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SOME DYNAMIC INEQUALITIES APPLICABLE TO PARTIAL INTEGRODIFFERENTIAL EQUATIONS ON TIME SCALES

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Abstract. The main objective of the paper is to study explicit bounds of certain dynamic integral inequalities on time scales. Using these inequalities we prove the uniqueness of some partial integrodifferential equations on time scales.

1. Introduction

The study time scale calculus was initiated in 1989 by Stefan Hilger [5] a German Mathematician in his Ph.D dissertation. Since then many authors have applied time scales calculus for various applications in Mathematics. Mathematical inequalities on time scales plays very important role. Recently many authors have studied various properties of dynamic inequalities on time scales [2, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Let $\mathbb{R}$ denotes the real number $\mathbb{Z}$ the set of integers and $\mathbb{T}$ denotes the arbitrary time scales. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales, and let $\Omega = \mathbb{T}_1 \times \mathbb{T}_2$. The rd-continuous function is denoted by C_{rd}. We denote the partial delta derivative of $w(x,y)$ with respect to x, y and xy for $(x,y) \in \Omega$ by $w^{\Delta_1}(x,y)$, $w^{\Delta_2}(x,y)$, $w^{\Delta_1\Delta_2}(x,y) = w^{\Delta_2\Delta_1}(x,y)$. The basic information about time scales calculus can be found in [1, 3, 4, 5].

2. Main results

In [7] the author has obtained some estimates of the some Gronwall like inequalities while in [6, 14] the authors have studied some nonlinear dynamic integral inequalities on time scales. Motivated by the above research work in this paper we obtain some explicit bounds of certain dynamic inequalities on time scales.

Theorem 2.1. *Let $u(t)$, $h(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}_+)$, $p(\tau, s) \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$ defined for τ, $s \in \mathbb{T}$, and $c \geq 0$ be a constant. If*

$$u(t) \leq c + \int_{t_0}^{t} \Big[h(\tau) u(\tau) + \int_{t_0}^{\tau} p(\tau, s) u(s) \Delta s \Big] \Delta \tau \,, \tag{2.1}$$

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for $t \in \mathbb{T}$. *Then*

$$u(t) \le c e_{H_1}(t, t_0)\,, \tag{2.2}$$

where

$$H_1(t) = h(t) + \int_{t_0}^{t} p(t,\tau)\Delta\tau, \tag{2.3}$$

Proof. Let $c > 0$ and define a function $w(t)$ by right hand side of (2.1). Then $w(t) > 0$ and non decreasing for $w(t_0) = c$, $u(t) \le w(t)$ and

$$\begin{aligned} w^{\Delta}(t) &= h(t)\,u(t) + \int_{t_0}^{t} p(t,\tau)\,u(\tau)\Delta\tau \\ &\le h(t)\,w(t) + \int_{t_0}^{t} p(t,\tau)\,w(\tau)\Delta\tau \\ &\le w(t)\left[h(t) + \int_{t_0}^{t} p(t,\tau)\Delta\tau\right], \end{aligned}$$

$$\frac{w^{\Delta}(t)}{w(t)} \le h(t) + \int_{t_0}^{t} p(t,\tau)\Delta\tau\,. \tag{2.4}$$

Integrating (2.4) we get

$$w(t) \le c e_{H_1}(t, t_0)\,. \tag{2.5}$$

Since $u(t) \le w(t)$ we get (2.4). □

Theorem 2.2. *Let* $u(t)$, $h(t)$, $p(t,\tau)$ *be as in Theorem 2.1 and* $f(t)$, $g(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}_+)$, $p(\tau, s) \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$. *If*

$$u(t) \le f(t) + g(t) \int_{t_0}^{t} \left[h(\tau)\,u(\tau) + \int_{t_0}^{t} p(\tau,s)\,u(s)\,\Delta s\right]\Delta t\,, \tag{2.6}$$

for $t \in \mathbb{T}$, *then*

$$u(t) \le f(t) + g(t)\,K_2(t)\,e_{H_2}(t, t_0)\,, \tag{2.7}$$

where

$$H_2(t) = h(t)\,g(t) + \int_{t_0}^{t} p(t,\tau) g(\tau)\,\Delta\tau\,, \tag{2.8}$$

and

$$K_2(t) = \int_{t_0}^{t} \left[h(\tau)\,f(\tau) + \int_{t_0}^{t} p(\tau,s)\,f(s)\,\Delta s\right]\Delta\tau\,. \tag{2.9}$$

Proof. Define a function $w(t)$ by right hand side of equation (2.6)

$$w(t) = \int_{t_0}^{t} \left[h(\tau)\,u(\tau) + \int_{t_0}^{t} p(\tau,s)\,u(s)\,\Delta s\right]\Delta t\,, \tag{2.10}$$

then (2.6) becomes

$$u(t) \le f(t) + g(t) w(t)\,. \tag{2.11}$$

From (2.10) and (2.11) we get

$$w(t) \le \int_{t_0}^{t} \Big[h(\tau)\{f(\tau)+g(\tau)w(\tau)\} + \int_{t_0}^{\tau} p(\tau,s)\{f(s)+g(s)w(s)\}\Delta s\Big]\Delta\tau$$
$$= \int_{t_0}^{t} \Big[h(\tau)f(\tau) + \int_{t_0}^{t} p(\tau,s)f(s)\Delta s\Big]\Delta\tau$$
$$+ \int_{t_0}^{t} \Big[h(\tau)\, g(\tau)\, w(\tau) + \int_{t_0}^{t} p(\tau,s)\, g(s)\, w(s)\,\Delta s\Big]\Delta\tau$$
$$\le n(t) + \int_{t_0}^{t} \Big[h(\tau)\, g(\tau)\, w(\tau) + \int_{t_0}^{t} p(\tau,s)\, g(s)\, w(s)\,\Delta s\Big]\Delta\tau\,, \tag{2.12}$$

where

$$n(t) = \epsilon + \int_{t_0}^{t} \Big[h(\tau)\, f(\tau) + \int_{t_0}^{t} p(\tau,s)\, f(s)\,\Delta s\Big]\Delta\tau\,,$$

where ϵ is arbitrarily small constant.

From (2.12) we have

$$\frac{w(t)}{n(t)} \le 1 + \int_{t_0}^{t} \Big[h(\tau)\, g(\tau)\,\frac{w(\tau)}{n(\tau)} + \int_{t_0}^{t} p(\tau,s)\, g(s)\,\frac{w(s)}{n(s)}\Delta s\Big]\Delta\tau\,. \tag{2.13}$$

Now applying Theorem (2.1) to (2.13) yields

$$w(t) \le n(t)\, e_{H_2}(t,t_0)\ . \tag{2.14}$$

Using (2.14) in (2.11) and letting $\epsilon \to 0$, we get (2.7). □

Theorem 2.3. *Let $a_i \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}_+)$, $i=1,2$ satisfying*

$$0 \le a_i(t,u) - a_i(t,v) \le b_i(t,v)(u-v)\ , \tag{2.15}$$

for $u \ge v \ge 0$ where $b_i(t,v)$ are non negative rd-continuous function. If

$$u(t) \le f(t) + g(t)\int_{t_0}^{t} \Big[h(\tau)\, a_1(\tau,u(\tau)) + \int_{t_0}^{t} p(\tau,s)\, a_2(s,u(s))\,\Delta s\Big]\Delta\tau\,, \tag{2.16}$$

$t \in \mathbb{T}$ then

$$u(t) \le f(t) + g(t)K_3(t)\, e_{H_3}(t,t_0)\ , \tag{2.17}$$

$$H_3(t) = h(t)\, a_1(t,f(t)) + \int_{t_0}^{t} p(t,\tau)\, a_2(t,f(\tau))\,\Delta\tau\,, \tag{2.18}$$

and

$$K_3(t) = h(t)\, a_1(t,f(t))\, g(t) + \int_{t_0}^{t} p(t,\tau)\, b_2(t,f(\tau))\, g(\tau)\,\Delta\tau\,, \tag{2.19}$$

for $t \in \mathbb{T}$.

Proof. Define a function $w(t)$ by

$$w(t) = \int_{t_0}^{t} \Big[h(\tau)\, a_1(\tau,u(\tau)) + \int_{t_0}^{s} p(\tau,s)\, a_2(\tau,u(s))\Big]\Delta\tau\,, \tag{2.20}$$

then from (2.16) we have

$$u(t) \leq f(t) + g(t)\, w(t)\ , \tag{2.21}$$

From (2.20), (2.21) and (2.15) we have

$$\begin{aligned}
w(t) \leq & \int_{t_0}^{t} \Big[h(\tau)\{a_1(\tau, f(\tau) + g(\tau)w(\tau)) - a_1(\tau, f(\tau) + a_1(\tau, f(\tau)))\}\Delta\tau \\
& + \int_{t_0}^{s} p(\tau, s\{a_2\big(s, f(s) + g(s)w(s)\big) - a_2(s, f(s) + a_2(s, f(s)))\}\Delta s\Big]\Delta\tau \\
\leq & \int_{t_0}^{t} \Big[h(\tau)b_1(\tau, f(\tau))g(\tau)w(\tau) + \int_{t_0}^{s} p(\tau, s)b_2(s, f(s))g(s)w(s)\Delta s\Big]\Delta\tau \\
& + \int_{t_0}^{t} \Big[h(\tau)a_1(\tau, f(\tau)) + \int_{t_0}^{s} p(\tau, s)a_2(s, f(s))\Delta s\Big]\Delta\tau \\
\leq & N(t) + \int_{t_0}^{t} \Big[h(\tau)b_1\big(\tau, f(\tau)\big)g(\tau)w(\tau) \\
& + \int_{t_0}^{s} p(\tau, s)b_2(s, f(s))g(s)w(s)\Delta s\Big]\Delta\tau\,,
\end{aligned} \tag{2.22}$$

where

$$N(t) = \epsilon + \int_{t_0}^{t} \Big[h(\tau)a_1\big(\tau, f(\tau)\big) + \int_{t_0}^{s} p(\tau, s)a_2\big(s, f(s)\big)\Delta s\Big]\Delta\tau\,, \tag{2.23}$$

where $\epsilon > 0$ is an arbitrary small constant. Using the same steps as in Theorem 2.2 we get the result. □

Theorem 2.4. *Let $u(x, y), h(x, y) \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$ be nonnegative functions defined for $x,\ y,\ s,\ t \in \mathbb{T}$ and $p(x, y, s, t) \in C_{\mathrm{rd}}(\Omega \times \Omega, \mathbb{R}_+)$. If*

$$\begin{aligned}
u(x, y) \leq c + & \int_{x_0}^{x}\int_{x_0}^{y} \Big[h(s, t)u(s, t) \\
& + \int_{x_0}^{s}\int_{x_0}^{t} p(s, t, \xi, \eta)u(s, t)\Delta\eta\, \Delta\xi\Big]\Delta t\, \Delta s\,,
\end{aligned} \tag{2.24}$$

for $(x, y) \in \Omega$ then

$$u(x, y) \leq c e_{\overline{H}_1}(x, x_0)\,, \tag{2.25}$$

where

$$\overline{H}_1(x, y) = \int_{x_0}^{y} \Big[h(x, t) + \int_{x_0}^{x}\int_{x_0}^{t} p(x, t, \xi, \eta)\Delta\eta\, \Delta\xi\Big]\,. \tag{2.26}$$

Proof. Let $c > 0$ and define a function $w(x,y)$ by right hand side of (2.24) then $w(x,y) > 0$, $w(x_0,y) = w(x,x_0) = c$, $u(x,y) \le w(x,y)$ and

$$\begin{aligned} w^{\Delta}(x,y) &= \int_{x_0}^{y} \Big[h(x,t)u(x,t) + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)u(\xi,\eta)\Delta\eta\,\Delta\xi \Big]\Delta t \\ &\le \int_{x_0}^{y} \Big[h(x,t)w(x,t) + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)w(\xi,\eta)\Delta\eta\,\Delta\xi \Big]\Delta t \\ &\le w(x,y)\int_{x_0}^{y} \Big[h(x,t) + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)\Delta\eta\,\Delta\xi \Big]\Delta t\,, \end{aligned}$$

i.e

$$\frac{w^{\Delta}(x,y)}{w(x,y)} \le \int_{x_0}^{y} \Big[h(x,t) + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)\Delta\eta\,\Delta\xi \Big]\Delta t\,. \tag{2.27}$$

Keeping y fixed in (2.27) , set $x = s$ and integrating it with respect to s from x_0 to x we get

$$w\,(x,y) \le c e_{\overline{H}_1}\,(x,x_0)\ . \tag{2.28}$$

Using (2.28) in $u(x,y) \le w(x,y)$ we get the result in (2.25). ☐

Theorem 2.5. *Let $u(x,y)$, $h(x,y)$, $p(x,y,s,t)$, c as in Theorem 2.4 and $f(x,y)$, $g(x,y) \in C_{\mathrm{rd}}(\Omega,\mathbb{R}_+)$. If*

$$\begin{aligned} u(x,y) \le f(x,y) + g(x,y) &\int_{x_0}^{x}\int_{x_0}^{y} \Big[h(s,t)u(s,t) \\ &+ \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)u(\xi,\eta)\Delta\eta\,\Delta\xi \Big]\Delta t\,\Delta s\,, \end{aligned} \tag{2.29}$$

for $(x,y) \in \Omega$ then

$$u\,(x,y) \le f\,(x,y) + g\,(x,y)\,\overline{K}_2\,(x,y)\,e_{\overline{H}_2}\,(x,x_0)\ , \tag{2.30}$$

$$\overline{H}_2(x,y) = \int_{x_0}^{y} \Big[h(x,t)g(s,t) + \int_{x_0}^{x}\int_{x_0}^{t} p(x,t,\xi,\eta)g(\xi,\eta)\Delta\eta\,\Delta\xi \Big]\Delta s\,, \tag{2.31}$$

$$\overline{K}_2(x,y) = \int_{x_0}^{x}\int_{x_0}^{y} \Big[h(x,t) + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)u(\xi,\eta)\Delta\eta\,\Delta\xi \Big]\Delta t\,\Delta s\,. \tag{2.32}$$

Proof. Define a function $w(x,y)$ by

$$w(x,y) = \int_{x_0}^{x}\int_{x_0}^{y} \Big[h(x,t)u(s,t) + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)u(\xi,\eta)\Delta\eta\,\Delta\xi \Big]\Delta t\,\Delta s\,, \tag{2.33}$$

then (2.29) we have

$$u\,(x,y) \le f\,(x,y) + g\,(x,y)\,w\,(x,y)\ . \tag{2.34}$$

From (2.33) and (2.34) we have

$$
\begin{aligned}
w(x,y) \leq & \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)\big\{f(s,t)+g(s,t)w(s,t)\big\} \\
& + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)\big\{f(s,t)g(s,t)w(s,t)\big\}\Delta\eta\,\Delta\xi\Big]\,\Delta t\,\Delta s \\
\leq & \int_{x_0}^{x}\int_{x_0}^{y}\Big[\Big[h(s,t)f(s,t) + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)f(s,t)\Delta\eta\,\Delta\xi\Big]\,\Delta t\,\Delta s \\
& + \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)g(s,t)w(s,t) \\
& + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)g(\xi,\eta)w(\xi,\eta)\Delta\eta\,\Delta\xi\Big]\,\Delta t\,\Delta s \\
\leq & \,\overline{n}(x,y) + \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)g(s,t)w(s,t) \\
& + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)g(\xi,\eta)w(\xi,\eta)\Delta\eta\,\Delta\xi\Big]\,\Delta t\,\Delta s\,,
\end{aligned}
\tag{2.35}
$$

where

$$
\begin{aligned}
\overline{n}(x,y) = \varepsilon + & \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)f(s,t) \\
& + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)f(\xi,\eta)\Delta\eta\,\Delta\xi\Big]\,\Delta t\,\Delta s\,,
\end{aligned}
\tag{2.36}
$$

and $\epsilon > 0$ is an arbitrary small constant $\overline{n}\,(x,y)$ is positive, rd-continuous and nondecreasing. From (2.35) we have

$$
\begin{aligned}
\frac{w(x,y)}{\overline{n}(x,y)} \leq 1 + & \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)g(s,t)\frac{w(s,t)}{\overline{n}(s,t)} \\
& + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)g(\xi,\eta)\frac{w(s,t)}{\overline{n}(s,t)}\,\Delta\eta\,\Delta\xi\Big]\,\Delta t\,\Delta s\,.
\end{aligned}
\tag{2.37}
$$

Now an application of Theorem 2.4 to (2.37) we have

$$
w\,(x,y) \leq \overline{n}\,(x,y)\,e_{\overline{H}_2}\,(x,x_0)\;. \tag{2.38}
$$

Using (2.38), (2.34) and letting $\epsilon \to 0$ we get (2.30). □

Theorem 2.6. *Let $u(x,y)$, $f(x,y)$, $g(x,y)$, $h(x,y)$, $p(x,y,s,t)$ be as in Theorem 3.2. Let $A_i \in C_{\mathrm{rd}}\,(\Omega\times\mathbb{T},\mathbb{R}_+)$, $I=1,2$ such that*

$$
0 \leq A_i\,(x,y,u) - A_i\,(x,y,v) \leq B_i\,(x,y,v)\,(u-v)\;, \tag{2.39}
$$

for $0 \le v \le u$ *where* B_i *nonnegative* rd-*continuous. If*

$$u(x,y) \le f(x,y) + g(x,y) \int_{x_0}^{x} \int_{x_0}^{y} \Big[(s,t) A_1\big(x, y, u(s,t)\big)$$

$$+ \int_{x_0}^{s} \int_{x_0}^{t} p(s,t,\xi,\eta) A_1\big(x, y, u(\xi,\eta)\big) \Delta\eta \, \Delta\xi \Big] \Delta t \, \Delta s \,, \tag{2.40}$$

for $(x,y) \in \omega$ *then*

$$u\,(x,y) \le f\,(x,y) + g\,(x,y)\, \overline{K}_3\,(x,y)\, e_{\overline{H}_3}\,(x,x_0) \tag{2.41}$$

where

$$\overline{H}_3\,(x,y) = \int_{x_0}^{x} \int_{x_0}^{y} \Big[h(s,t) B_1\big(s,t,f(s,t)\big) g(s,t)$$

$$+ \int_{x_0}^{s} \int_{x_0}^{t} p(s,t,\xi,\eta) B_2\big(\xi,\eta,f(\xi,\eta)\big) g(\xi,\eta) \Delta\eta \, \Delta\xi \Big] \Delta t \, \Delta s \,, \tag{2.42}$$

and

$$\overline{K}_3(x,y) = \int_{x_0}^{x} \int_{x_0}^{y} \Big[h(s,t) A_1\big(s,t,f(s,t)\big) g(s,t)$$

$$+ \int_{x_0}^{s} \int_{x_0}^{t} p(s,t,\xi,\eta) A_2\big(\xi,\eta,f(\xi,\eta)\big) g(\xi,\eta) \Delta\eta \, \Delta\xi \Big] \Delta t \, \Delta s \,, \tag{2.43}$$

for $(x,y) \in \Omega$.

Proof. Define a function $w(x,y)$ by

$$w(x,y) = \int_{x_0}^{x} \int_{x_0}^{y} \Big[h(s,t) A_1\big(s,t,u(s,t)\big)$$

$$+ \int_{x_0}^{s} \int_{x_0}^{t} p(s,t,\xi,\eta) A_2\big(\xi,\eta,u\big(\xi,\eta\big)\big) \Delta\eta \, \Delta\xi \Big] \Delta t \, \Delta s \,, \tag{2.44}$$

then we have (2.44)

$$w\,(x,y) \le f\,(x,y) + g\,(x,y)\, w\,(x,y)\ . \tag{2.45}$$

From (2.43) and (2.44) we have

$$w\,(x,y) \le \int_{x_0}^{x} \int_{x_0}^{y} \Big[h(s,t) A_1\big(s,t,f(x,y) + g(x,y) w(x,y)\big)$$

$$+ \int_{x_0}^{s} \int_{x_0}^{t} p(s,t,\xi,\eta) A_2\big(\xi,\eta,f(\xi,\eta) + g(\xi,\eta) w(\xi,\eta)\big) \Delta\eta \, \Delta\xi \Big] \Delta t \, \Delta s$$

$$
\begin{aligned}
&= \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)A_1\big(s,t,f(x,y)\big) \\
&\quad + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)A_2\big(\xi,\eta,f(\xi,\eta)\big)\Delta\eta\,\Delta\xi\Big]\Delta t\,\Delta s \\
&\quad + \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)A_1\big(s,t,g(x,y)w(x,y)\big) \\
&\quad + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)A_2\big(\xi,\eta,g(\xi,\eta)w(\xi,\eta)big)\Delta\eta\,\Delta\xi\Big]\Delta t\,\Delta s \\
&\leq \overline{N}(x,y) + \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)A_1\big(s,t,g(x,y)w(x,y)\big) \\
&\quad + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)A_2\big(\xi,\eta,g(\xi,\eta)w(\xi,\eta)\big)\Delta\eta\,\Delta\xi\Big]\,\Delta t\,\Delta s\,, \qquad (2.46)
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{N}(x,y) &= \varepsilon + \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)A_1\big(s,t,f(x,y)\big) \\
&\quad + \int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)A_2\big(\xi,\eta,f(\xi,\eta)\big)\Delta\eta\,\Delta\xi\Big]\,\Delta t\,\Delta s\,, \qquad (2.47)
\end{aligned}
$$

in which ϵ arbitrary small constant.

Clearly $\overline{N}\,(x,y)$ is positive, rd-continuous and nondecreasing. Remaining part of the proof can be done similarly as in Theorem 2.5. □

Remark. Theorem 2.2 and Theorem 2.5 are special cases of Theorem 2.3 and Theorem 2.6 respectively with $a(t,u)=u$.

3. Applications

In this section we give some applications of inequality proved in theorem to obtain the uniqueness and bound on the solution for dynamic partial integrodifferential equation of the form

$$
V^{\Delta_2\Delta_1}(x,y) = F\big(x,y,V(x,y)\big) + \int_{x_0}^{x}\int_{x_0}^{y} G\big(x,y,\xi,\eta,V(\xi,\eta)\big)\Delta\eta\,\Delta\xi\,, \qquad (3.1)
$$

$$
w\,(x,x_0) = k_1\,(x)\;, \quad w\,(x_0,y) = k_2\,(y)\;, \quad k_1\,(x_0) = k_2\,(x_0) = 0\,, \qquad (3.2)
$$

where $F \in C_{\mathrm{rd}}\left(\Omega^2 \times \mathbb{R}_+ \to \mathbb{R}_+\right)$, $G \in C_{\mathrm{rd}}\left(\Omega^2 \times \Omega^2 \times \mathbb{R}_+ \to \mathbb{R}_+\right)$.

Now we give the bounds on (3.1) and (3.2).

Theorem 3.1. *Suppose that*

$$|F(x,y,V(x,y))| \le h(x,y)\,|V(x,y)|\ , \tag{3.3}$$

$$|G(x,y,\xi,\eta,V(\xi,\eta))| \le p(x,y,\xi,\eta)\,|V(\xi,\eta)|\ , \tag{3.4}$$

$$|k_1(x)+k_2(x)| \le c\,, \tag{3.5}$$

where h, p, c are as defined in Theorem 2.2. If $w(x,y)$ is solution of (3.1) *and* (3.2) *then*

$$|w(x,y)| \le c e_{\overline{H}_1}(x,x_0)\ , \tag{3.6}$$

for $(x,y)\in\Omega$ where $\overline{H}_1$ is given by (2.26).

Proof. The solution $V(x,y)$ of (3.1)–(3.2) satisfy the equation

$$\begin{aligned} V(x,y) = k_1(x)+k_2(y) + \int_{x_0}^{x}\int_{x_0}^{y}\Big[F\big(s,t,V(s,t)\big) \\ + \int_{x_0}^{x}\int_{x_0}^{y} G\big(x,y,\xi,\eta,V(\xi,\eta)\big)\Delta\eta\,\Delta\xi\Big]\Delta t\,\Delta s\,. \end{aligned} \tag{3.7}$$

Now for (3.3)–(3.5) and (3.7) we get

$$\begin{aligned} V(x,y) \le c + \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)\,|V(s,t)| \\ + \int_{x_0}^{s}\int_{x_0}^{t} p(x,y,\xi,\eta)\,|V(\xi,\eta)|\,\Delta\eta\,\Delta\xi\Big]\Delta t\,\Delta s\,. \end{aligned} \tag{3.8}$$

Now we apply Theorem 2.4 to (3.8) gives (3.6). The right hand side of (3.8) gives the bounds of solution of (3.1)–(3.2). □

Now in the next theorem we give the uniqueness of solution of (3.1)–(3.2).

Theorem 3.2. *Suppose*

$$\big|F\big(x,y,V(x,y)\big) - F\big(x,y,\overline{V}(x,y)\big)\big| \le h(x,y)\big|V(x,y)-\overline{V}(x,y)\big|\,, \tag{3.9}$$

$$\begin{aligned} &\big|G\big(x,y,\xi,\eta,V(\xi,\eta)\big) - G\big(x,y,\xi,\eta,\overline{V}(\xi,\eta)\big)\big| \\ &\qquad \le p(x,y,\xi,\eta)\big|V(\xi,\eta)-\overline{V}(\xi,\eta)\big|\,, \end{aligned} \tag{3.10}$$

where h, p are as defined in Theorem 2.4. Then (3.1)–(3.2) *has at most one solution for $(x,y)\in\Omega$.*

Proof. Let $V(x,y)$ and $\overline{V}(x,y)$ be two solution of (3.1)–(3.2) then

$$\begin{aligned} &V(x,y)-\overline{V}(x,y) \\ &= \int_{x_0}^{x}\int_{x_0}^{y}\Big[\big\{F\big(s,t,V(s,t)\big) - F\big(s,t,\overline{V}(s,t)\big)\big\} \\ &+ \int_{x_0}^{s}\int_{x_0}^{t}\big\{G\big(s,t,\xi,\eta,V(\xi,\eta)\big) - G\big(s,t,\xi,\eta,\overline{V}(\xi,\eta)\big)\big\}\Delta\eta\,\Delta\xi\Big]\Delta t\,\Delta s\,. \end{aligned} \tag{3.11}$$

From (3.9), (3.10) and (3.11) we have

$$\begin{aligned} \big|V(x,y)-\overline{V}(x,y)\big| \le \int_{x_0}^{x}\int_{x_0}^{y}\Big[h(s,t)\big|V(s,t)-\overline{V}(s,t)\big| \\ +\int_{x_0}^{s}\int_{x_0}^{t} p(s,t,\xi,\eta)\big|V(\xi,\eta)-\overline{V}(\xi,\eta)\big|\Delta\eta\,\Delta\xi\Big]\Delta t\,\Delta s\,. \end{aligned} \tag{3.12}$$

Now applying Theorem 2.4 with k=0 gives $\left|V\left(x,y\right)-\overline{V}\left(x,y\right)\right|\le 0$ giving $V\left(x,y\right)=\overline{V}\left(x,y\right)$ for $(x,y)\in\Omega$. Thus there is at most one solution of (3.1)–(3.2). □

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