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HEAPS AND UNPOINTED STABLE HOMOTOPY THEORY

Lukáš Vokřínek

Abstract. In this paper, we show how certain "stability phenomena" in *unpointed* model categories provide the sets of homotopy classes with a canonical structure of an abelian heap, i.e. an abelian group without a choice of a zero. In contrast with the classical situation of stable (pointed) model categories, these sets can be empty.

1. Introduction

In stable homotopy theory, the set of homotopy classes of maps admits a structure of an abelian group. In this paper, we study the corresponding situation for "unpointed stable homotopy theory". It turns out to be quite similar with one exception: the zero of the abelian group structure is non-canonical, i.e. the homotopy classes form an abelian heap. However, instead of defining and dealing with unpointed spectra, we study the stability directly in terms of spaces through an unpointed version of the Freudenthal suspension theorem. This theorem may seem somewhat obvious for spaces themselves but turns out not to be all that trivial for spaces equipped with a further structure, such as an action of a group G or a map to a fixed base space B, or for objects of a general model category.

This paper grew out of an attempt to understand the appearance of non-canonical abelian group structures on sets of equivariant fibrewise homotopy classes of maps under certain stability restrictions (dimension vs. connectivity as in the Freudenthal theorem), utilized in [3] for the algorithmic computation of these sets. A complementary result [2] shows that unstably, even the existence of a map is undecidable and thus, the abelian group structures are essential for algorithmic computations.

In [9], it was realized that the non-canonical abelian group structure could be replaced by a canonical abelian heap structure that, in fact, comes from an "up to homotopy" abelian heap structure on the stable part of every fibrewise space. This structure was constructed from the Moore–Postnikov tower that is rather specific to spaces. The present paper describes a more conceptual approach, phrased in terms of model categories.

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The non-canonicality of the group structure comes from the absence of basepoints – classically, the constant map onto the basepoint serves as the zero element of this group, while in [3], basepoints do not exist in general, e.g. for spaces equipped with a free action of a fixed group G or spaces over a fixed base space B that do not admit any section. In such situations, there is no canonical choice of a zero element and the best that one could hope for is a structure of an abelian heap that, in addition, could also be empty.[1]

We are ready to state the main result of this paper. It uses the notions of a d-connected object and an n-dimensional object that will only be explained later, but which in many cases, most notably those mentioned above, have a straightforward interpretation. Heaps are defined formally after the statement of the theorem – they are essentially groups without a choice of a zero.

Theorem 1.1. *Let $\mathcal{M}$ be a simplicial model category. Then the set $[X, Y]$ of homotopy classes of maps from an n-dimensional cofibrant object X to a d-connected fibrant object Y with $n \leq 2d$ admits a canonical structure of a (possibly empty) abelian heap.*

When $[X, Y]$ is non-empty, it is possible to make $[X, Y]$ into an abelian group: this amounts to picking a zero element $0 \in [X, Y]$. However, such a choice is non-canonical and thus, there is in fact a whole *family* of abelian group structures on $[X, Y]$, one for each homotopy class of maps $X \to Y$. On the other hand, the abelian heap structure is completely canonical.

In the last section of the paper, we outline a construction of a category of finite spectra in the spirit of Spanier and Whitehead.

Heaps. Here we define heaps, discuss their relationship to groups and give an Eckman–Hilton argument; this covers all that is needed in the paper. For further information, we suggest either [7] or [1]. A *Mal'cev operation* on a set S is a ternary operation

$$t \colon S \times S \times S \to S$$

satisfying the following two Mal'cev conditions:

$$t(x, x, y) = y, \qquad t(x, y, y) = x.$$

It is said to be

- *asssociative* if $t(x, r, t(y, s, z)) = t(t(x, r, y), s, z)$;
- *commutative* if $t(x, r, y) = t(y, r, x)$.

A set equipped with an associative Mal'cev operation is called a *heap*. It is said to be an *abelian heap* if in addition, the operation is commutative. We remark that traditionally, heaps are assumed to be non-empty. Since it is possible to have $[X, Y] = \emptyset$ in Theorem 1.1, it will be more convenient to drop this convention.

[1]The set of G-equivariant homotopy classes of maps $* \to Y$ is empty when Y has no fixed-point, e.g. when $Y = EG$. The set of homotopy classes over B of maps $B \to Y$ is empty when Y admits no section.

The relation of heaps and groups works as follows. Every group becomes a heap if the Mal'cev operation is defined as $t(x,r,y) = x - r + y$. On the other hand, by fixing an element $0 \in S$ of a heap S, we may define the addition and the inverse

$$x + y = t(x, 0, y), \quad -x = t(0, x, 0)\,.$$

It is simple to verify that this makes S into a group with neutral element 0. In both passages, commutativity of heaps corresponds exactly to the commutativity of groups.

Finally, we will need an *Eckman–Hilton argument* for heaps. If t_0, t_1 are two heap operations on the same set S such that t_1 is a heap homomorphism with respect to t_0 (the structures distribute over each other) then

$$\begin{aligned} t_1(x,r,y) &= t_1(t_0(x,r,r), t_0(r,r,r), t_0(r,r,y)) \\ &= t_0(t_1(x,r,r), t_1(r,r,r), t_1(r,r,y)) = t_0(x,r,y) \end{aligned}$$

and also

$$\begin{aligned} t_1(x,r,y) &= t_1(t_0(r,r,x), t_0(r,r,r), t_0(y,r,r)) \\ &= t_0(t_1(r,r,y), t_1(r,r,r), t_1(x,r,r)) = t_0(y,r,x). \end{aligned}$$

That is, the two structures are equal and commutative.

2. Suspensions and loop spaces in unpointed model categories

The proof of Theorem 1.1 follows from abstract versions of theorems of Freudenthal and Whitehead. For the Freudenthal theorem, we need to introduce suspsensions and loop spaces or rather their unpointed versions. To get some intuition, we will describe these constructions for unpointed spaces. Here the suspension is the usual unreduced suspension, thought of, however, as a space equipped with *two basepoints*. The loop space of a space equipped with two basepoints is the space of paths from the first basepoint to the second. In this paper, loop spaces will be dealt with in this manner. We will now proceed with formal definitions.

We work in a simplicial model category $\mathcal{M}$ with the enriched hom-set denoted by $\mathsf{map}(X,Y)$, tensor by $K \otimes X$ and cotensor by Y^K. Examples that we have in mind are G-spaces over a fixed G-space B, or diagrams of such, see Section 4. We denote by I the simplicial set $\bullet\!\longleftarrow\!\bullet\!\longrightarrow\!\bullet$ formed by two standard 1-simplices glued along their initial vertices, and by ∂I its obvious "boundary" composed of the two terminal vertices. Further, we denote by $\mathbb{I}^0$ the cofibrant fibrant replacement of the terminal object. The standard references for simplicial sets, model categories and homotopy colimits are [4, 5]. We use the standard Bousfield–Kan simplicial models for homotopy (co)limits.

We will now define a Quillen adjunction $\Sigma \dashv \Omega$ composed of the *suspension* and *loop space* functors[2]

$$\Sigma\colon \mathcal{M}/\mathbb{I}^0 \rightleftarrows \mathbb{I}^0 \sqcup \mathbb{I}^0/\mathcal{M} : \Omega$$

[2]One may also use $\mathcal{M}/\mathbb{I}^0 \times \mathbb{I}^0$ instead of $\mathcal{M}/\mathbb{I}^0$ (later replacing equalizer–cokernel pair by pullback–pushout), thus producing a more symmetric adjunction. However, the non-symmetric version is easier to generalize to higher suspensions – these will be needed later.

where $\mathcal{M}/\mathbb{I}^0$ is the slice category of objects over $\mathbb{I}^0$ while $\mathbb{I}^0 \sqcup \mathbb{I}^0/\mathcal{M}$ is the slice category of objects under $\mathbb{I}^0 \sqcup \mathbb{I}^0$. The functor Σ is defined on $p\colon X \to \mathbb{I}^0$ as a homotopy pushout

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "p"'] & \mathbb{I}^0 \arrow[d, "j"] \\
\mathbb{I}^0 \arrow[r, "i"'] \arrow[ur, phantom, "\ulcorner h"] & \Sigma X
\end{tikzcd}$$

with the two components i, j of the universal cone making ΣX into an object under $\mathbb{I}^0 \sqcup \mathbb{I}^0$. In other words, the maps i, j form the homotopy cokernel pair of p. The loop space functor is defined on $[i,j]\colon \mathbb{I}^0 \sqcup \mathbb{I}^0 \longrightarrow Y$ as the homotopy equalizer of i and j. The Bousfield–Kan models for homotopy (co)limits translate these into the following pushout/pullback squares:

$$\begin{tikzcd}
\partial I \otimes X \arrow[r, "\mathrm{id} \otimes p"] \arrow[d, "\mathrm{incl}"'] & \partial I \otimes \mathbb{I}^0 \cong \mathbb{I}^0 \sqcup \mathbb{I}^0 \arrow[d, "{[i,j]}"] \\
I \otimes X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & Y^I \arrow[d, "\mathrm{res}"] \\
\mathbb{I}^0 \arrow[r, "{(i,j)}"'] & Y^{\partial I}
\end{tikzcd}$$

From this restatement, it follows rather easily that $\Sigma \dashv \Omega$ is a Quillen adjunction.

Since the unique map $\mathbb{I}^0 \to 1$ to the terminal object is a weak equivalence between fibrant objects, it is easy to see that there is a Quillen equivalence $\mathcal{M}/\mathbb{I}^0 \simeq_Q \mathcal{M}$. Thus, one may think of Σ as being defined on $\mathcal{M}$ while Ω is defined on objects equipped with a "pair of basepoints" (and ΩY is then the space of paths from the first basepoint to the second).

For a cofibrant object Y, we consider the *derived unit* $\eta_Y\colon Y \to \Omega(\Sigma Y)^{\mathrm{fib}}$, where the superscript "fib" denotes the fibrant replacement of ΣY. To state an abstract version of a Freudenthal suspension theorem, we need a notion of a d-equivalence.

Abstract theorems of Freudenthal and Whitehead. We say that a cofibrant object $D \in \mathcal{M}$ is *excisive* if the right derived functor of $\mathsf{map}(D, -)$ preserves homotopy pushouts in the following sense: when $Y\colon \mathcal{S} \to \mathcal{M}$ is a diagram consisting of fibrant objects, indexed by the span category $\mathcal{S} = \bullet \longleftarrow \bullet \longrightarrow \bullet$, then the composition

$$\operatorname{hocolim}_{\mathcal{S}} \mathsf{map}(D, Y-) \longrightarrow \mathsf{map}(D, \operatorname{hocolim}_{\mathcal{S}} Y) \longrightarrow \mathsf{map}(D, (\operatorname{hocolim}_{\mathcal{S}} Y)^{\mathrm{fib}})$$

is a weak equivalence.

Let us fix a collection $\mathcal{D} \subseteq \mathcal{M}^{\mathrm{cof}}$ of cofibrant excisive objects.

- We say that Y is *d-connected* if for each $D \in \mathcal{D}$, $\mathsf{map}(D, Y^{\mathrm{fib}})$ is d-connected.
- We say that a map $f\colon Y \to Z$ is a *d-equivalence* if for each $D \in \mathcal{D}$, the map $f_*\colon \mathsf{map}(D, Y^{\mathrm{fib}}) \to \mathsf{map}(D, Z^{\mathrm{fib}})$ is a d-equivalence of simplicial sets.

Theorem 2.1 (generalized Freudenthal)**.** *Let Y be a d-connected cofibrant object. Then the canonical map $\eta_Y\colon Y \to \Omega(\Sigma Y)^{\mathrm{fib}}$ is a $(2d+1)$-equivalence.*

Let $\mathcal{I}_n$ denote the following collection of maps:

$$\mathcal{I}_n = \{\partial\Delta^k \otimes D \to \Delta^k \otimes D \mid k \leq n,\, D \in \mathcal{D}\}\,.$$

We say that X has *dimension* at most n if the unique map $0 \to X$ from the initial object is an $\mathcal{I}_n$-cell complex (i.e. it is obtained from $\mathcal{I}_n$ by pushouts and transfinite compositions); we write $\dim X \le n$. More generally, if $A \to X$ is an $\mathcal{I}_n$-cell complex, we write $\dim_A X \le n$.

Remark. It is also possible to add to $\mathcal{I}_n$ all trivial cofibrations – this does not change the homotopy theoretic nature of $\mathcal{I}_n$-cell complexes.

Theorem 2.2 (generalized Whitehead)**.** *Let X, Y, Z be objects of $\mathcal{M}$ and $f\colon Y \to Z$ a d-equivalence. If $\dim X \le d$, then the induced map*

$$f_*\colon [X, Y] \to [X, Z]$$

is surjective. If $\dim X < d$, the induced map is a bijection.

The usefullness of the above theorems is limited by the existence of a class $\mathcal{D}$ of excisive objects for which the resulting notions of connectivity and dimension are interesting. Examples of such classes are provided in Section 4. We continue with the proof of Theorem 1.1 assuming Theorems 2.1 and 2.2 – these are proved in Section 3.

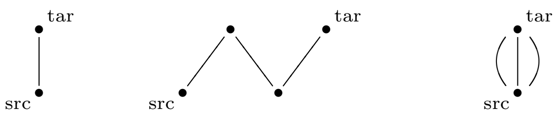
Proof of Theorem 1.1. We will denote by $[\ ,\ ]^A$ the set of homotopy classes in $A/\mathcal{M}$, i.e. homotopy classes of maps under A, and by $[\ ,\ ]_B$ the set of homotopy classes in $\mathcal{M}/B$, i.e. homotopy classes of maps over B.

It follows from the Quillen adjunction $\Sigma \dashv \Omega$, Theorems 2.1 and 2.2 that for $\dim X \le 2 \operatorname{conn} Y$, we have

$$[X, Y] \cong [X, Y]_{I\!I^0} \cong [X, \Omega(\Sigma Y)^{\mathrm{fib}}]_{I\!I^0} \cong [\Sigma X, \Sigma Y]^{\partial I\!I},$$

where we denote $\partial I\!I = \partial I \otimes I\!I^0 \cong I\!I^0 \sqcup I\!I^0$. It is rather straightforward to equip $\Sigma X \in \partial I\!I/\mathcal{M}$ with a "weak co-Malcev cooperation" – this comes from such a structure on $I \in \partial I/\mathsf{sSet}$ given by the zig-zag

$$I \xleftarrow{\ \sim\ } \widetilde{I} \longrightarrow I \sqcup_{\partial I} I \sqcup_{\partial I} I \tag{1}$$



(both maps take the copies of I in $\widetilde{I}$ onto the corresponding copies of I in the target; for the second map, they are the left, the middle and the right copy). Tensor-multiplying by X and collapsing the source and target copies of X to $I\!I^0$'s, one gets

$$\Sigma X \xleftarrow{\ \sim\ } \widetilde{\Sigma} X \longrightarrow \Sigma X \sqcup_{\partial I\!I} \Sigma X \sqcup_{\partial I\!I} \Sigma X .$$

On homotopy classes, it induces the map t in the following diagram.

$$\begin{tikzcd}
{[\Sigma X, \Sigma Y]^{\partial I\!I} \times [\Sigma X, \Sigma Y]^{\partial I\!I} \times [\Sigma X, \Sigma Y]^{\partial I\!I}} \arrow[d, "\cong"] \arrow[rrd, bend left, "t"] & & \\
{[\Sigma X \sqcup_{\partial I\!I} \Sigma X \sqcup_{\partial I\!I} \Sigma X, \Sigma Y]^{\partial I\!I}} \arrow[r] & {[\widetilde{\Sigma} X, \Sigma Y]^{\partial I\!I}} & {[\Sigma X, \Sigma Y]^{\partial I\!I}} \arrow[l, "\cong"']
\end{tikzcd}$$

To prove the Mal'cev conditions, consider the homotopy commutative diagram

$$\begin{tikzcd}
\widetilde{\Sigma} X \arrow[r] \arrow[d, "\sim"'] \arrow[dr, phantom, "\text{htpy}"] & \Sigma X \sqcup_{\partial \mathbb{I}} \Sigma X \sqcup_{\partial \mathbb{I}} \Sigma X \arrow[d, "{[\mathrm{in}_{\mathrm{left}}, \mathrm{in}_{\mathrm{left}}, \mathrm{in}_{\mathrm{right}}]}"] \\
\Sigma X \arrow[r, "\mathrm{in}_{\mathrm{right}}"'] & \Sigma X \sqcup_{\partial \mathbb{I}} \Sigma X
\end{tikzcd}$$

with the map on the right restricting to the indicated maps on the three copies of ΣX in the domain – they are the inclusions of ΣX as the left or right copy in $\Sigma X \sqcup_{\partial \mathbb{I}} \Sigma X$. Easily, this yields in $[\Sigma X, \Sigma Y]^{\partial \mathbb{I}}$ the identity $t(x, x, y) = y$ and a symmetric diagram gives $t(x, y, y) = x$. Thus, t is a Mal'cev operation. The associativity is equally simple to verify.

Higher suspensions and commutativity. In order to get commutativity, we introduce higher suspensions. Let ∂I^k be the obvious boundary of $I^k = I \times \cdots \times I$ and denote $\partial \mathbb{I}^k = \partial I^k \otimes \mathbb{I}^0$ and $\mathbb{I}^k = I^k \otimes \mathbb{I}^0$. We assume that $\mathcal{M}$ is right proper (otherwise, one would have to fibrantly replace $\mathbb{I}^k$ and make the implied adjustments in the constructions below). The higher suspensions are naturally defined on the category $\partial \mathbb{I}^k / \mathcal{M} / \mathbb{I}^k$ of composable pairs of maps $\partial \mathbb{I}^k \xrightarrow{i} X \xrightarrow{p} \mathbb{I}^k$, whose composition is the canonical inclusion. Then $\Sigma^\ell X$ is the pushout in

$$\begin{tikzcd}
\partial I^\ell \otimes X \arrow[r, "\mathrm{id} \otimes p"] \arrow[d, "\mathrm{in} \otimes \mathrm{id}"'] & \partial I^\ell \otimes \mathbb{I}^k \arrow[d] \arrow[dr, bend left, "\mathrm{in} \otimes \mathrm{id}"] & \\
I^\ell \otimes X \arrow[r] \arrow[rr, bend right, "\mathrm{id} \otimes p"'] & \Sigma^\ell X \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed] & I^\ell \otimes \mathbb{I}^k
\end{tikzcd}$$

This makes $\Sigma^\ell X$ into an object over $\mathbb{I}^{\ell+k}$. The map i then induces

$$\partial \mathbb{I}^{\ell+k} = \Sigma^\ell \partial \mathbb{I}^k \to \Sigma^\ell X,$$

making Σ^ℓ into a functor $\Sigma^\ell \colon \partial \mathbb{I}^k / \mathcal{M} / \mathbb{I}^k \to \partial \mathbb{I}^{k+\ell} / \mathcal{M} / \mathbb{I}^{k+\ell}$. As such, Σ^ℓ is a left Quillen functor. Moreover, it is clear that $\Sigma^{\ell_0} \Sigma^{\ell_1} \cong \Sigma^{\ell_0 + \ell_1}$.

The right properness of $\mathcal{M}$ implies $\partial \mathbb{I}^k / \mathcal{M} / \mathbb{I}^k \simeq_Q \partial \mathbb{I}^k / \mathcal{M}$ and we may think of the suspensions as defined on $\partial \mathbb{I}^k / \mathcal{M}$. By an obvious generalization of Theorem 2.1, we obtain for $\dim X \leq 2 \operatorname{conn} Y$ bijections

$$[X, Y] \cong [\Sigma X, \Sigma Y]^{\partial \mathbb{I}} \cong [\Sigma^2 X, \Sigma^2 Y]^{\partial \mathbb{I}^2}.$$

"Squaring" (1) yields the following diagram

$$\begin{tikzcd}
\partial I^2 \arrow[d] & \partial \widetilde{I}^2 \arrow[l, "h", "\sim"'] \arrow[r, "h", "\sim"'] \arrow[d] & \partial I^2 \arrow[d] \\
I^2 & \widetilde{I}^2 \arrow[l, "\sim"] \arrow[r, shift left] \arrow[r, shift right] & I^2 \sqcup_{\partial I^2} I^2 \sqcup_{\partial I^2} I^2
\end{tikzcd}$$

with the two parallel arrows denoting two possible ways of folding a square into three squares – horizontally and vertically. Thus, the diagram takes place in $\partial \widetilde{I}^2 / \mathsf{sSet}$.

Denoting $\partial\widetilde{\mathbb{I}}^2 = \partial\widetilde{I}^2 \otimes \mathbb{I}^0$, we obtain two heap structures on $[\Sigma^2 X, \Sigma^2 Y]^{\partial\widetilde{\mathbb{I}}^2}$ that distribute over each other. Since the Eckman–Hilton argument holds for heaps, these structures are identical and commutative. Because h is a weak equivalence, the canonical map

$$[\Sigma^2 X, \Sigma^2 Y]^{\partial\mathbb{I}^2} \xrightarrow{\cong} [\Sigma^2 X, \Sigma^2 Y]^{\partial\widetilde{\mathbb{I}}^2}$$

is a bijection and it may be used to transport the abelian heap structure to $[\Sigma^2 X, \Sigma^2 Y]^{\partial\mathbb{I}^2}$. □

3. Proofs of the generalized Freudenthal theorem and the generalized Whitehead theorem

Proof of Theorem 2.1. The following diagram commutes

$$\begin{tikzcd} & \Omega(\Sigma\,\mathsf{map}(D,Y))^{\mathrm{fib}} \arrow[d, "\sim"] \\ \mathsf{map}(D,Y) \arrow[ur, bend left, "\eta_{\mathsf{map}(D,Y)}"] \arrow[r, bend right, "\eta_{Y*}"'] & \mathsf{map}(D, \Omega(\Sigma Y)^{\mathrm{fib}}) \end{tikzcd}$$

and the vertical map is a weak equivalence since $\mathsf{map}(D,-)$ commutes with homotopy limits such as Ω in general and it commutes with the homotopy pushout Σ by our assumption of D being excisive. The map $\eta_{\mathsf{map}(D,Y)}$ is a $(2d+1)$-equivalence since the Freudenthal suspension theorem holds in simplicial sets and $\mathsf{map}(D,Y)$ is d-connected. □

Proof of Theorem 2.2. Denoting $\iota\colon \partial\Delta^k \otimes D \to \Delta^k \otimes D$, we will first show that the square

$$\begin{tikzcd} \mathsf{map}(\Delta^k \otimes D, Y) \arrow[r, "f_*"] \arrow[d, two heads, "\iota^*"'] & \mathsf{map}(\Delta^k \otimes D, Z) \arrow[d, two heads, "\iota^*"] \\ \mathsf{map}(\partial\Delta^k \otimes D, Y) \arrow[r, "f_*"'] & \mathsf{map}(\partial\Delta^k \otimes D, Z) \end{tikzcd} \tag{2}$$

is $(d-k)$-cartesian, i.e. that the map from the top left corner to the homotopy pullback is a $(d-k)$-equivalence. Equivalently, the induced map of the homotopy fibres of the two vertical maps is a $(d-k)$-equivalence for all possible choices of basepoints.

The square (2) is isomorphic to

$$\begin{tikzcd} \mathsf{map}(D,Y)^{\Delta^k} \arrow[r, "f_*"] \arrow[d, two heads, "\iota^*"'] & \mathsf{map}(D,Z)^{\Delta^k} \arrow[d, two heads, "\iota^*"] \\ \mathsf{map}(D,Y)^{\partial\Delta^k} \arrow[r, "f_*"'] & \mathsf{map}(D,Z)^{\partial\Delta^k} \end{tikzcd}$$

This square maps to the square on the left of the following diagram via evaluation at any vertex of Δ^k in such a way that the corresponding homotopy fibres over

$\varphi\colon D \to Y$ and $\psi = f\varphi\colon D \to Z$ are organized in the right square:

$$\begin{tikzcd} \mathsf{map}(D,Y) \arrow[r, "f_*"] \arrow[d, "\mathrm{id}"'] & \mathsf{map}(D,Z) \arrow[d, "\mathrm{id}"] \\ \mathsf{map}(D,Y) \arrow[r, "f_*"'] & \mathsf{map}(D,Z) \end{tikzcd} \qquad \begin{tikzcd} \text{contractible} \arrow[r] \arrow[d] & \text{contractible} \arrow[d] \\ \Omega^{k-1}_\varphi \mathsf{map}(D,Y) \arrow[r, "f_*"'] & \Omega^{k-1}_\psi \mathsf{map}(D,Z) \end{tikzcd}$$

(the loop spaces are the usual loop spaces based at the indicated points). The square on the left is ∞-cartesian and in the one on the right, the map of the homotopy fibres of the vertical maps is $f_*\colon \Omega^k_\varphi \mathsf{map}(D,Y) \to \Omega^k_\psi \mathsf{map}(D,Z)$ which is indeed a $(d-k)$-equivalence.

It follows easily from the properties of $(d-n)$-cartesian squares that for all $\mathcal{I}_n$-cell complexes $\iota\colon A \to X$, the square

$$\begin{tikzcd} \mathsf{map}(X,Y) \arrow[r, "f_*"] \arrow[d, two heads, "\iota^*"'] & \mathsf{map}(X,Z) \arrow[d, two heads, "\iota^*"] \\ \mathsf{map}(A,Y) \arrow[r, "f_*"'] & \mathsf{map}(A,Z) \end{tikzcd}$$

is also $(d-n)$-cartesian. In particular, when $n \leq d$ and $A = 0$, we obtain a surjection on the components of the spaces at the top, i.e. $f_*\colon [X,Y] \to [X,Z]$ is surjective. For $n < d$, it is a bijection (and the induced map on π_1 is still surjective). $\square$

4. Examples

We will now show how to produce examples of collections of excisive objects.

a) **Spaces.** In the category of simplicial sets, $\mathcal{D} = \{\Delta^0\}$, i.e. the collection containing only the standard 0-simplex (the one point space), consists of excisive objects. The resulting notions of d-equivalences and n-dimensional objects are the standard ones.

In the following examples, we assume that $\mathcal{D} \subseteq \mathcal{M}$ is a collection of excisive objects in a right proper model category $\mathcal{M}$.

b) **Diagram categories.** Let $\mathcal{C}$ be a small (simplicial) category. When $\mathcal{M}$ is cofibrantly generated, then the diagram category $\mathcal{M}^{\mathcal{C}}$, i.e. the category of (simplicial) functors $\mathcal{C} \to \mathcal{M}$, admits a projective model structure. The collection

$$\mathcal{D}' \stackrel{\mathrm{def}}{=} \{\mathcal{C}(c,-) \otimes D \mid c \in \mathcal{C},\ D \in \mathcal{D}\}$$

also consists of excisive objects – this follows from the Yoneda lemma

$$\mathsf{map}(\mathcal{C}(c,-) \otimes D, Y) \cong \mathsf{map}(D, Yc)$$

and the fact that homotopy colimits in $\mathcal{M}^{\mathcal{C}}$ are computed pointwise.

In this way, a map $p\colon Y \to Z$ in $\mathcal{M}^{\mathcal{C}}$ is a d-equivalence if and only if each component $p_c\colon Yc \to Zc$ is a d-equivalence.

c) **Equivariant categories.** Let G be a group and $\mathcal{M} = \mathsf{sSet}$ the model category of simplicial sets or, more generally, a model category satisfying the conditions of [8]. These conditions provide a model structure on the category $\mathcal{M}^G$ of objects of $\mathcal{M}$ equipped with a G-action and a Quillen equivalence $\mathcal{M}^{\mathcal{O}_G^{\mathrm{op}}} \simeq_Q \mathcal{M}^G$ so that the equivariant case reduces to that of the diagram categories. The resulting collection of excisive objects is

$$\mathcal{D}' \overset{\mathrm{def}}{=} \{G/H \otimes D \mid H \leq G,\, D \in \mathcal{D}\}.$$

In this way, a map $p\colon Y \to Z$ in $\mathcal{M}^G$ is a d-equivalence if and only if all fixed point maps $p^H\colon Y^H \to Z^H$ are d-equivalences. In sSet^G, dimension has the usual meaning.

d) **Fibrewise categories.** Let $B \in \mathcal{M}$ be an object. Then in the category $\mathcal{M}/B$, the collection

$$\mathcal{D}' \overset{\mathrm{def}}{=} \{f\colon D \to B \mid D \in \mathcal{D},\, f \text{ arbitrary}\}$$

also consists of excisive objects. This follows from the fact that for a homotopy pushout square Y of fibrant objects over B, the resulting homotopy pushout square $\mathsf{map}(D, Y-)$ maps to the constant square $\mathsf{map}(D, B)$ and the corresponding square of fibres $\mathsf{map}_B(D, Y-)$, i.e. the square of mapping spaces in $\mathcal{M}/B$, is a homotopy pushout by Mather's cube theorem, see [6].

In this way, a map $p\colon Y \to Z$ in $\mathcal{M}/B$ is a d-equivalence if and only if it is a d-equivalence in $\mathcal{M}$. Also, $\dim X \leq n$ in $\mathcal{M}/B$ if and only if the same is true in $\mathcal{M}$.

e) **Relative categories.** Let $A \in \mathcal{M}$ be an object. Then in the category $A/\mathcal{M}$, the collection

$$\mathcal{D}' \overset{\mathrm{def}}{=} \{\mathrm{in}\colon A \to A \sqcup D \mid D \in \mathcal{D}\}$$

of coproduct injections also consists of excisive objects. This follows from the fact that $\mathsf{map}^A(A \sqcup D, Y) \cong \mathsf{map}(D, Y)$ and homotopy pushouts in $A/\mathcal{M}$ are computed essentially as in $\mathcal{M}$ (more precisely, the homotopy pushout of $X_1 \leftarrow X_0 \rightarrow X_2$ in $A/\mathcal{M}$ is obtained from that in $\mathcal{M}$ by collapsing the copy of $I \otimes A$ to A; the identification map is a weak equivalence).

In this way, a map $p\colon Y \to Z$ in $A/\mathcal{M}$ is a d-equivalence if and only if it is a d-equivalence in $\mathcal{M}$. The dimension of an object $X \in A/\mathcal{M}$ equals $\dim_A X$.

5. The Spanier–Whitehead category of spectra

With the unpointed suspension and loop space as a tool, we will outline a construction of a category $\mathsf{Sp}_{\mathcal{M}}$ of spectra. For simplicity, and since we do not have any particular applications in mind, we will only deal with finite spectra in the spirit of Spanier and Whitehead.

As before, we assume that $\mathcal{M}$ is right proper. We say that $X \in \mathcal{M}$ is a finite complex if it is a finite $\left(\bigcup_{n\geq 0} \mathcal{I}_n\right)$-cell complex.

The objects of $\mathsf{Sp}_{\mathcal{M}}$ are formal (de)suspensions $\Sigma^\ell X$ – these are simply pairs (ℓ, X) such that

- $\ell \in \mathbb{Z}$ is an arbitrary integer,
- for some $k \geq -\ell$, $X \in \partial I\!I^k/\mathcal{M}^{\mathrm{fin}}$ is an arbitrary finite complex.

We say that X is of *degree* $d = \ell + k \geq 0$. If $\Sigma^{\ell_0} X_0$, $\Sigma^{\ell_1} X_1$ are two objects of the same degree d, we define the set $[\Sigma^{\ell_0} X_0, \Sigma^{\ell_1} X_1]$ as the colimit

$$\operatorname*{colim}_{i \geq \max\{-\ell_0, -\ell_1\}} [\Sigma^{\ell_0+i} X_0, \Sigma^{\ell_1+i} X_1]^{\partial I\!I^{d+i}}.$$

There are obvious functors $J_k \colon \mathsf{Ho}(\partial I\!I^k/\mathcal{M}^{\mathrm{fin}}) \to \mathsf{Sp}_{\mathcal{M}}$ given by $X \mapsto \Sigma^0 X$. We have the following diagram that commutes up to a natural isomorphism

$$\begin{tikzcd}
\mathsf{Ho}(\partial I\!I^k/\mathcal{M}^{\mathrm{fin}}) \arrow[r, "J_k"] \arrow[d, "\mathbb{L}\Sigma"'] & \mathsf{Sp}_{\mathcal{M}} \arrow[d, "\Sigma"] \\
\mathsf{Ho}(\partial I\!I^{k+1}/\mathcal{M}^{\mathrm{fin}}) \arrow[r, "J_{k+1}"'] & \mathsf{Sp}_{\mathcal{M}}
\end{tikzcd}$$

where $\mathbb{L}\Sigma$ denotes the total left derived functor of Σ and where the suspension functor on the right is $\Sigma^\ell X \mapsto \Sigma^{\ell+1} X$; it is clearly an equivalence onto its image. Thus, the suspension functor in $\mathcal{M}^{\mathrm{fin}}$ is turned into an equivalence in $\mathsf{Sp}_{\mathcal{M}}$.

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References

[1] Bergman, G.M., Hausknecht, A.O., *Cogroups and co-rings in categories of associative rings*, Math. Surveys Monogr., vol. 45, American Mathematical Society, Providence, RI, 1996.

[2] Čadek, M., Krčál, M., Matoušek, J., Vokřínek, L., Wagner, U., *Extendability of continuous maps is undecidable*, Discrete Comput. Geom. **51** (2014), 24–66.

[3] Čadek, M., Krčál, M., Vokřínek, L., *Algorithmic solvability of the lifting-extension problem*, Preprint, `arXiv:1307.6444`, 2013.

[4] Goerss, P.G., Jardine, J.F., *Simplicial homotopy theory*, Birkhäuser Verlag, 1999.

[5] Hirschhorn, P.S., *Model categories and their localizations*, Math. Surveys Monogr., vol. 99, American Mathematical Society, 2003.

[6] Mather, M., *Pullbacks in homotopy theory*, Canad. J. Math. **28** (1976), 225–263.

[7] *nLab entry on "heap",* `http://ncatlab.org/nlab/show/heap`.

[8] Stephan, M., *Elmendorfs theorem for cofibrantly generated model categories*, Preprint, `arXiv:1308.0856`, 2013.

[9] Vokřínek, L., *Computing the abelian heap of unpointed stable homotopy classes of maps*, Arch. Math. (Brno) **49** (5) (2013), 359–368.

Department of Mathematics and Statistics, Masaryk University,
Kotlářská 2, CZ 611 37 Brno, Czech Republic
E-mail: `koren@math.muni.cz`