e.g. in [19, p. 101] given by

$$(E\mathsf{C}_2)_n = C^0(\Delta^n, \mathsf{C}_2)$$

one can deduce that $E\mathsf{C}_2$ has effective homology [18].

The unreduced suspension $SE\mathsf{C}_2$ of $E\mathsf{C}_2$ has two distinct points which we denote $+1, -1$ where the action of C_2 is fixed and the action is free everywhere else. Further $SE\mathsf{C}_2$ has effective homology and we can construct a contraction to one of the points, say $+1$. We take a countable wedge sum of copies of $SE\mathsf{C}_2$ on the basepoints $+1$. The resulting space $\bigvee_{\mathbb{N}} SE\mathsf{C}_2$ again has effective homology.

On the other hand, the space $\mathsf{colim}_{\mathsf{C}_2} SE\mathsf{C}_2$ has different homology. Factoring by the action of C_2, we get $\mathsf{colim}_{\mathsf{C}_2} SE\mathsf{C}_2 = SB\mathsf{C}_2$ the unreduced suspension of the space $B\mathsf{C}_2$. As $B\mathsf{C}_2 = \mathbb{RP}^\infty$ has nontrivial homology, the space $SB\mathsf{C}_2$ has nontrivial homology, namely for each $n \geq 1$ we have

$$H_{2n-1}(\mathsf{colim}_{\mathsf{C}_2} SE\mathsf{C}_2) \cong \mathbb{Z}/2\mathbb{Z} = \mathsf{C}_2\,.$$

It follows that $H_{2n-1}(\mathsf{colim}_{\mathsf{C}_2} \bigvee_{\mathbb{N}} SE\mathsf{C}_2) = \bigoplus_{\mathbb{N}} H_{2n-1}(\mathsf{colim}_{\mathsf{C}_2} SE\mathsf{C}_2)$ is an infinitely generated group, so it cannot have effective homology.

Perturbation lemmas for diagrams. We now define perturbation and give perturbation lemmas for diagrams of chain complexes:

Definition 2.14. Let C, $C' \colon \mathcal{I} \to \mathsf{Ch}_+$. Notice that the differential ∂ on C can be seen as a natural transformation $C \to C[1]$ satisfying $\partial\partial = 0$. Here $C[1]$ is diagram of chain complexes C with all the chain complexes moved up by one dimension. We call a collection of maps $\delta \colon C \to C[1]$ *perturbation* if the sum $\partial + \delta$ is also a differential.

We now formulate the lemmas.

Lemma 2.15 (Easy Perturbation Lemma). *Let $(\alpha, \beta, \eta) \colon (C, \partial) \Rightarrow (C', \partial')$ be a reduction of diagrams of chain complexes . Suppose δ' is a perturbation of differential ∂'. Then there is a reduction $(\alpha, \beta, \eta) \colon (C, \partial + \beta\delta'\alpha) \Rightarrow (C', \partial' + \delta)$.*

Lemma 2.16 (Basic Perturbation Lemma). *Let $(\alpha, \beta, \eta) \colon (C, \partial) \Rightarrow (C, \partial')$ be a reduction of diagrams of chain complexes. Suppose δ is a perturbation of differential ∂ and further for every $i \in \mathcal{I}$ and every $c \in C(i)$ there is some $k \in \mathbb{N}$ such that we get $(\eta\delta)^k(c) = 0$. Then there is a reduction of diagrams of chain complexes $(\alpha', \beta', \eta') \colon (C, \partial + \delta) \Rightarrow (C', \partial' + \delta')$.*

Both lemmas can be proven in the same way as classical perturbation lemmas. We have concrete description of the new reductions and they are given as sums of compositions of α, β, η, ∂, δ. Therefore it is enough to check the previously described formulas.

Lemma 2.17.

(1) *Let $X, Y \colon \mathcal{I} \to \mathsf{sSet}$ be diagrams of simplicial sets with pointwise effective homology. Then the diagram $(X \times Y) \colon \mathcal{I} \to \mathsf{sSet}$ has pointwise effective homology.*

(2) *Let $X \colon \mathcal{I} \to \mathsf{sSet}$, $Y \colon \mathcal{J} \to \mathsf{sSet}$ be diagrams with effective homology. Then $X \mathbin{\widehat{\times}} Y \colon \mathcal{I} \times \mathcal{J} \to \mathsf{sSet}, X \mathbin{\widehat{\times}} Y(i,j) = X(i) \times Y(j)$, has effective homology.*

(3) *Let $C \colon \mathcal{I} \to \mathsf{Ch}_+$, $C' \colon \mathcal{I} \to \mathsf{Ch}_+$ be diagrams of chain complexes with effective homology. Then $C \oplus C'$ has effective homology.*

Proof. It is enough to show that for each $i \in \mathcal{I}$, $C((X \times Y)(i))$ has effective homology and this is proven by the second part of Lemma 2.2.

In the second part we use the functoriality of Eilenberg–Zilber reduction (see [11, Theorem 2.1a]). The diagram $C(X \times Y)$ is strongly equivalent to diagram $C^{\mathrm{ef}}(X) \otimes C^{\mathrm{ef}}(Y)$ and it remains to show the latter is effective.

Let x_α be the finite basis for $C^{\mathrm{ef}}(X)$ and y_β be the basis of $C^{\mathrm{ef}}(Y)$. It is well-known that the basis of tensor product is formed by tensor products of basis elements, so the basis of $C^{\mathrm{ef}}(X) \otimes C^{\mathrm{ef}}(Y)$ is generated by the set

$$\{f_* x_\alpha \otimes g_* y_\beta \mid f \in \mathcal{I}(i_\alpha, i), g \in \mathcal{J}(j_{\beta,i})\} = \\ \{(f,g)_* x_\alpha \otimes y_\beta \mid (f,g) \in \mathcal{I} \mathbin{\widehat{\times}} \mathcal{J}\big((i_\alpha, j_\beta), (i,j)\big)\}\,.$$

The last part is trivial. □

Corollary 2.18. *Let $C \colon \mathcal{I} \to \mathsf{Ch}_+$ be a diagram with effective homology and let C' be a chain complex with effective homology. Then $C' \otimes C \colon \mathcal{I} \to \mathsf{Ch}_+$ has effective homology.*

Proof. We can see C' as a diagram $C' \colon * \to \mathsf{Ch}_+$. The diagram $C' \otimes C$ is strongly equivalent to some ${C'}^{\mathrm{ef}} \otimes C^{\mathrm{ef}} \colon \mathcal{I} \to \mathsf{Ch}_+$. Lemma 2.17 (2) then gives the result. □

In what follows we are going to apply a general lemma about filtered diagrams of chain complexes. Let $C \colon \mathcal{I} \to Ch_+$. We consider a filtration F on diagram C of chain complexes:

$$0 = F_{-1}C \subseteq F_0 C \subseteq F_1 C \subseteq \cdots$$

such that $C = \bigcup_k F_k C$. We further assume that each $F_k C$ is a cellular subcomplex, i.e. it is generated by a subset of the given basis of C and that the filtration is locally finite, i.e. for each n we have $F_k C_n = C_n$ for $k \gg 0$.

Lemma 2.19 ([6], Lemma 7.3)**.** *Let C be a diagram of chain complexes with filtration F satisfying properties as above. If each filtration quotient $G_k C = F_k C / F_{k-1} C$ is a diagram with effective homology then so is C.*

Proof. We define $G = \bigoplus_{k \geq 0} G_k$. The sum is not finite, but it is *locally finite*: By the properties of F, we get that $G_k C_n = 0$ for $k \gg 0$. Thus for each n, we get a direct sum of diagrams with effective homology $G_k C_n \colon \mathcal{I} \to \mathsf{Ch}_+$ and it follows that G has effective homology.

The diagram C differs from G only by a perturbation of its differential. This perturbation decreases the filtration degree. If we take a direct sum of given strong equivalences $G_k \Leftarrow \widehat{G_k} \Rightarrow G_k^{\mathrm{ef}}$, we obtain a strong equivalence $G \Leftarrow \widehat{G} \Rightarrow G^{\mathrm{ef}}$. All the chain complexes are equipped with a filtration degree. Since the perturbation on G decreases the filtration degree, while the homotopy operator preserves it, we can apply the perturbation lemmas 2.15, 2.16 to obtain a strong equivalence $C \Leftarrow \widehat{C} \Rightarrow C^{\mathrm{ef}}$. □

3. Proofs of main results

We remark, that both homotopy colimit $\mathsf{hocolim}\, X$ and cofibrant replacement X^{cof} of a diagram $X \colon \mathcal{I} \to \mathsf{sSet}$ can be seen as cases of *homotopy left Kan extension*

$\mathsf{holan}_p X$ ([17], [21, chapter 8]), where p is a functor $\mathcal{I} \to \mathcal{J}$. We picture the situation in the following diagram:

$$\begin{tikzcd} \mathcal{I} \arrow[rr, "X"] \arrow[dr, "p"'] & & \mathsf{sSet} \\ & \mathcal{J} \arrow[ur, dotted, "\mathsf{holan}_p X"'] & \end{tikzcd}$$

For the homotopy colimit one chooses $\mathcal{J} = *$ and p the unique functor to the terminal category and for the cofibrant replacement we set $\mathcal{J} = \mathcal{I}$ and $p = \mathrm{id}$ [17], [20, chapter 5]. We will make use of the following Bousfield–Kan formula for $\mathsf{holan}_p X$ (see [1, chapter 11], [7, 17]):

(4) $(\mathsf{holan}_p X)(j) =$

$$\coprod_n \coprod_{i_0,\cdots,i_n} \Delta^n \times X(i_0) \times \mathcal{I}(i_0, i_1) \times \cdots \times \mathcal{I}(i_{n-1}, i_n) \times \mathcal{J}(p(i_n), j)/\sim ,$$

here the relation $\sim$ is given as

$$\begin{aligned} &(d^k t, x, f_1, f_2, \dots f_n, g) \sim (t, x, f_1, f_2, \dots, f_{k+1} f_k, \dots, f_{n-1}, f_n, g)\,, && 0 < k < n\,; \\ &(d^k t, x, f_1, f_2, \dots f_n, g) \sim (t, x, f_1, f_2, \dots, f_{n-2}, f_{n-1}, g p(f_n))\,, && k = n\,; \\ &(d^k t, x, f_1, f_2, \dots f_n, g) \sim (t, f_1(x), f_2, \dots, f_{n-1}, f_n, g)\,, && k = 0\,. \end{aligned}$$

where $d^k \colon \Delta^{n-1} \to \Delta^n$ is the inclusion into the k-th face.

For the choices of $\mathcal{J}$, p we get corresponding formulae for homotopy colimit and cofibrant replacement namely

$$\mathsf{hocolim}\, X = \coprod_n \coprod_{i_0,\cdots,i_n} \Delta^n \times X(i_0) \times \mathcal{I}(i_0, i_1) \times \cdots \times \mathcal{I}(i_{n-1}, i_n)/\sim$$

$$X^{\mathsf{cof}}(-) = \coprod_n \coprod_{i_0,\cdots,i_n} \Delta^n \times X(i_0) \times \mathcal{I}(i_0, i_1) \times \cdots \times \mathcal{I}(i_{n-1}, i_n) \times \mathcal{I}(i_n, -)/\sim$$

with the obvious specified relations.

We now formulate a theorem regarding effective homology of the Bousfield–Kan model of $\mathsf{holan}_p X$ and we obtain both Proposition 1.1 and 1.2 as straightforward corollaries.

Theorem 3.1. *Let $X \colon \mathcal{I} \to \mathsf{sSet}$ be a pointwise effective diagram, $p \colon \mathcal{I} \to \mathcal{J}$ a functor between finite categories. Then $\mathsf{holan}_p X \colon \mathcal{J} \to \mathsf{sSet}$ is a diagram with effective homology.*

Proof. For any category $\mathcal{I}$ there is a simplicial set $N\mathcal{I}$, the nerve of $\mathcal{I}$. The simplicial set $N\mathcal{I}$ can be seen as a homotopy colimit of the diagram consisting of points. Then there is a projection $q \colon \mathsf{holan}_p X \to N\mathcal{I}$ given as a projection onto

$$\coprod_n \coprod_{i_0,\cdots,i_n} \Delta^n \times \mathcal{I}(i_0, i_1) \times \cdots \times \mathcal{I}(i_{n-1}, i_n)/\sim$$

and we define the skeleton of $\mathsf{holan}_p X$:

$$\mathsf{sk}_k \mathsf{hoLan}_p X = q^{-1}(sk_k N\mathcal{I})\,.$$

Let $C\colon \mathsf{sSet} \to \mathsf{Ch}_+$ denote the standard chain complex functor. We want to use Lemma 2.19 to prove that the diagram $C(\mathsf{hoLan}_p X)\colon \mathcal{I} \to \mathsf{Ch}_+$ has effective homology. Therefore, we first have to introduce a filtration F on the diagram of chain complexes $C(\mathsf{hoLan}_p X)$. We define F as follows:

$$F_k C(\mathsf{hoLan}_p X) = C(\mathsf{sk}_k \mathsf{hoLan}_p X)\,.$$

Denoting $G_k = F_k/F_{k-1}$, we get

$$G_k\big(C(\mathsf{hoLan}_p X)\big) = \bigoplus_{\substack{i_0 \to \cdots \to i_k \\ \text{nondeg.}}} C\big(\Delta^k \times X(i_0) \times \mathcal{J}\big(p(i_k), -\big), \partial\Delta^k \times X(i_0) \times \mathcal{J}(p(i_k), -)\big)\,.$$

The sum is taken over chains of morphisms in $\mathcal{I}$ that do not contain identity as those are canceled out when computing $G_k = F_k/F_{k-1}$. By the finiteness of $\mathcal{I}$, the number of nondegenerate chains of morphisms of length k is finite, so the sum is finite.

Using the Eilenberg–Zilber reduction we get that $G_k(C(\mathsf{hoLan}_p X))$ is strongly equivalent to

$$\bigoplus_{\substack{i_0 \to \cdots \to i_k \\ \text{nondeg.}}} C(\Delta^k, \partial\Delta^k) \otimes C\big(X(i_0)\big) \otimes \mathbb{Z}\mathcal{J}\big(p(i_k), -\big) \cong \bigoplus_{\substack{i_0 \to \cdots \to i_k \\ \text{nondeg.}}} s^k C\big(X(i_0)\big) \otimes \mathbb{Z}\mathcal{J}\big(p(i_k), -\big)\,, \tag{5}$$

where s denotes the suspension. To finish the proof, it remains to show that under the assumptions of Theorem 3.1 the diagrams G_k have effective homology.

In Example 2.10, we have shown that $\mathbb{Z}\mathcal{J}(p(i_k), -)$ is effective diagram of chain complexes. Therefore it has effective homology. Further $s^k C(X(i_0))$ is a chain complex with effective homology. Using Corollary 2.18 we get that $s^k C(X(i_0)) \otimes \mathbb{Z}\mathcal{J}(p(i_k), -)$ is diagram of chain complexes with effective homology.
As $G_k(C(\mathsf{hoLan}_p X))$ is strongly equivalent to a finite direct sum of chain complexes with effective homology, it has effective homology. Now we can apply Lemma 2.19 to complete the proof. $\square$

4. Application

In paper [4] the authors presented algorithm computing the set of homotopy classes of maps $[X, Y]$[1] where X, Y are simplicial sets satisfying certain conditions. One of the steps in their construction based on building a Postnikov tower for Y was to compute $[X, K(\pi, n)]$ where X is a simplicial set with effective homology and

[1]We remark that all through this section, we are using the standard notation from model categories, so by $[X, Y]$ we in fact mean $[X^{\mathsf{cof}}, Y^{\mathsf{fib}}]$ where Y^{fib} is a fibrant replacement of Y.

$K(\pi, n)$ is a simplicial model for the Eilenberg–MacLane space with *fully effective abelian* group π. In paper [6] this result was extended to computing the homotopy classes of equivariant maps $[X, Y]_G$ where G is a finite group with free actions on X and Y. Their construction also involved computing $[X, K(\pi, n)]_G$ where π is an abelian group with an action of G.

In this section we generalize the computation of $[X, K(\pi, n)]$ for X and $K(\pi, n)$ being diagrams of simplicial sets. As a corollary we will further get an algorithm computing the cohomology operations $[K_G(\pi, n), K_G(\rho, k)]_G$ where G is a finite group that acts on Eilenberg–MacLane G-simplicial sets $K_G(\pi, n), K_G(\rho, k)$. This can be considered as a generalization of some constructions in [10] and [11].

We again assume the projective model structure on the category of functors $\mathcal{I} \to \mathsf{sSet}$. Given a diagram of abelian groups $\pi\colon \mathcal{I} \to \mathsf{Ab}$ and a diagram of simplicial sets $X\colon \mathcal{I} \to \mathsf{sSet}$, we define the cochain complex $C^*(X;\pi) = \mathsf{Hom}(C_*X, \pi)$. A group $H^n(X;\pi)$ the n-th *cohomology group* of X with coefficients in π is then given by the homology of $C^*(X;\pi)$.

Further, π induces an *Eilenberg–MacLane object* $K(\pi, n)$ (see [9]), which is in fact a diagram of Eilenberg–MacLane spaces $K(\pi(i), n)$. If not stated otherwise, by $K(\pi, n)$ we will always mean the following model which is due to [19, Theorem 23.9]:

$$K(\pi, n)_q = Z^n(\Delta^q, \pi)\,.$$

Here $\Delta^q \in \mathsf{sSet}$ is seen as a trivial diagram $\mathcal{I} \to \mathsf{sSet}$ and Z^n denotes the cocycles. The following proposition uses the notion of fully effective abelian group. We give the proper definition later, for now we only remark that a fully effective abelian group A means there exists a computable description of all generators and relations of A.

Proposition 4.1. *Let $\mathcal{I}$ be a finite category and $X\colon \mathcal{I} \to \mathsf{sSet}$ a functor. Suppose that $X(i)$ has effective homology for all $i \in \mathcal{I}$ and that all maps in the diagram X are computable. Let $\pi\colon \mathcal{I} \to \mathsf{Ab}$ be a diagram of fully effective abelian groups. Then there is an algorithm which computes $[X, K(\pi, n)]$.*

We postpone the proof until later and formulate a corollary regarding G-simplicial sets. Let G be a finite abelian group and let X be a simplicial set with a G action. Consider the category sSet^G of G-simplicial sets. This category can be equipped with a model structure, called the *fixed point* model structure, see [12, 24].

Further, there is a category of orbits $\mathcal{O}_G$, where the objects are orbits G/H for $H \leq G$ and the morphisms are equivariant maps $G/H_1 \to G/H_2$. By Elemendorf's theorem [12] the categories sSet^G with fixed point model structure and $\mathsf{sSet}^{\mathcal{O}_G^{\mathsf{op}}}$ with projective model structure are Quillen equivalent. We remark, that the right Quillen fully faithful functor $\Phi\colon \mathsf{sSet}^G \to \mathsf{sSet}^{\mathcal{O}_G^{\mathsf{op}}}$ is defined by

$$\Phi(X)(G/H) = X^H = \{x \in X \mid hx = x, \forall h \in H\}\,.$$

Using this functor, we define the Bredon cohomology $H^*_G(X;\pi)$ of some $X \in \mathsf{sSet}^G$ with coefficients in a diagram $\rho\colon \mathcal{O}_G^{\mathsf{op}} \to \mathsf{Ab}$ as $H^*_G(X;\rho) = H^*(\Phi X;\rho)$.

Using the left Quillen functor $\Psi\colon \mathsf{sSet}^{\mathcal{O}_G^{\mathsf{op}}} \to \mathsf{sSet}^G$ given by $\Psi(T) = (T^{\mathsf{cof}})(G/e)$, we define the Eilenberg–MacLane G-simplicial set $K_G(\rho, n) = \Psi K(\rho, n)$. Now we can state:

Corollary 4.2. *Let G be a finite group and $\pi, \rho\colon \mathcal{O}_G^{\mathsf{op}} \to Ab$ diagrams of fully effective abelian groups. Then there is an algorithm computing $[K_G(\pi, n), K_G(\rho, k)]_G$.*

Proof. Using the Elmendorf's theorem [12, 20, 24], we get that

$$[K_G(\pi, n), K_G(\rho, k)] \cong [K(\pi, n), K(\rho, k)]\,,$$

where $K(\pi, n)$ is a diagram with pointwise effective homology [5, 18]. The proof is finished using Proposition 4.1. Notice that we have computed $H_G^k(K_G^{\mathsf{cof}}(\pi, n); \rho)$. □

We now sum up the necessary information about fully effective abelian groups. For more detailed information we refer the reader to [4].

Fully effective abelian groups. A *fully effective abelian* group A consists of

- a set of representatives $\mathcal{A}$ which we imagine are stored in a computer. The element represented by an $\alpha \in \mathcal{A}$ is denoted $[\alpha]$,
- algorithms that provide us with a representative for neutral element, product and inverse. In more detail we can compute $0 \in \mathcal{A}$ such that $[0] = e$, given any $\alpha, \beta \in \mathcal{A}$ we compute $\gamma \in \mathcal{A}$ such that $[\gamma] = [\alpha] + [\beta]$ and for any $\alpha \in \mathcal{A}$ we can compute $\beta \in \mathcal{A}$ such that $[\beta] = -[\alpha]$,
- a list of generators $a_1, \ldots, a_r$ (given by the representatives) and numbers $q_1, \ldots, q_r \in \{2, 3, \ldots\} \cup \{0\}$, where a_i is seen as a generator of $\mathbb{Z}/q_i$ and $\mathbb{Z}/0 \cong \mathbb{Z}$,
- an algorithm that given $\alpha \in \mathcal{A}$ computes integers $z_1, \ldots, z_r$ such that $[\alpha] = \sum_{i=1}^r z_i a_i$.

We call a mapping $f\colon A \to B$ of fully effective abelian groups *computable homomorphism* if there is a computable mapping of sets $\phi\colon \mathcal{A} \to \mathcal{B}$ such that $f([\alpha] = [\phi(\alpha)])$.

In other words, the above mentioned structure of fully effective abelian group A gives us a computable isomorphism (together with its inverse) $A \cong \mathbb{Z}/q_1 \oplus \cdots \oplus \mathbb{Z}/q_r$.

The proof of the following lemma can be found in [4].

Lemma 4.3. *Let $f\colon A \to B$ be a computable homomorphism of fully effective abelian groups. Then both* $\operatorname{Ker}(f)$ *and* $\operatorname{coker}(f)$ *can be represented as fully effective abelian groups.*

Let $\mathcal{I}$ be a finite category and let $\pi\colon \mathcal{I} \to \mathsf{Ab}$ be a diagram such that every $\pi(i)$ is fully effective abelian and every morphism is computable homomorphism. We will call such diagram a *diagram of fully effective abelian groups.* As a consequence of the previous lemma, we get

Lemma 4.4. *Let $\mathcal{I}$ be a finite category and let $\pi, \rho\colon \mathcal{I} \to \mathsf{Ab}$ be a diagrams of fully effective abelian groups. Then* $\mathsf{Hom}(\pi, \rho)$ *is a fully effective abelian group.*

Proof. First notice that each $\mathsf{Hom}(\pi(i),\rho(i'))$ is a fully effective abelian group. Secondly, we can see that $\mathsf{Hom}(\pi,\rho) \le \prod_{i\in\mathcal{I}} \mathsf{Hom}(\pi(i),\rho(i))$. We define a homomorphism

$$F\colon \prod_{i\in\mathcal{I}} \mathsf{Hom}(\pi(i),\rho(i)) \to \prod_{f\colon i\to i'} \mathsf{Hom}(\pi(i),\rho(i'))$$

for any $g \in \prod_{i\in\mathcal{I}} \mathsf{Hom}(\pi(i),\rho(i))$ as follows:

$$F(g) = (\rho(f)g(i) - g(i')\pi(f))_{f\colon i\to i'}\,.$$

Then the desired $\mathsf{Hom}(\pi,\rho)$ is equal to $\operatorname{Ker} F$. As $\mathcal{I}$ is a finite category, $\prod_{i\in\mathcal{I}} \mathsf{Hom}(\pi(i),\rho(i))$, $\prod_{f\colon i\to i'} \mathsf{Hom}(\pi(i),\rho(i'))$ are both fully effective abelian groups. Because F is computable, Lemma 4.3 gives us that $\operatorname{Ker} F$ is fully effective. $\square$

Proof of Proposition 4.1. We remark that $K(\pi,n)$ is already a fibrant object, and by Proposition 1.2, X^{cof} has effective homology. We must now compute $[X^{\mathsf{cof}}, K(\pi,n)]$ Thanks to the model we are using, we can use the constructions in [19], Section 24, as presented in [14], p. 345, and we get an isomorphism $H^n(X^{\mathsf{cof}};\pi) \cong [X^{\mathsf{cof}}, K(\pi,n)]$. By Proposition 1.2, the chain complex C_*X^{cof} has effective homology and this in particular implies that each C_nX^{cof} is a diagram of fully effective abelian groups. By Lemma 4.4 each $\mathsf{Hom}(C_kX^{\mathsf{cof}},\pi)$ is a fully effective abelian group and using Lemma 4.3, we get that $H^n\mathsf{Hom}(C_*X^{\mathsf{cof}},\pi)$ is an effective abelian group, which concludes the proof. $\square$

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