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ON INVERSE CATEGORIES WITH SPLIT IDEMPOTENTS

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Abstract. We present some special properties of inverse categories with split idempotents. First, we examine a Clifford-Leech type theorem relative to such inverse categories. The connection with right cancellative categories with pushouts is illustrated by simple examples. Finally, some basic properties of inverse categories with split idempotents and kernels are studied in terms of split idempotents which generate (right or left) principal ideals of annihilators.

1. Introduction

A category is *inverse* if for each morphism f there is a unique morphism f^{-1} such that $ff^{-1}f = f$ and $f^{-1}ff^{-1} = f^{-1}$. Inverse categories are the natural extension of inverse monoids. Many basic properties of inverse semigroups have been applied to morphisms in inverse categories. In fact, for each object A of an inverse category $\mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(A,A)$ of all morphisms from A to A is an inverse monoid. Quite analogously to the case of inverse semigroups, the concepts left (right) principal ideals, the Green's relations ($\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ etc.), the natural ordering $\leq$, etc., can be introduced for arbitrary inverse category (see, for example, [2, 3, 7]).

In many familiar categories $\mathcal{C}$, a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ (denoted also $f\colon A \to B$) admits a canonical factorization $A \overset{h}{\twoheadrightarrow} X \overset{u}{\rightarrowtail} B$ (h is an epimorphism, u is a monomorphism, and $f = uh$) called an epi-mono factorization of f (epimorphisms are denoted in text by $\twoheadrightarrow$ and monomorphism by $\rightarrowtail$). In many algebraic categories, X is the set-theoretic image of f in the usual sense. The epi-mono factorizations plays an important role in the theory of exact categories. Exact category has assumed different meanings in category theory. In Mitchell's book [6] an exact category is a category with kernels and cokernels, it is normal (i.e., any monomorphism is a kernel of a morphism) and conormal (i.e., any epimorphism is a cokernel of a morphism), and every morphism has an epi-mono factorization. In [6] abelian categories are defined as additive exact categories. An idempotent i of a category is said to be split idempotent if i admits an epi-mono factorization. Inverse categories with kernels and with closed and split idempotents (for the definition of closed idempotents see Section 4) are exact categories ([7]).

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The theory of bisimple inverse monoids created by Clifford is one of the most fundamental contribution to the classical theory of inverse semigroups. A semigroup is bisimple if it consists of a single $\mathcal{D}$-class. Clifford's theorem describes the structure of bisimple inverse monoids in terms of their right unit subsemigroup. The Clifford result on bisimple inverse monoid was interpreted by Leech [5] categorically and it was extended to arbitrary inverse monoids. Section 2 contains an extension of these results from inverse monoids to inverse categories with split idempotents (see also [2, Section 2.1]). The illustrative example described in Section 3 involves a pair of subcategories of the category of pointed sets. Finally, in Section 4 basic concepts and properties are considered in inverse categories with kernels and split idempotents. In a forthcoming paper we will utilize all of these for continuing the study of exact inverse categories. The notations and terminologies are standard and we presume elementary basic knowledge of category theory.

2. Inverse categories and Clifford's theorem

The origins of this section lie in the fundamental result of Clifford [1] on bisimple inverse monoids.

Theorem 2.1 (Clifford [1]).
(i) *Let S be a bisimple inverse monoid. Then the $\mathcal{R}$-class $\mathbb{R}(S)$ of S containing the identity, $\mathbb{R}(S) = \{s \in S \mid ss^{-1} = 1\}$, is a right cancellative monoid in which the set of principal left ideals is closed under finite intersections.*

(ii) *If R is a right cancellative monoid in which the set of principal left ideals is closed under finite intersections then $\mathbb{S}(R) = R \times R/\varrho$, where $(s_1,t_1)\varrho(s_2,t_2)$ iff $(s_1,t_1) = (us_2,ut_2)$ for some unit $u \in R$, can be equipped with a multiplication $\cdot$ such that $(\mathbb{S}(R),\cdot)$ becomes a bisimple inverse monoid. This multiplication on $\mathbb{S}(R) = R \times R/\varrho$ is defined by:*

$$[s_1,t_1]\cdot[s_2,t_2] = [ps_1,qt_2]\,,$$

where $Rt_1 \cap Rs_2 = Rr$ and $pt_1 = qs_2 = r$ for some $p,q,r \in R$ ($[s,t]$ being the equivalence class on $R\times R$ generated by (s,t)).

(iii) $S \cong \mathbb{S}(\mathbb{R}(S))$ *(S is isomorphic to $\mathbb{S}(\mathbb{R}(S))$).*

(iv) $R \cong \mathbb{R}(\mathbb{S}(R))$ *(R is isomorphic to $\mathbb{R}(\mathbb{S}(R))$).*

Leech's interpretation is given in terms of a right cancellative category with pushouts and weakly initial object (i.e., an object I such that $\mathrm{Hom}(I,A)$ is non-empty for any object A), and it is extended to arbitrary inverse monoid.

Theorem 2.2 (Leech [5]).
(i) *Let S be an inverse monoid. Define $\mathbb{C}(S)$ by*
- $\mathrm{Ob}\,\mathbb{C}(S) = E(S)$ *(where $E(S)$ is the set of idempotents of S);*
- $\mathrm{Hom}_{\mathbb{C}(S)}(e,f) = \{(s,e)\in S\times E(S) | es^{-1}s = s^{-1}s$ and $ss^{-1} = f\}$; *and*
- *the composition of two morphisms $(s,e)\colon e \to f$ and $(t,f)\colon f\to g$ being given by $(t,f)\cdot(s,e) = (ts,e)$.*

Then $\mathbb{C}(S)$ *is a right cancellative category with pushouts and with 1 a weakly initial object of* $\mathbb{C}(S)$*.*

(ii) *If* $(\mathcal{C}, I)$ *is a pair where* $\mathcal{C}$ *is a right cancellative category with pushouts,* I *being a weakly initial object in* $\mathcal{C}$ *then* $\mathbb{S}(\mathcal{C}, I) = M/\varrho$*, where* $M = \{(\alpha,\beta) \in \operatorname{Mor}\mathcal{C} \times \operatorname{Mor}\mathcal{C} \mid \operatorname{Dom}\alpha = \operatorname{Dom}\beta = I$ *and* $\operatorname{Codom}\alpha = \operatorname{Codom}\beta\}$ *and* (α_1,β_1) $\varrho(\alpha_2,\beta_2)$ *iff* $(\alpha_1,\beta_1) = (\iota\alpha_2, \iota\beta_2)$ *for some isomorphism* $\iota \in \operatorname{Mor}\mathcal{C}$*, can be equipped with a multiplication* $\cdot$ *such that* $(\mathbb{S}(\mathcal{C}, I), \cdot)$ *becomes an inverse monoid. This multiplication on* $\mathbb{S}(\mathcal{C}, I) = M/\varrho$ *is defined by:*

$$[\alpha_1,\beta_1]\cdot[\alpha_2,\beta_2] = [\gamma\alpha_1, \delta\beta_2],$$

where $\{\alpha_2, \beta_1, \gamma, \delta\}$ *is a pushout in* $\mathcal{C}$ *and* $[\alpha,\beta]$ *is the equivalence class generated by* $(\alpha,\beta) \in M$*.*

(iii) $S \cong \mathbb{S}(\mathbb{C}(S), 1)$ $($*S is isomorphic to* $\mathbb{S}(\mathbb{C}(S), 1))$*.*

(iv) $\mathcal{C} \approx \mathbb{C}(\mathbb{S}(\mathcal{C}, I))$ $($*there is an equivalence of categories* $F\colon \mathcal{C} \to \mathbb{C}(\mathbb{S}(\mathcal{C}, I))$ *such that* $F(I) = 1)$*.*

It may be interesting to consider above the possibility of replacing inverse monoids with inverse categories. The essential steps in this direction were taken by Jones and Lawson [2, Propositions 2.2, 2.3, 2.4]. We will follow some of their ideas, however there are some major differences between our technique and that in [2]. Kawahara's [4] construction of relations in categories with pullbacks will be very useful in our development. In order to do this we prove the following lemma.

Lemma 2.1. *Let* $\mathcal{I}$ *be an inverse category with split idempotents. Given two morphisms* $\alpha\colon A \to B$ *and* $\beta\colon A \to C$ *with a common domain, and* $\alpha^{-1}\alpha\beta^{-1}\beta = \gamma\gamma^{-1}$ *(Diagram 1),*

$$\begin{tikzcd} A \arrow[rr, "\alpha^{-1}\alpha\beta^{-1}\beta"] \arrow[dr, "\gamma^{-1}"'] & & A \\ & Y \arrow[ur, "\gamma"'] & \end{tikzcd}$$

Diagram 1

is an epi-mono factorization of the idempotent $\alpha^{-1}\alpha\beta^{-1}\beta$ *then the following diagram (Diagram 2)*

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C \arrow[d, "\gamma^{-1}\alpha^{-1}\alpha\beta^{-1}"] \\ B \arrow[r, "\gamma^{-1}\beta^{-1}\beta\alpha^{-1}"] & Y \end{tikzcd}$$

Diagram 2

is a pushout in $\mathcal{I}$ *if* α *and* β *are epimorhisms.*

Proof. Diagram 2 is commutative:

$$(\gamma^{-1}\beta^{-1}\beta\alpha^{-1})\alpha = \gamma^{-1}\beta^{-1}\beta\alpha^{-1}\alpha = \gamma^{-1}\alpha^{-1}\alpha\beta^{-1}\beta = (\gamma^{-1}\alpha^{-1}\alpha\beta^{-1})\beta$$

Now, if Diagram 3 is a commutative diagram in $\mathcal{I}$,

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C \arrow[d, "\beta_1"] \\ B \arrow[r, "\alpha_1"] & X \end{tikzcd}$$

Diagram 3

and

$$x = \alpha_1\alpha\gamma = \beta_1\beta\gamma \qquad (x\colon Y \to X)\,,$$

then

$$\begin{aligned} x\gamma^{-1}\beta^{-1}\beta\alpha^{-1} &= \alpha_1\alpha\gamma\gamma^{-1}\beta^{-1}\beta\alpha^{-1} = \alpha_1\alpha\alpha^{-1}\alpha\beta^{-1}\beta\beta^{-1}\beta\alpha^{-1} \\ &= \alpha_1\alpha\beta^{-1}\beta\alpha^{-1} = \beta_1\beta\beta^{-1}\beta\alpha^{-1} = \beta_1\beta\alpha^{-1}\,. \end{aligned}$$

Since α is epimorphism, that is $\alpha\alpha^{-1} = 1_B$, the equality $\alpha_1\alpha = \beta_1\beta$ implies $\beta_1\beta\alpha^{-1} = \alpha_1$. Thus

$$x\gamma^{-1}\beta^{-1}\beta\alpha^{-1} = \alpha_1\,.$$

Analogously,

$$x\gamma^{-1}\alpha^{-1}\alpha\beta^{-1} = \beta_1\,.$$

If $y\colon Y \to X$ is a morphism in $\mathcal{I}$ such that $y\gamma^{-1}\beta^{-1}\beta\alpha^{-1} = \alpha_1$ and $y\gamma^{-1}\alpha^{-1}\alpha\beta^{-1} = \beta_1$, then

$$x = \alpha_1\alpha\gamma = y\gamma^{-1}\beta^{-1}\beta\alpha^{-1}\alpha\gamma = y\gamma^{-1}\gamma\gamma^{-1}\gamma = y\,.$$

□

Theorem 2.3.
(i) *Let $\mathcal{I}$ be an inverse category with split idempotents. Then the subcategory $\mathbb{E}(\mathcal{I})$ consisting of all epimorphisms of $\mathcal{I}$ (hence $\operatorname{Mor}\mathbb{E}(\mathcal{I}) = \{\alpha \in \operatorname{Mor}\mathcal{I} | \alpha\alpha^{-1} = 1_{\operatorname{Codom}\alpha}\}$) is a right cancellative category with pushouts.*

(ii) *If $\mathcal{E}$ is a right cancellative category with pushouts then define the category $\mathbb{I}(\mathcal{E})$ by*

- $\operatorname{Ob}\mathbb{I}(\mathcal{E}) = Ob\mathcal{E}$*;*
- $\operatorname{Hom}_{\mathbb{I}(\mathcal{E})}(A,B) = \{(\alpha,\beta) \in \operatorname{Hom}_{\mathcal{E}}(A,\bullet) \times \operatorname{Hom}_{\mathcal{E}}(B,\bullet) | \operatorname{Codom}\alpha = \operatorname{Codom}\beta\}/\varrho\,,$ *where $(\alpha_1,\beta_1)\varrho(\alpha_2,\beta_2)$ iff $(\alpha_1,\beta_1) = (\iota\alpha_2,\iota\beta_2)$ for some isomorphism $\iota \in \operatorname{Mor}\mathcal{E}$;*
- *the composition of two morphisms $[\alpha_1,\beta_1]\colon A \to B$ and $[\alpha_2,\beta_2]\colon B \to C$ is given by*

$$[\alpha_2,\beta_2][\alpha_1,\beta_1] = [\gamma\alpha_1,\delta\beta_2]\,,$$

where $\{\alpha_2,\beta_1,\gamma,\delta\}$ is a pushout in E and $[\alpha,\beta]$ is the equivalence class generated by (α,β).

Then $\mathbb{I}(\mathcal{E})$ is an inverse category with split idempotents.

(iii) $\mathcal{I} \cong \mathbb{I}(\mathbb{E}(\mathcal{I}))$.

(iv) $\mathcal{E} \cong \mathbb{E}(\mathbb{I}(\mathcal{E}))$.

Proof.

(i) By Lemma 2.1, Diagram 2 is a pushout in $\mathcal{I}$ if α and β are epimorhisms. It follows that $\gamma^{-1}\beta^{-1}\beta\alpha^{-1}\alpha$ and $\gamma^{-1}\alpha^{-1}\alpha\beta^{-1}\beta$ are also epimorphisms. Since $x\gamma^{-1}\beta^{-1}\beta\alpha^{-1} = \alpha_1$ and $x\gamma^{-1}\alpha^{-1}\alpha\beta^{-1} = \beta_1$ (using notation from the proof of Lemma 2.1), it follows that x is also epimorphism if α_1 and β_1 are epimorphisms. Thus $\mathbb{E}(\mathcal{I})$ is a category with pushouts and obviously it is right cancellative.

(ii) This construction of $\mathbb{I}(\mathcal{E})$ is the dual construction of Kawahara's [3] category of relations (relative to the subcategory of isomorphisms) for a category with pullbacks. For $[\alpha,\beta] \in \mathrm{Hom}_{\mathbb{I}(\mathcal{E})}(A,B)$, define $[\alpha,\beta]^* = [\beta,\alpha] \in Hom_{\mathbb{I}(\mathcal{E})}(A,B)$; then we have: $[\alpha,\beta]^{**} = [\alpha,\beta]$ and $([\alpha_2,\beta_2][\alpha_1,\beta_1])^* = [\alpha_1,\beta_1]^*[\alpha_2,\beta_2]^*$. Now, it is an embedding of $\mathcal{E}$ in $\mathbb{I}(\mathcal{E})$, $\Gamma\colon \mathcal{E} \to \mathbb{I}(\mathcal{E})$ defined by

$$A \in \mathrm{Ob}\,\mathcal{E} \rightsquigarrow \Gamma(A) = A \in \mathrm{Ob}\,\mathbb{I}(\mathcal{E})\,;$$

$$\alpha \in \mathrm{Hom}_{\mathcal{E}}(A,B) \rightsquigarrow \Gamma(\alpha) = [\alpha, 1_B] \in \mathrm{Hom}_{\mathbb{I}(\mathcal{E})}(A,B)$$

such that

$$[\alpha,\beta] = \Gamma(\beta)^*\Gamma(\alpha)\,.$$

Since $\mathcal{E}$ is right cancellative it follows that $\{\alpha,\alpha,1_{\mathrm{Codom}\,\alpha},1_{\mathrm{Codom}\,\alpha}\}$ is a pushout in $\mathcal{E}$ for any morphism α, and therefore

$$\Gamma(\alpha)\Gamma(\alpha)^* = [1_B,1_B] \quad \text{for any morphism } \alpha \in \mathrm{Hom}_{\mathcal{E}}(A,B)\,.$$

So, for any morphism $[\alpha,\beta]$ we have: $[\alpha,\beta] = \Gamma(\beta)^*\Gamma(\alpha)$ is an epi-mono factorization in $\mathbb{I}(\mathcal{E})$. It remains to prove that $\mathbb{I}(\mathcal{E})$ is an inverse category. We have:

$$[\alpha,\beta][\alpha,\beta]^*[\alpha,\beta] = \Gamma(\beta)^*\Gamma(\alpha)\Gamma(\alpha)^*\Gamma(\beta)\Gamma(\beta)^*\Gamma(\alpha) = \Gamma(\beta)^*\Gamma(\alpha) = [\alpha,\beta]\,.$$

The set $\{[\alpha,\alpha] \,|\, \mathrm{Dom}\,\alpha = A\}$ is the set of idempotents in $Hom_{\mathbb{I}(\mathcal{E})}(A,A)$, and $[\alpha,\alpha][\beta,\beta] = [\beta,\beta][\alpha,\alpha]$ if $[\alpha,\alpha]$ and $[\beta,\beta]$ are two idempotents in $\mathrm{Hom}_{\mathbb{I}(\mathcal{E})}(A,A)$. So, $\mathbb{I}(\mathcal{E})$ is a regular category and its idempotents commute. This proved that $\mathbb{I}(\mathcal{E})$ is an inverse category with split idempotents.

(iii) The functors $F\colon \mathcal{I} \to \mathbb{I}(\mathbb{E}(\mathcal{I}))$ and $G\colon \mathbb{I}(\mathbb{E}(\mathcal{I})) \to \mathcal{I}$ defined by

$$F(A) = A\,; \qquad F(\alpha) = [h, u^{-1}]$$

(where $\alpha = uh$ is an epi-mono factorization of α), and

$$G(A) = A\,; \qquad G([\alpha,\beta]) = \beta^{-1}\alpha$$

respectively, are mutually inverse to each other; i.e. $G \cdot F$ is the identity functor on $\mathcal{I}$ and $F \cdot G$ is the identity functor on $\mathbb{I}(\mathbb{E}(\mathcal{I}))$.

(iv) It is straightforward to check that $\mathbb{E}(\mathbb{I}(\mathcal{E}))$ is the image of the functor Γ (i.e., the smallest subcategory of $\mathbb{I}(\mathcal{E})$ which contains all image morphisms $\Gamma(\alpha)$). So, $\mathcal{E}$ and $\mathbb{E}(\mathbb{I}(\mathcal{E}))$ are isomorphic through Γ. □

Now, it is straightforward to see that:

Proposition 2.1. *If the right cancellative category $\mathcal{E}$ with pushouts has a weakly initial object I, then the inverse monoid* $\mathrm{Hom}_{\mathbb{I}(\mathcal{E})}(I,I)$ *is just Leech's inverse monoid* $\mathbb{S}(\mathcal{E},I)$.

Let $(P,\leq)$ be a join-semilattice. The join-semilattice $(P,\leq)$ is viewed as a category in which each Hom-set has at most one element: $\mathrm{Ob}(P,\leq)=P$, and $\mathrm{Hom}_{(P,\leq)}(a,b)$ is a singleton if and only if $a\leq b$. The category $(P,\leq)$ is a right cancellative category with pushouts. The inverse category $\mathbb{I}(P,\leq)$ is defined by:

- $\mathrm{Ob}\,\mathbb{I}(P,\leq)=P$;
- $\mathrm{Hom}_{\mathbb{I}(P,\leq)}(a,b)=\{(a,x,b)\in P^3 | a,b\leq x\}$; and
- the composition of two morphisms $(a,x,b)\colon a\to b$ and $(b,y,c)\colon b\to c$ is given by
$$(b,y,c)(a,x,b)=(a,x\vee y,c)\,.$$

Every endomorphism in $\mathbb{I}(P,\leq)$ is an idempotent, (a,a,a) is the identity morphism in $\mathrm{Hom}_{\mathbb{I}(P,\leq)}(a,a)$, $(a,x,b)^{-1}=(b,x,a)$, and the morphism (a,x,b) is an epimorphism (monomorphism) if and only if $x=b$ ($x=a$). The factorization $(a,x,b)=a\overset{(a,x,x)}{\twoheadrightarrow}x\overset{(x,x,b)}{\rightarrowtail}b$ is an epi-mono factorization of (a,x,b). If $(P,\leq)$ is a join-semilattice with a least element 0, then the category $(P,\leq)$ is right cancellative with pushouts and with the weakly initial object 0. In this case Leech's inverse monoid $(\mathbb{S}((P,\leq),0),\cdot)$ is just the semilattice $(P,\vee)$.

The following example is much more complicated. Let $\mathcal{C}_a$ the category of integer affine maps (see [9, Section 2]) defined by

- $\mathrm{Ob}\,\mathcal{C}_a=\mathbb{Z}^+$;
- $\mathrm{Hom}_{\mathcal{C}_a}(k,n)=$
$=\{(f,k)|\ f:\mathbb{R}\to\mathbb{R},\ f(x)=ax+b,\ a,b\in\mathbb{Z},\ a>0,\ b\geq 0,\ \text{and } f(k)=n\}$; and
- the composition $(g,n)(f,k)$ of two morphisms $(f,k)\colon k\to n$ and $(g,n)\colon n\to m$ is given by $(g,n)(f,k)=(g\circ f,k)$, where $g\circ f$ is the usual composition of maps.

The category $\mathcal{C}_a$ is a right cancellative category with pushouts. A morphism (f,k) will be denoted by f if the domain k (and therefore the codomain $f(k)$ also) is implied. If $f=ax+b$ and $g=cx+d$ are two morphisms in C_a with a common domain k then Diagram 4 is a pushout,

$$\begin{tikzcd}
k \arrow[r, "g"] \arrow[d, "f"'] & g(k) \arrow[d, "h_2=m_2x+m-m_2g(k)"] \\
f(k) \arrow[r, "h_1=m_1x+m-m_1f(k)"] & \max\{m_1f(k),m_2g(k)\}=m
\end{tikzcd}$$

Diagram 4

where $m_1 = \frac{l.c.m.\{a,c\}}{a}$ and $m_2 = \frac{l.c.m.\{a,c\}}{c}$. A morphism f is an isomorphism in $\mathcal{C}_a$ if and only if it is an identity. Therefore the morphisms of the inverse category $\mathbb{I}(\mathcal{C}_a)$ are pairs of morphisms of $\mathcal{C}_a$ with common codomain. We have: $\operatorname{Ob}\mathbb{I}(\mathcal{C}_a) = \mathbb{N}^*$ and $\operatorname{Hom}_{\mathbb{I}(\mathcal{C}_a)}(k_1, k_2) = \{(f, g) \in \operatorname{Hom}_{\mathcal{C}_a}(k_1, f(k_1)) \times \operatorname{Hom}_{\mathcal{C}_a}(k_2, g(k_2)) \mid f(k_1) = g(k_2)\}$. The composition $(f_2, g_2)(f_1, g_1)$ of two morphisms $(f_1, g_1)\colon k_1 \to k_2$, $(f_2, g_2)\colon k_2 \to k_3$, $f_i = a_i x + b_i$ $(i = 1, 2)$, $g_i = c_i x + d_i$ $(i = 1, 2)$ is given by

$$(a_2x + b_2, c_2x + d_2)(a_1x + b_1, c_1x + d_1) = (m_1a_1(x - k_1) + m, m_2c_2(x - k_3) + m)\,,$$

where $m_1 = \frac{l.c.m.\{c_1,a_2\}}{c_1}$, $m_2 = \frac{l.c.m.\{c_1,a_2\}}{a_2}$ and $m = \max\{m_1f_1(k_1), m_2g_2(k_3)\}$ (see Diagram 5),

$$\begin{tikzcd}
k_1 \arrow[dr, "f_1"'] & & k_2 \arrow[dl, "g_1"] \arrow[dr, "f_2"] & & k_3 \arrow[dl, "g_2"] \\
& f_1(k_1) = g_1(k_2) \arrow[dr, "h_1=m_1x+m-m_1f_1(k_1)"'] & & f_2(k_2) = g_2(k_3) \arrow[dl, "h_2=m_2x+m-m_2g_2(k_3)"] & \\
& & \max\{m_1f_1(k_1), m_2g_2(k_3)\} = m & &
\end{tikzcd}$$

Diagram 5

Note that the inverse monoid $\operatorname{Hom}_{\mathbb{I}(\mathcal{C}_a)}(1, 1)$ is just the Dirichlet analogue of the free monogenic inverse semigroup (see [9]).

We end this section with a simple example. Let $\mathcal{I}_{\mathbb{Q}^+}$ be the category defined by:

- $\operatorname{Ob}\mathcal{I}_{\mathbb{Q}^+} = \mathbb{Q}^+$;
- $\operatorname{Hom}_{\mathcal{I}_{\mathbb{Q}^+}}(x, y) = \{(x, a, b) | a, b \in \mathbb{Z}^+,\ x\frac{b}{a} = y\}$; and
- the composition $(y, c, d)(x, a, b)$ of two morphisms $(x, a, b)\colon x \to y$ and $(y, c, d)\colon y \to z$ is given by $(y, c, d)(x, a, b) = \left(x, \frac{am}{b}, \frac{md}{c}\right)$, where $m = l.c.m.\{b, c\}$.

All endomorphisms are idempotents and the morphism $(x, 1, 1)$ is the identity on $\operatorname{Hom}_{\mathcal{I}_{\mathbb{Q}^+}}(x, x)$. The category $\mathcal{I}_{\mathbb{Q}^+}$ is inverse in which $(x, a, b)^{-1} = \left(x\frac{b}{a}, b, a\right)$ and $(x, a, b) = x \overset{(x,a,1)}{\twoheadrightarrow} x\frac{1}{a} \overset{(x\frac{1}{a},1,b)}{\rightarrowtail} x\frac{b}{a}$ is an epi-mono factorization in $\mathcal{I}_{\mathbb{Q}^+}$ of the morphism (x, a, b). The category $\mathbb{E}(\mathcal{I}_{\mathbb{Q}^+})$ is nothing but the partially ordered set $(\mathbb{Q}^+, \preceq)$ where $x \preceq y$ if and only if $\frac{x}{y} \in \mathbb{Z}^+$.

We continue in the next section with the presentation of a rich variety of examples.

3. Two subcategories of the category of pointed sets

The objects of the category of pointed sets Set_* are pairs (A, a) consisting of a nonempty set A together with a designated element $a \in A$. Morphisms in Set_* are mappings that preserve the designated points. The composition of morphisms in Set_* are the standard composition of mappings. The identity mapping on A is the identity morphism $1_{(A,a)}$. A singleton in Set_*, that is $(\{*\},*)$, is a zero object (both initial and terminal).

Let $\mathscr{S}_1$ be the subcategory of Set_* defined by: $\mathrm{Ob}\,\mathscr{S}_1 = \mathrm{Ob}\,\mathrm{Set}_*$; the morphisms from (A,a) to (B,b) are surjective maps f from A to B such that $f(a)=b$ and $f(x)=f(y)\neq b$ implies $x=y$. More precisely, the morphisms in $\mathscr{S}_1$ from (A,a) to (B,b) are such morphisms f of Set_* ($f\colon A\to B$ with $f(a)=b$) which establish a bijection between $A-A_{f,b}$ and B, where $A_{f,b}$ denotes the set $\{x\in A-\{a\}|f(x)=b\}$.

Proposition 3.1. *The category $\mathscr{S}_1$ is right cancellative with pushouts.*

Proof. It is straightforward to see that $\mathscr{S}_1$ is right cancellative.

If $f\in \mathrm{Hom}_{\mathscr{S}_1}((A,a),(B,b))$ and $\overline{f}\colon B\to A$ is the map defined by

$$\overline{f}(y)=\begin{cases} x & \text{if} \quad y\in B-\{b\} \text{ and } f(x)=y\\ a & \text{if} \quad y=b\,,\end{cases}$$

then $\overline{f}$ is a morphism in $\mathscr{S}_1$ from (B,b) to $(A-A_{f,b},a)$ such that $f_{/(A-A_{f,b})}\overline{f} = 1_{(B,b)}$, where $f_{/(A-A_{f,b})}$ is the restriction of f to $A-A_{f,b}$. The morphism $\overline{f}f$ in $\mathscr{S}_1$ from (A,a) to $(A-A_{f,b},a)$ is such that $\overline{f}f(x)=x$ for all $x\in A-A_{f,b}$ and $\overline{f}f(x)=a$ for all $x\in A_{f,b}$ (that is, $A_{\overline{f}f,a}=A_{f,b}$). Now, it is a routine matter to verify that for any morphisms $f\in\mathrm{Hom}_{\mathscr{S}}((A,a),(B,b))$ and $g\in\mathrm{Hom}_{\mathscr{S}}((A,a),(C,c))$, the Diagram 6 is a pushout in $\mathscr{S}_1$.

$$\begin{tikzcd}
(A,a) \arrow[r, "g"] \arrow[d, "f"'] & (C,c) \arrow[d, "\overline{f}f\overline{g}"] \\
(B,b) \arrow[r, "\overline{g}g\overline{f}"] & (A-(A_{f,b}\cup A_{g,c}),a)
\end{tikzcd}$$

Diagram 6

□

Now, it is clear that $f\in\mathrm{Hom}_{\mathscr{S}_1}((A,a),(B,b))$ is an isomorphism in $\mathscr{S}_1$ if and only if $\overline{f}f=1_{(A,a)}$, that is if and only if $A_{f,b}=\emptyset$. More precisely, $f\in Hom_{\mathscr{S}_1}((A,a),(B,b))$ is an isomorphism in $\mathscr{S}_1$ if and only if $f:A\to B$ is a bijection with $f(a)=b$. The inverse category $\mathbb{I}(\mathscr{S}_1)$ is defined by:

- $\mathrm{Ob}\,\mathbb{I}(\mathscr{S}_1)=\mathrm{Ob}\,\mathscr{S}_1$;
- $\mathrm{Hom}_{\mathbb{I}(\mathscr{S}_1)}((A,a),(B,b)) =$
  $=\{[f,g \mid \mathrm{Codom}\, f=\mathrm{Codom}\, g, f\in\mathrm{Hom}_{\mathscr{S}_1}((A,a),\bullet), g\in\mathrm{Hom}_{\mathscr{S}_1}((B,b),\bullet)\}\,,$
  where $[f,g]$ is the $\overline{\varrho}$-equivalence class generated by (f,g), $\overline{\varrho}$ being defined on the set $\mathrm{Hom}_{\mathscr{S}_1}((A,a),\bullet)\times\mathrm{Hom}_{\mathscr{S}_1}((B,b),\bullet)$ by $(f_1,g_1)\overline{\varrho}(f_2,g_2)$ iff $(f_1,g_1)=(\iota f_2,\iota g_2)$ for some isomorphism ι of $\mathscr{S}_1$;
- the composition of two morphisms $[f_1,g_1]\colon(A,a)\to(B,b)$ and $[f_2,g_2]\colon(B,b)\to(C,c)$ (see Diagram 7) is given by

$$[f_2,g_2][f_1,g_1]=[\overline{f}_2f_2\overline{g}_1f_1,\overline{g}_1g_1\overline{f}_2g_2]\,,$$

where $\{g_1, f_2, \overline{f}_2 f_2\overline{g}_1, \overline{g}_1 g_1\overline{f_2}\}$ is a pushout in $\mathscr{S}_1$.

$$\begin{tikzcd}
(A,a) \arrow[dr, "f_1"'] & & (B,b) \arrow[dl, "g_1"] \arrow[dr, "f_2"'] & & (C,c) \arrow[dl, "g_2"] \\
& (D,d) \arrow[dr, "\overline{f}_2{,}f_2\overline{g}_1"'] & & (E,e) \arrow[dl, "\overline{g}_1 g_1\overline{f}_2"] & \\
& & (B-(B_{g_1,d}\cup B_{f_2,e}),b) & &
\end{tikzcd}$$

Diagram 7

Let us consider a second subcategory $\mathscr{S}_2$ of the category Set_*: $\mathrm{Ob}\,\mathscr{S}_2 = \mathrm{Ob}\,\mathrm{Set}_*$; the morphisms from (A,a) to (B,b) are maps f from A to B such that $f(a) = b$ and $f(x) = f(y) \neq b$ implies $x = y$. Obviously, $\mathscr{S}_1$ is a subcategory of $\mathscr{S}_2$.

Proposition 3.2. *The categories $\mathbb{I}(\mathscr{S}_1)$ and $\mathscr{S}_2$ are isomorphic.*

Proof. If $g \in \mathrm{Hom}_{\mathscr{S}_1}((B,b),(C,c))$ then $\overline{g}$, which is a morphism in $\mathscr{S}_1$ from (C,c) to $(B - B_{g,c}, b)$, is also a morphism in $\mathscr{S}_2$ from (C,c) to (B,b). So, if $f \in \mathrm{Hom}_{\mathscr{S}_1}((A,a),(C,c))$ then $\overline{g}f \in \mathrm{Hom}_{\mathscr{S}_2}((A,a),(B,b))$. More, if

$$(f_1, g_1) \in \mathrm{Hom}_{\mathscr{S}_1}((A,a),(C,c)) \times \mathrm{Hom}_{\mathscr{S}_1}((B,b),(C,c))$$

and

$$(f_1, g_1) \in \mathrm{Hom}_{\mathscr{S}_1}((A,a),(D,d)) \times \mathrm{Hom}_{\mathscr{S}_1}((B,b),(D,d))$$

then

$$(f_1, g_1)\varrho(f_2, g_2) \quad \text{if and only if} \quad \overline{g}_1 f_1 = \overline{g}_2 f_2\,.$$

So the correspondence

$$[f,g] \rightsquigarrow \overline{g}f$$

is well defined, and it gives rise to an isomorphism between $\mathbb{I}(\mathscr{S}_1)$ and $\mathscr{S}_2$. □

Thus $\mathscr{S}_1$ and $\mathscr{S}_2$ are two subcategories of Set_* which correspond to each other in the above described connection (Section 2) between right cancellative categories with pushouts and inverse categories with split idempotents. In the inverse category $\mathscr{S}_2$ the inverse f^{-1} of a morphism $f \in \mathrm{Hom}_{\mathscr{S}_2}((A,a),(B,b))$ is given by

$$(\forall y \in B) \qquad f^{-1}(y) = \begin{cases} x & \text{if} \quad y \in f(A) - \{b\} \quad \text{and} \quad f(x) = y \\ a & \text{otherwise.} \end{cases}$$

An endomorphism $f \in \mathrm{Hom}_{\mathscr{S}_2}((A,a),(A,a))$ is an idempotent in $\mathscr{S}_2$ if and only if $f(x) = x$ for all $x \in A - A_{f,a}$. The factorization $f = gh$ is a mono-epi factorization of the idempotent $f \in \mathrm{Hom}_{\mathscr{S}_2}((A,a),(A,a))$ if $g \in Hom_{\mathscr{S}_2}((f(A),a),(A,a))$, $g(x) = x$ for all $x \in f(A)$, and $h \in \mathrm{Hom}_{\mathscr{S}_2}((A,a),(f(A),a))$, $h(x) = f(x)$ for all $x \in A$. Note that the category $\mathscr{S}_2$ is a category with kernels: if f is a morphism in

$\mathscr{S}_2$ from (A,a) to (B,b) then $((A_{f,b},a),u)$ is the kernel of f, where $u(x)=x$ for all $x\in A_{f,b}$.

4. Kernels in inverse categories with split idempotents

Proposition 4.1. *Let $\mathcal{I}$ be an inverse category with split idempotents and with a zero object* 0*. The following assertions are equivalent:*

(1) *$\mathcal{I}$ is a category with kernels;*

(2) *for any morphism f in $\mathcal{I}$ the right annihilators of f form a principal right ideal generated by an idempotent;*

(3) *$\mathcal{I}$ is a category with cokernels;*

(4) *for any morphism f in $\mathcal{I}$ the left annihilators of f form a principal left ideal generated by an idempotent.*

Proof. (1) $\Rightarrow$ (2). Let f be a morphism in $\mathcal{I}$, $u=\ker f$, and $i=uu^{-1}$. If g is a morphism in $\mathcal{I}$ such that $fg=0$, then $g=uh$ for some morphism h in $\mathcal{I}$, and therefore $g=uu^{-1}uh\in i\mathcal{I}$.

Conversely, if $g\in i\mathcal{I}$ then $fg=0$, since $u=\ker f$.

(2) $\Rightarrow$ (3). Let f be a morphism in $\mathcal{I}$ and $j\mathcal{I}$ the principal right ideal (generated by the idempotent j) of the right annihilators of f^{-1}. If $j=uu^{-1}$ is an epi-mono factorization of the idempotent j, then $u^{-1}=\operatorname{coker} f$.

(3) $\Rightarrow$ (4). Let f be a morphism in $\mathcal{I}$, $h=\operatorname{coker} f$, and $i=h^{-1}h$. Then $\mathcal{I}i$ is the principal left ideal (generated by the idempotent i) of the left annihilators of f.

(4) $\Rightarrow$ (1). Let f be a morphism in $\mathcal{I}$ and $\mathcal{I}j$ the principal left ideal (generated by the idempotent j) of the left annihilators of f^{-1}. If $j=uu^{-1}$ is an epi-mono factorization of the idempotent j, then $u=\ker f$. $\square$

For any morphism f of the inverse category $\mathcal{I}$ with split idempotents and kernels, the idempotent i (resp. j) which generates the principal right (resp. left) ideal of right (resp. left) annihilators of f is uniquely determined by f, and we will denote it by $(f)'$ (resp. $'(f)$). It is straightforward to check that for any morphism f of $\mathcal{I}$ we have $(f)'={}'(f^{-1})$ and therefore $(i)'={}'(i)$ for any idempotent i. Much more, for any idempotent i of $\mathcal{I}$ we have $i\le((i)')'$, and an idempotent i will be called closed if $i=((i)')'$.

Now in the inverse category with split idempotents $\mathscr{S}_2$ the object (A,a) is a zero object if and only if A is a singleton: $A=\{a\}$. The category $\mathscr{S}_2$ is a category with kernels and closed idempotents (see [8]). For any morphism $f\in \operatorname{Hom}_{\mathscr{S}_2}((A,a),(B,b)$ we have:

$$(\forall x\in A)\qquad (f)'(x)=\begin{cases} x & \text{if}\quad x\in A_{f,b}-\{a\}\\ a & \quad\text{otherwise.}\end{cases}$$

and

$$(\forall y\in B)\qquad {}'(f)(y)=\begin{cases} y & \text{if}\quad y\notin f(A)\\ b & \quad\text{otherwise.}\end{cases}$$

An exact category in the sense of Mitchell [6] is a normal category (i.e. any monomorphism is a kernel of a morphism) and conormal category (i.e. any epimorphism is a cokernel of a morphism) with kernels and cokernels, and every morphism has an epi-mono factorization. A category is called Boolean if the set of subobjects of any object forms a Boolean algebra.

Theorem 4.1 ([7]). *An inverse category with split idempotents and kernels in which all idempotents are closed is an exact Boolean category. The Boolean algebra of the subobjects of an object A is isomorphic to the Boolean algebra of all idempotents of* $\mathrm{Hom}(A,A)$*, where $(i)'$ is the complement of the idempotent i and the intersection and union operations are defined by*

$$i \wedge j = ij\,, \qquad i \vee j = ((i)'(j)')'\,.$$

In what follows $\overline{\mathcal{I}}$ will be denote an inverse category with split idempotents and kernels in which all idempotents are closed. By [8, Lemma 1] in a such category for any morphism f we have:

$$(f)'' = f^{-1}f \quad \text{and} \quad ''(f) = ff^{-1}\,,$$

where $(f)'' = ((f)')'$ and $''(f) = \ '('(f))$.

Proposition 4.2. *Let $f \in \mathrm{Hom}_{\overline{\mathcal{I}}(A,B)}$ and $g \in \mathrm{Hom}_{\overline{\mathcal{I}}(B,C)}$ be two morphisms in the category $\overline{\mathcal{I}}$. The following assertions are equivalent:*

(1) *the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is a semi-exact sequence in $\overline{\mathcal{I}}$;*

(2) *the sequence $B \xrightarrow{''(f)} B \xrightarrow{(g)''} B$ is a semi-exact sequence in $\overline{\mathcal{I}}$;*

(3) *there is an idempotent $i \in Hom_{\overline{\mathcal{I}}}(B,B)$ such that $if = f$ and $gi = 0$;*

(4) $''(f) \leq (g)'$,

where $\leq$ is the natural partial order relation on the inverse monoid $\mathrm{Hom}_{\overline{\mathcal{I}}}(B,B)$.

Proof. (1) $\Rightarrow$ (2) $\quad [(g)''][''(f)] = g^{-1}gff^{-1} = 0$.

(2) $\Rightarrow$ (3) $\quad ''(f)f = f$ and $g[''(f)] = g[(g)''][''(f)] = 0$.

(3) $\Rightarrow$ (4) $\quad gi = 0$ implies that $i \in (g)'\mathcal{I}$ and therefore $i \leq (g)'$; and $if = f$ implies that $i[''(f)] = \ ''(f)$ and therefore $''(f) \leq i$.

(4) $\Rightarrow$ (1) $\quad ''(f) \leq (g)'$, that is $[(g)'][''(f)] = \ ''(f)$, implies that $gf = g[''(f)]f = g[(g)'][''(f)]f = 0$. □

Proposition 4.3. *The sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence in $\overline{\mathcal{I}}$ if and only if $''(f) = (g)'$.*

Proof First we show that if $u\colon X \rightarrowtail A$ and $v\colon Y \rightarrowtail A$ are two subobjects of A, then they are isomorphic subobjects of A if and only if $''(u) = \ ''(v)$. If $u = v\gamma$, where γ is an isomorphism, then $''(u) = uu^{-1} = v\gamma\gamma^{-1}v^{-1} = vv^{-1} = \ ''(v)$. Conversely if $''(u) = \ ''(v)$ then $u = uu^{-1}u = vv^{-1}u$ and $v = uu^{-1}v$, where $(v^{-1}u)(u^{-1}v) = 1_Y$ and $(u^{-1}v)(v^{-1}u) = 1_X$. Therefore $u\colon X \rightarrowtail A$ and $v\colon Y \rightarrowtail A$ are isomorphic subobjects.

Now, if $f = uh$ $(A \xrightarrow{f} B = A \overset{h}{\twoheadrightarrow} X \overset{u}{\rightarrowtail} B)$ is an epi-mono factorization of the morphism f, then $u\colon X \rightarrowtail B$ is the (epimorphic) image of f and it is straightforward to see that $''(f) = \ ''(u)$. If $(g)' = vv^{-1}$ $(B \xrightarrow{(g)'} B = B \overset{v^{-1}}{\twoheadrightarrow} Y \overset{v}{\rightarrowtail} B)$ is an epi-mono factorization of $(g)'$, by the proof of (4) $\Rightarrow$ (1) in Proposition 4.1, taking into account that $(g)' = \ '(g^{-1}))$, it follows that $v\colon Y \rightarrowtail B$ is the kernel of g, and clearly $(g)' = \ ''(v)$. Since the subobjects $u\colon X \rightarrowtail B$ and $v\colon Y \rightarrowtail B$ are isomorphic if and only if $''(u) = \ ''(v)$, it follows that the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $\ ''(f) = (g)'$. □

Remark 4.1. *If the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is a semi-exact sequence in $\overline{\mathcal{I}}$ then the interval $[''(f), (g)']$ in the semilattice of idempotents of* $\mathrm{Hom}_{\overline{\mathcal{I}}}(B, B)$ *serves as a measure of the deviation from exactness.*

The exact category in the sense of Mitchell [6] is designed to encapsulate, in the first stage, the properties of exact (semi-exact) sequences and the Noether isomorphism theorems. As in the case of exactness (Propositions 4.2 and 4.3) the exact category $\overline{\mathcal{I}}$ brings with it specific elements related to idempotents of the category. The following two propositions are the expression of the Noether isomorphism theorems ([6, Corollaries 16.2 and 16.7]) in terms of split idempotents.

Proposition 4.4. *Let $A \overset{u}{\rightarrowtail} B \overset{v}{\rightarrowtail} C$ in the category $\overline{\mathcal{I}}$, and let $'(v) = C \overset{q}{\twoheadrightarrow} X \overset{q^{-1}}{\rightarrowtail} C$ and $\ '(vu)['(v)] = C \overset{h}{\twoheadrightarrow} Z \overset{h^{-1}}{\rightarrowtail} C$ be epi-mono factorizations of the idempotents $\ '(v)$ and $\ '(vu)['(v)]$, respectively. Then $h['(vu)]q^{-1}\colon X \xrightarrow{\sim} Z$ is an isomorphism in $\overline{\mathcal{I}}$ (Diagram 8).*

$$\begin{tikzcd}
& X \arrow[dr, "q^{-1}"] & & & & Z \arrow[dr, "h^{-1}"] & \\
C \arrow[ur, "q"] \arrow[rr, "'(v)"] & & C \arrow[rr, "'(vu)"] \arrow[dr, "p"'] & & C \arrow[ur, "h"] \arrow[rr, "'(vu)['(v)]"] & & C \\
& & & Y \arrow[ur, "p^{-1}"'] & & &
\end{tikzcd}$$

Diagram 8

Proof. If $'(vu) = C \overset{p}{\twoheadrightarrow} Y \overset{p^{-1}}{\rightarrowtail} C$ is an epi-mono factorization of $\ '(vu)$, then by replacing α by p and β by q in Diagram 2, it follows by Lemma 2.1 that Diagram 9 is a pushout.
Now, since $p = \mathrm{coker}(vu)$ and $q = \mathrm{coker}(v)$, the isomorphism of the upright arrow $h['(vu)]q^{-1}$ follows taking into account [6, Proposition 16.5] and the Diagram from the proof of [6, Corollary 16.2].

$$\begin{tikzcd}
C \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "h['(vu)]q^{-1}"] \\
Y \arrow[r, "h['(v)]p^{-1}"] & Z
\end{tikzcd}$$

Diagram 9

□

Proposition 4.5. *Let $A \overset{u_1}{\rightarrowtail} C$ and $B \overset{u_2}{\rightarrowtail} C$ be two subobjects of C in the category $\overline{\mathcal{I}}$. If $C = A \cup B$ then* $\mathrm{Im}({}'(u_1)[{}'(u_2)]) = 0$.

Proof. If $C \overset{h}{\twoheadrightarrow} \overline{C}$ is the cointersection of the quotient objects $C \overset{p}{\twoheadrightarrow} X$ and $C \overset{q}{\twoheadrightarrow} Y$, where $p = \operatorname{coker} u_1$ and $q = \operatorname{coker} u_2$, then the sequence $0 \to A \cup B \overset{v}{\rightarrowtail} C \overset{h}{\twoheadrightarrow} \overline{C} \to 0$ is exact in $\overline{\mathcal{I}}$. Now, if $C = A \cup B$ then $v = 1_{A\cup B}$ and $\overline{C} = 0$. By [6, Proposition 16.5], taking into account the Diagram what precedes the Second Noether Theorem in [6], the following diagram (Diagram 10) is a pushout:

$$\begin{tikzcd}
A \cup B \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d] \\
Y \arrow[r] & 0
\end{tikzcd}$$

Diagram 10

By Lemma 2.1 it follows that $C \twoheadrightarrow 0 \rightarrowtail C$ is an epi-mono factorization of the idempotent ${}'(u_1)[{}'(u_2)]$, that is $\mathrm{Im}({}'(u_1)[{}'(u_2)]) = 0$. $\square$

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