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GLOBAL BEHAVIOR OF THE DIFFERENCE EQUATION $x_{n+1} = \frac{ax_{n-3}}{b+cx_{n-1}x_{n-3}}$

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Abstract.
In this paper, we introduce an explicit formula and discuss the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-3}}{b+cx_{n-1}x_{n-3}}\,, \qquad n = 0, 1, \dots$$

where a, b, c are positive real numbers and the initial conditions x_{-3}, x_{-2}, x_{-1}, x_0 are real numbers.

1. Introduction

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Recently, there has been a great interest in studying properties of nonlinear and rational difference equations. One can see [3, 5, 8, 9, 11, 12, 13, 14, 15, 19, 18] and the references therein.
In [4], the authors discussed the global behavior of the difference equation

$$x_{n+1} = \frac{Ax_{n-2r-1}}{B+Cx_{n-2l}x_{n-2k}}\,, \quad n = 0, 1, \dots$$

where A, B, C are nonnegative real numbers and r, l, k are nonnegative integers such that $l \leq k$ and $r \leq k$.

In [2] we have discussed global asymptotic stability of the difference equation

$$x_{n+1} = \frac{A+Bx_{n-1}}{C+D{x_n}^2}\,, \qquad n = 0, 1, \dots$$

where A, B are nonnegative real numbers and C, $D > 0$.
We have also discussed in [1] the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{Bx_{n-2k-1}}{C+D\prod_{i=l}^{k} x_{n-2i}}\,, \quad n = 0, 1, \dots$$

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In [17], D. Simsek et al. introduced the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}, \quad n = 0, 1, \ldots$$

where x_{-3}, x_{-2}, x_{-1}, $x_0 \in (0, \infty)$.
Also in [16], D. Simsek et al. introduced the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots$$

with positive initial conditions.
R. Karatas et al. [10] discussed the positive solutions and the attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}, \quad n = 0, 1, \ldots$$

where the initial conditions are nonnegative real numbers.
In [6], E.M. Elsayed discussed the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2}x_{n-5}}, \quad n = 0, 1, \ldots$$

where the initial conditions are nonzero real numbers with $x_{-5}x_{-2} \neq 1$, $x_{-4}x_{-1} \neq 1$ and $x_{-3}x_0 \neq 1$. Also in [7], E.M. Elsayed determined the solutions to some difference equations. He obtained the solution to the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots$$

where the initial conditions are nonzero positive real numbers.
In this paper, we introduce an explicit formula and discuss the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-3}}{b + cx_{n-1}x_{n-3}}, \qquad n = 0, 1, \ldots \tag{1.1}$$

where a, b, c are positive real numbers and the initial conditions x_{-3}, x_{-2}, x_{-1}, x_0 are real numbers.

2. Solution of equation (1.1)

In this section, we establish the solutions of equation (1.1).
From equation (1.1), we can write

$$x_{2n+1} = \frac{ax_{2n-3}}{b + cx_{2n-1}x_{2n-3}}, \qquad n = 0, 1, \ldots \tag{2.1}$$

$$x_{2n+2} = \frac{ax_{2n-2}}{b + cx_{2n}x_{2n-2}}, \qquad n = 0, 1, \ldots \tag{2.2}$$

Using the substitution $y_{2n-1} = \frac{1}{x_{2n-1}x_{2n-3}}$, equation (2.1) is reduced to the linear nonhomogeneous difference equation

$$y_{2n+1} = \frac{b}{a}y_{2n-1} + \frac{c}{a}, \quad y_{-1} = \frac{1}{x_{-1}x_{-3}}, \qquad n = 0, 1, \ldots \tag{2.3}$$

Note that for the backward orbits, the product reciprocals $v_{2k-1} = \frac{1}{x_{2k-1}x_{2k-3}}$ satisfy the equation

$$v_{2k+1} = \frac{a}{b}v_{2k-1} - \frac{c}{b}\,, \quad v_{-1} = \frac{1}{x_{-1}x_{-3}} = -\frac{c}{b}\,, \qquad k = 0, 1, \ldots$$

Therefore,

$$x_{2n-1}x_{2n-3} = -\frac{b}{c\sum_{r=0}^{n}(\frac{a}{b})^r}\,.$$

By induction on n we can show that for any $n \in \mathbb{N}$, if $x_{2n-1}x_{2n-3} = -\frac{b}{c\sum_{r=0}^{n}(\frac{a}{b})^r}$, then $x_{-1}x_{-3} = -\frac{b}{c}$.
The same argument can be done for equation (2.2) and will be omitted.

Now we are ready to give the following lemma.

Lemma 2.1. *The forbidden set F of equation* (1.1) *is*
$F = \bigcup_{n=0}^{\infty}\left\{(u_0, u_{-1}, u_{-2}, u_{-3}) : u_{-3} = -\left(\frac{b}{c\sum_{l=0}^{n}(\frac{a}{b})^i}\right)\frac{1}{u_{-1}}\right\} \cup \bigcup_{m=0}^{\infty}\left\{(u_0, u_{-1}, u_{-2}, u_{-3}) : u_{-2} = -\left(\frac{b}{c\sum_{l=0}^{m}(\frac{a}{b})^i}\right)\frac{1}{u_0}\right\}$.

Clear that the forbidden set F is a sequence of hyperbolas contained entirely in the interiors of the 2^{nd} and the 4^{th} quadrant of the planes u_0u_{-2} and $u_{-1}u_{-3}$ of the four dimensional Euclidean space

$$\mathbb{R}^4 = \{(u_0, u_{-1}, u_{-2}, u_{-3}), u_{-i} \in \mathbb{R},\ i = 0, 1, 2, 3\}\,.$$

That is the forbidden set is a sequence of hyperbolas contained entirely in the set

$$\{(u_0, u_{-1}, u_{-2}, u_{-3}), u_{-1}u_{-3} < 0\} \cup \{(u_0, u_{-1}, u_{-2}, u_{-3}), u_0u_{-2} < 0\}\,.$$

We define $\alpha_i = x_{-2+i}x_{-4+i}$, $i = 1, 2$.

Theorem 2.2. *Let x_{-3}, x_{-2}, x_{-1} and x_0 be real numbers such that $(x_0, x_{-1}, x_{-2}, x_{-3}) \notin F$. If $a \neq b$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation* (1.1) *is*

$$x_n = \begin{cases} x_{-3}\prod_{j=0}^{\frac{n-1}{4}} \frac{(\frac{b}{a})^{2j}\theta_1+c}{(\frac{b}{a})^{2j+1}\theta_1+c}\,, & n = 1, 5, 9, \ldots \\ x_{-2}\prod_{j=0}^{\frac{n-2}{4}} \frac{(\frac{b}{a})^{2j}\theta_2+c}{(\frac{b}{a})^{2j+1}\theta_2+c}\,, & n = 2, 6, 10, \ldots \\ x_{-1}\prod_{j=0}^{\frac{n-3}{4}} \frac{(\frac{b}{a})^{2j+1}\theta_1+c}{(\frac{b}{a})^{2j+2}\theta_1+c}\,, & n = 3, 7, 11, \ldots \\ x_0\prod_{j=0}^{\frac{n-4}{4}} \frac{(\frac{b}{a})^{2j+1}\theta_2+c}{(\frac{b}{a})^{2j+2}\theta_2+c}\,, & n = 4, 8, 12, \ldots \end{cases} \tag{2.4}$$

where $\theta_i = \frac{a-b-c\alpha_i}{\alpha_i}$, $\alpha_i = x_{-2+i}x_{-4+i}$, and $i = 1, 2$.

Proof. We can write the given solution as

$$x_{4m+1} = x_{-3}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1 + c}{(\frac{b}{a})^{2j+1}\theta_1 + c}\,, \quad x_{4m+2} = x_{-2}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_2 + c}{(\frac{b}{a})^{2j+1}\theta_2 + c}\,,$$

$$x_{4m+3} = x_{-1}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j+1}\theta_1 + c}{(\frac{b}{a})^{2j+2}\theta_1 + c}\,, \quad x_{4m+4} = x_0\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j+1}\theta_2 + c}{(\frac{b}{a})^{2j+2}\theta_2 + c}, \quad m = 0, 1, \ldots$$

It is easy to check the result when $m=0$. Suppose that the result is true for $m>0$.

Then

$$x_{4(m+1)+1} = \frac{ax_{4m+1}}{b+cx_{4m+1}x_{4m+3}} = \frac{ax_{-3}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1+c}{(\frac{b}{a})^{2j+1}\theta_1+c}}{b+cx_{-3}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1+c}{(\frac{b}{a})^{2j+1}\theta_1+c}x_{-1}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j+1}\theta_1+c}{(\frac{b}{a})^{2j+2}\theta_1+c}}$$

$$= \frac{ax_{-3}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1+c}{(\frac{b}{a})^{2j+1}\theta_1+c}}{b+cx_{-3}(\prod_{j=0}^{m}(\frac{b}{a})^{2j}\theta_1+c)x_{-1}\prod_{j=0}^{m}\frac{1}{(\frac{b}{a})^{2j+2}\theta_1+c}}$$

$$= \frac{ax_{-3}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1+c}{(\frac{b}{a})^{2j+1}\theta_1+c}}{b+cx_{-1}x_{-3}(\theta_1+c)(\frac{1}{(\frac{b}{a})^{2m+2}\theta_1+c})}$$

$$= \frac{ax_{-3}((\frac{b}{a})^{2m+2}\theta_1+c)\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1+c}{(\frac{b}{a})^{2j+1}\theta_1+c}}{b((\frac{b}{a})^{2m+2}\theta_1+c)+c\alpha_1(\theta_1+c)}$$

$$= \frac{ax_{-3}((\frac{b}{a})^{2m+2}\theta_1+c)\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1+c}{(\frac{b}{a})^{2j+1}\theta_1+c}}{b((\frac{b}{a})^{2m+2}\theta_1+c)+c(a-b)}$$

$$= \frac{x_{-3}((\frac{b}{a})^{2m+2}\theta_1+c)\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1+c}{(\frac{b}{a})^{2j+1}\theta_1+c}}{\frac{b}{a}((\frac{b}{a})^{2m+2}\theta_1+c)+\frac{c}{a}(a-b)}$$

$$= x_{-3}\frac{(\frac{b}{a})^{2m+2}\theta_1+c)}{((\frac{b}{a})^{2m+3}\theta_1+c)}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1+c}{(\frac{b}{a})^{2j+1}\theta_1+c}$$

$$= x_{-3}\prod_{j=0}^{m+1}\frac{(\frac{b}{a})^{2j}\theta_1+c}{(\frac{b}{a})^{2j+1}\theta_1+c}.$$

Similarly we can show that

$$x_{4(m+1)+2} = x_{-2}\prod_{j=0}^{m+1}\frac{(\frac{b}{a})^{2j}\theta_2+c}{(\frac{b}{a})^{2j+1}\theta_2+c}, \qquad x_{4(m+1)+3} = x_{-1}\prod_{j=0}^{m+1}\frac{(\frac{b}{a})^{2j+1}\theta_1+c}{(\frac{b}{a})^{2j+2}\theta_1+c}$$

and

$$x_{4(m+1)+4} = x_0\prod_{j=0}^{m+1}\frac{(\frac{b}{a})^{2j+1}\theta_2+c}{(\frac{b}{a})^{2j+2}\theta_2+c}.$$

This completes the proof.

$\square$

3. GLOBAL BEHAVIOR OF EQUATION (1.1)

In this section, we investigate the global behavior of equation (1.1) with $a \neq b$, using the explicit formula of its solution.
We can write the solution of equation (1.1) as

$$x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \beta(j,t,i)\,,$$

where $\beta(j,t,i) = \frac{(\frac{b}{a})^{2j+t}\theta_i + c}{(\frac{b}{a})^{2j+t+1}\theta_i + c}$, $t \in \{0,1\}$ and $i \in \{1,2\}$.

In the following theorem, suppose that $\alpha_i \neq \frac{a-b}{c}$ for all $i \in \{1,2\}$.

Theorem 3.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation* (1.1) *such that $(x_0, x_{-1}, x_{-2}, x_{-3}) \notin F$. Then the following statements are true.*

(1) *If $a < b$, then $\{x_n\}_{n=-3}^{\infty}$ converges to* 0.

(2) *If $a > b$, then $\{x_n\}_{n=-3}^{\infty}$ converges to a period-4 solution.*

Proof.

(1) If $a < b$, then $\beta(j,t,i)$ converges to $\frac{a}{b} < 1$ as $j \to \infty$, for all $t \in \{0,1\}$ and $i \in \{1,2\}$. So, for every pair $(t,i) \in \{0,1\} \times \{1,2\}$ we have for a given $0 < \epsilon < 1$ that, there exists $j_0(t,i) \in \mathbb{N}$ such that, $\mid \beta(j,t,i) \mid < \epsilon$ for all $j \geq j_0(t,i)$. If we set $j_0 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_0(t,i)$, then for all $t \in \{0,1\}$ and $i \in \{1,2\}$ we get

$$\begin{aligned} |x_{4m+2t+i}| &= |x_{-4+2t+i}|\,|\prod_{j=0}^{m} \beta(j,t,i)| \\ &= |x_{-4+2t+i}|\,|\prod_{j=0}^{j_0-1} \beta(j,t,i)|\,|\prod_{j=j_0}^{m} \beta(j,t,i)| \\ &< |x_{-4+2t+i}|\,|\prod_{j=0}^{j_0-1} \beta(j,t,i)|\epsilon^{m-j_0+1}\,. \end{aligned}$$

As m tends to infinity, the solution $\{x_n\}_{n=-3}^{\infty}$ converges to 0.

(2) If $a > b$, then $\beta(j,t,i) \to 1$ as $j \to \infty$, $t \in \{0,1\}$ and $i \in \{1,2\}$. This implies that, for every pair $(t,i) \in \{0,1\} \times \{1,2\}$ there exists $j_1(t,i) \in \mathbb{N}$ such that, $\beta(j,t,i) > 0$ for all $j \geq j_1(t,i)$. If we set $j_1 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_1(t,i)$, then for all $t \in \{0,1\}$ and $i \in \{1,2\}$ we get

$$\begin{aligned} x_{4m+2t+i} &= x_{-4+2t+i} \prod_{j=0}^{m} \beta(j,t,i) \\ &= x_{-4+2t+i} \prod_{j=0}^{j_1-1} \beta(j,t,i) \exp\Big(\sum_{j=j_1}^{m} \ln\big(\beta(j,t,i)\big) \Big)\,. \end{aligned}$$

We shall test the convergence of the series $\sum_{j=j_1}^{\infty} |\ln\big(\beta(j,t,i)\big)|$. Since for all $t \in \{0,1\}$ and $i \in \{1,2\}$ we have $\lim_{j\to\infty} \left|\frac{\ln(\beta(j+1,t,i)}{\ln(\beta(j,t,i)}\right| = \frac{0}{0}$, using L'Hospital's rule we obtain

$$\lim_{j\to\infty} \left|\frac{\ln\beta(j+1,t,i)}{\ln\beta(j,t,i)}\right| = \left(\frac{b}{a}\right)^2 < 1\,.$$

It follows from the ratio test that the series $\sum_{j=j_1}^{\infty} |\ln\beta(j,t,i)|$ is convergent. This ensures that there are four positive real numbers ν_{ti}, $t \in \{0,1\}$ and $i \in \{1,2\}$ such that

$$\lim_{m\to\infty} x_{4m+2t+i} = \nu_{ti}\,, \quad t \in \{0,1\} \quad \text{and} \quad i \in \{1,2\}$$

where

$$\nu_{ti} = x_{-4+2t+i} \prod_{j=0}^{\infty} \frac{(\frac{b}{a})^{2j+t}\theta_i + c}{(\frac{b}{a})^{2j+t+1}\theta_i + c}\,, \quad t \in \{0,1\} \quad \text{and} \quad i \in \{1,2\}\,.$$

□

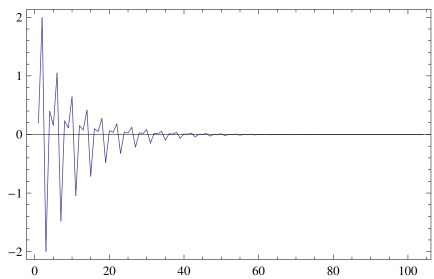

FIG. 1: $x_{n+1} = \frac{2x_{n-3}}{3+x_{n-1}x_{n-3}}$

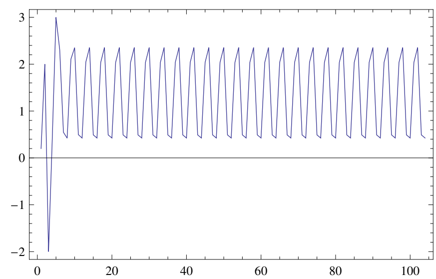

FIG. 2: $x_{n+1} = \frac{3x_{n-3}}{1+2x_{n-1}x_{n-3}}$

Example 1. Figure 1 shows that if $a = 2$, $b = 3$, $c = 1$ $(a < b)$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) with initial conditions $x_{-3} = 0.2$, $x_{-2} = 2$, $x_{-1} = -2$ and $x_0 = 0.4$ converges to zero.

Example 2. Figure 2 shows that if $a = 3$, $b = 1$, $c = 2$ $(a > b)$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) with initial conditions $x_{-3} = 0.2$, $x_{-2} = 2$, $x_{-1} = -2$ and $x_0 = 0.4$ converges to a period-4 solution.

4. CASE $a = b = c$

In this section, we investigate the behavior of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}x_{n-3}}\,, \qquad n = 0, 1, \ldots \tag{4.1}$$

Lemma 4.1. *The forbidden set G of equation* (1.1) *is*
$G=\bigcup_{n=0}^{\infty}\left\{(u_0,u_{-1},u_{-2},u_{-3}):u_{-3}=-\left(\frac{1}{n+1}\right)\frac{1}{u_{-1}}\right\}\cup\bigcup_{m=0}^{\infty}\left\{(u_0,u_{-1},u_{-2},u_{-3}):\right.$
$\left.u_{-2}=-\left(\frac{1}{m+1}\right)\frac{1}{u_0}\right\}.$

Theorem 4.2. *Let x_{-3},x_{-2},x_{-1} and x_0 be real numbers such that $(x_0,\ x_{-1},\ x_{-2},\ x_{-3})\notin G$. Then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation* (4.1) *is*

$$x_n=\begin{cases} x_{-3}\prod_{j=0}^{\frac{n-1}{4}}\frac{1+(2j)\alpha_1}{1+(2j+1)\alpha_1}\,, & n=1,5,9,\dots\\ x_{-2}\prod_{j=0}^{\frac{n-2}{4}}\frac{1+(2j)\alpha_2}{1+(2j+1)\alpha_2}\,, & n=2,6,10,\dots\\ x_{-1}\prod_{j=0}^{\frac{n-3}{4}}\frac{1+(2j+1)\alpha_1}{1+(2j+2)\alpha_1}\,, & n=3,7,11,\dots\\ x_{0}\prod_{j=0}^{\frac{n-4}{4}}\frac{1+(2j+1)\alpha_2}{1+(2j+2)\alpha_2}\,, & n=4,8,12\dots \end{cases} \tag{4.2}$$

Proof. The proof is similar to that of Theorem 2.2 and will be omitted. □

We can write the solution of equation (4.1) as

$$x_{4m+2t+i}=x_{-4+2t+i}\prod_{j=0}^{m}\gamma(j,t,i)\,,$$

where $\gamma(j,t,i)=\frac{1+(2j+t)\alpha_i}{1+(2j+t+1)\alpha_i}$, $t\in\{0,1\}$ and $i\in\{1,2\}$.

In the following theorem, suppose that $\alpha_i\neq 0$ for all $i\in\{1,2\}$.

Theorem 4.3. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation* (4.1) *such that $(x_0,x_{-1},x_{-2},x_{-3})\notin G$. Then $\{x_n\}_{n=-3}^{\infty}$ converges to* 0.

Proof. It is clear that $\gamma(j,t,i)\to 1$ as $j\to\infty$, $t\in\{0,1\}$ and $i\in\{1,2\}$. This implies that, for every pair $(t,i)\in\{0,1\}\times\{1,2\}$ there exists $j_2(t,i)\in\mathbb{N}$ such that, $\gamma(j,t,i)>0$ for all $j\geq j_2(t,i)$. If we set $j_2=\max_{0\leq t\leq 1,1\leq i\leq 2}j_2(t,i)$, then for all $t\in\{0,1\}$ and $i\in\{1,2\}$ we get

$$\begin{aligned} x_{4m+2t+i}&=x_{-4+2t+i}\prod_{j=0}^{m}\gamma(j,t,i)\\ &=x_{-4+2t+i}\prod_{j=0}^{j_2-1}\gamma(j,t,i)\exp\Big(-\sum_{j=j_2}^{m}\ln\frac{1}{\gamma(j,t,i)}\Big)\,. \end{aligned}$$

We shall show that $\sum_{j=j_2}^{\infty}\ln\frac{1}{\gamma(j,t,i)}=\sum_{j=j_2}^{\infty}\ln\frac{1+(2j+t+1)\alpha_i}{1+(2j+t)\alpha_i}=\infty$, by considering the series $\sum_{j=j_2}^{\infty}\frac{\alpha_i}{1+\alpha_i(2j+t)}$. As

$$\lim_{j\to\infty}\frac{1/\gamma(j,t,i)}{\alpha_i/(1+\alpha_i(2j+t))}=\lim_{j\to\infty}\frac{\ln\left((1+\alpha_i(2j+t+1))/(1+\alpha_i(2j+t))\right)}{\alpha_i/(1+\alpha_i(2j+t))}=1,$$

using the limit comparison test, we get $\sum_{j=j_2}^{\infty} \ln \frac{1}{\gamma(j,t,i)} = \infty$.
Therefore,

$$x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{j_2-1} \gamma(j,t,i) \exp\Big(- \sum_{j=j_2}^{m} \ln \frac{1}{\gamma(j,t,i)}\Big)$$

converges to zero as $m \to \infty$. □

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