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COMMENTS ON THE FRACTIONAL PARTS OF PISOT NUMBERS

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Abstract. Let $L(\theta,\lambda)$ be the set of limit points of the fractional parts $\{\lambda\theta^n\}$, $n=0,1,2,\dots$, where θ is a Pisot number and $\lambda\in\mathbb{Q}(\theta)$. Using a description of $L(\theta,\lambda)$, due to Dubickas, we show that there is a sequence $(\lambda_n)_{n\geq 0}$ of elements of $\mathbb{Q}(\theta)$ such that $\operatorname{Card}(L(\theta,\lambda_n)) < \operatorname{Card}(L(\theta,\lambda_{n+1}))$, $\forall\ n\geq 0$. Also, we prove that the fractional parts of Pisot numbers, with a fixed degree greater than 1, are dense in the unit interval.

1. Introduction

A well-known theorem, due to Pisot and Vijayaraghavan (see, e.g., [3]), asserts that if the sequence of fractional parts $\{\lambda\theta^n\}$, $n=0,1,2,\dots$, where λ is a non-zero real number and θ is an algebraic number greater than 1, has a finite number of limit points, then θ is a Pisot number and $\lambda\in\mathbb{Q}(\theta)$. A Pisot number is a real algebraic integer greater than 1 whose other conjugates are of modulus less than 1, and the set of such numbers is usually noted $\mathbb{S}$, [1].

From now on suppose $\theta\in\mathbb{S}$, $\lambda\in\mathbb{Q}(\theta)$ and $\lambda\neq 0$. Let $\sigma_1,\sigma_2,\dots,\sigma_d$ be the distinct embeddings of $\mathbb{Q}(\theta)$ into $\mathbb{C}$, and let

$$M_\theta(x)=(x-\theta_1)(x-\theta_2)\dots(x-\theta_d)=x^d-a_{d-1}x^{d-1}-\dots-a_0$$

be the minimal polynomial of θ, where $\theta_1=\sigma_1(\theta)=\theta$, $\theta_2=\sigma_2(\theta),\dots,\ \theta_d=\sigma_d(\theta)$.

Considering the inverse problem, Dubickas showed, in [4], many results about the set, say $L=L(\theta,\lambda)$, of limit points of the sequence $(\{\lambda\theta^n\})_{n\geq 0}$, and the first one implies immediately that $L(\theta,\lambda)$ is finite. Hence, the above mentioned theorem of Pisot and Vijayaraghavan is a characterization of the elements of the set $\{(\theta,\lambda)\mid\theta\in\mathbb{S},\lambda\in\mathbb{Q}(\theta)\text{ and}\lambda\neq 0\}$ among all pairs having a first coordinate which is a real algebraic number greater than 1, and a non-zero real second coordinate.

Also, Theorem 4 of [4], asserts that for a given θ, there is λ such that $\operatorname{Card}(L(\theta,\lambda))=1$, if and only if θ is a strong Pisot number or $|M_\theta(1)|\geq 2$. Recall that the Pisot number θ is said to be strong if $d=1$, or if $d\geq 2$ and θ has a real positive conjugate, say θ_2, which is greater than the absolute values of its $d-2$ remaining conjugates [2]; similarly as in [8], we denote by $\mathbb{S}_{\mathrm{st}}$ the set of strong Pisot

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numbers. The aim of this note is to collect some partial answers to the following question:

Let $\theta \in \mathbb{S}$ and $c \in \mathbb{N}$. Does there exist λ such that $\mathrm{Card}\,(L(\theta,\lambda)) = c$*?* (Q)

Clearly, by the above mentioned result of Dubickas we have a complete answer to (Q) when $c = 1$. Moreover, a simple calculation gives a positive answer to (Q), when the degree of θ is one, as asserted by the following proposition.

Proposition 1. *We have $L(2,1) = \{0\}$, and if $d = 1$ and $(\theta, c) \neq (2,1)$, then*

$$L(\theta, 1/(\theta^c - 1)) = \{1/(\theta^c - 1), \theta/(\theta^c - 1), \ldots, \theta^{c-1}/(\theta^c - 1)\}\,.$$

The corollaries below yield also some partial answers to (Q). These corollaries are deduced from our main result, presented in Theorem 1 together with Proposition 2, and proved in the next section. To state this result we shall follow the same scheme as in the introduction of [4], with some modifications. More precisely, for a fixed θ, we associate to each λ all pairs $(T = T(\theta,\lambda), m) \in \pounds_\theta \times \mathbb{N}$, where $\pounds_\theta$ is the group of linear recurrence sequences with rational integer terms and companion polynomial M_θ, satisfying $m\lambda \in \mathbb{Z}[\theta]/M'_\theta(\theta) = \{\beta/M'_\theta(\theta) \mid \beta \in \mathbb{Z}[\theta]\}$ ($M'_\theta(\theta)$ is the usual derivative of M_θ evaluated at θ), and $T = (\mathrm{Trace}\,(m\lambda\theta^n))_{n\geq 0}$.

Corollary 1. *The quantity* $\mathrm{Card}\,(L(\theta,\lambda))$ *takes infinitely many values when λ runs through the field $\mathbb{Q}(\theta)$.*

Another immediate consequence of Theorem 1, is obtained for strong Pisot numbers.

Corollary 2. *Suppose $\theta \in \mathbb{S}_{\mathrm{st}}$. Then, there exists λ such that* $\mathrm{Card}\,(L(\theta,\lambda)) = c$ *if and only if there is $((g_n)_{n\geq 0}, k) \in \pounds_\theta \times \mathbb{N}$ satisfying $s((g_n)_{n\geq 0}, k) = c$, where $s((g_n)_{n\geq 0}, k)$ is the number of distinct terms of the sequence $(g_n \bmod k)_{n\geq 0}$, occurring infinitely often.*

For example, if $M_\theta(x) = x^2 - 3x + 1$, then $\theta = (3+\sqrt{5})/2 \in \mathbb{S}_{\mathrm{st}}$, and $\forall$ $((g_n)_{n\geq 0}, k) \in \pounds_\theta \times \mathbb{N}$ the sequence $(g_n \bmod k)_{n\geq 0}$ is purely periodic, because θ is a unit (for more details see the next section). By considering, for instance, the element $(g_n)_{n\geq 0} \in \pounds_\theta$ defined by the relation $(g_0, g_1) = (2,3)$ (and $g_{n+2} = 3g_{n+1} - g_n$), we obtain $(g_n \bmod 1)_{n\geq 0} = \overline{0}$, $(g_n \bmod 2)_{n\geq 0} = \overline{011}$, $(g_n \bmod 3)_{n\geq 0} = \overline{2010}$, $(g_n \bmod 6)_{n\geq 0} = \overline{231053435013}$, $(g_n \bmod 7)_{n\geq 0} = \overline{23045403}$ and $(g_n \bmod 10)_{n\geq 0} = \overline{237873}$, where the equality $(g_n \bmod k)_{n\geq 0} = \overline{x_1 \ldots x_p}$ for some $p \in \mathbb{N}$, means that $g_{i+np} \bmod k = x_{i+1}$ $\forall$ $i \in \{0,\ldots,p-1\}$ and $\forall$ $n \geq 0$. Hence, $s((g_n)_{n\geq 0}, 1) = 1$, $s((g_n)_{n\geq 0}, 2) = 2$, $s((g_n)_{n\geq 0}, 3) = 3$, $s((g_n)_{n\geq 0}, 6) = 6$, $s((g_n)_{n\geq 0}, 7) = 5$, $s((g_n)_{n\geq 0}, 10) = 4$, and so by Corollary 2, for each $c \in \{1,\ldots,6\}$, there is λ such that $\mathrm{Card}\,(L(\theta,\lambda)) = c$. To obtain an element λ such that $\mathrm{Card}\,(L(\theta,\lambda)) = c$, where $c \geq 7$, we have to consider the sequence $(g_n \bmod k)_{n\geq 0}$, where $k \geq 11$, since $(g_n \bmod 4)_{n\geq 0} = \overline{233}$, $(g_n \bmod 5)_{n\geq 0} = \overline{23}$, $(g_n \bmod 8)_{n\geq 0} = \overline{237273}$, and $(g_n \bmod 9)_{n\geq 0} = \overline{237026762073}$, or another element $(g'_n)_{n\geq 0} \in \pounds_\theta$ (by modifying the initial conditions) having c residues $\bmod\, k$, for some $k \geq 7$. *We are unable to prove that for any $c \geq 7$ there is $((g'_n)_{n\geq 0}, k) \in \pounds_\theta \times \mathbb{N}$, such that sequence $(g'_n \bmod k)_{n\geq 0}$ has exactly c residues.*

We may also use Theorem 1 and Proposition 2, to obtain another proof of Theorem 4 of [4].

Corollary 3 (Dubickas)**.** *For a fixed Pisot number θ, the following assertions are equivalent.*

(i) *There is a non-zero element λ of the field $\mathbb{Q}(\theta)$ (or equivalently of the field $\mathbb{R}$) such that* $\mathrm{Card}\,(L(\theta,\lambda)) = 1$.

(ii) $\theta \in \mathbb{S}_{\mathrm{st}}$ *or* $|M_\theta(1)| \neq 1$.

It is easy to see, from the computation in the beginning of Section 2, that $L(\theta,\lambda) = \{0,1\}$ $\forall \lambda \in \mathbb{Z}[\theta]/M'_\theta(\theta)$, except when $\theta \in \mathbb{S}_{\mathrm{st}}$; in this last case we have, by Corollary 2, $\mathrm{Card}\,(L(\theta,\lambda)) = 1$. Moreover, we may deduce from Theorem 1 the following consequence.

Corollary 4. *If $\theta \in \mathbb{S}_{\mathrm{st}}$ and $|M_\theta(-1)| \geq 3$, then there is $\lambda \in \mathbb{Q}(\theta)$ such that* $\mathrm{Card}\,(L(\theta,\lambda)) = 2$.

To complete this last result we have to consider the case where $\theta \in \mathbb{S}_{\mathrm{st}}$ and $|M_\theta(-1)| \in \{1,2\}$, or to prove the inequality $|M_\theta(-1)| > 2$ for all $\theta \in \mathbb{S}_{\mathrm{st}}$. It is easy to verify that $|M_\theta(-1)| \geq 3$ when the degree of the strong Pisot number θ is less than 4, or when θ belongs to the families defined by Theorem 2 of [8]. *Also, we are unable to find $\theta \in \mathbb{S}_{\mathrm{st}}$ with $|M_\theta(-1)| \in \{1,2\}$.*

Among a large amount of the structure of the set of Pisot numbers is understood (see for instance [1]), the following result seems to be unknown.

Theorem 2. *Let d be a rational integer greater than 1. Then, the fractional parts of Pisot numbers with degree d are dense in the unit interval.*

A Salem number is a real algebraic integer greater than one whose other conjugates are of modulus at most 1, and with a conjugate with modulus one [2]. It is well known that the powers of a Salem number are dense modulo one [1]. Hence, *if θ is a Salem number with degree d, then the fractional parts of Salem numbers which belong to the field $\mathbb{Q}(\theta)$, with degree d, are dense in the unit interval.* In the proof of Theorem 2, we shall show an analogue of this last mentioned assertion only for quadratic Pisot numbers. This proof and the ones of the corollaries are, respectively, exhibited in the last and third sections.

2. The main result

As mentioned in the introduction for a fixed θ, we associate to each λ all elements $(T = T(\theta,\lambda), m)$ of the set $\pounds_\theta \times \mathbb{N}$, defined as follows. Let $m = m(\lambda,\theta) \in \mathbb{N}$ be such that $mM'_\theta(\theta)\lambda \in \mathbb{Z}[\theta]$. Then, $m\lambda \in \mathbb{Z}[\theta]/M'_\theta(\theta)$, and so by Lemma 2 of [7], we obtain

$$t_n := \mathrm{Trace}\,(m\lambda\theta^n) \in \mathbb{Z}\,, \quad \forall n \geq 0\,.$$

Since $\theta^{n+d} = a_{d-1}\theta^{n+d-1} + \cdots + a_0\theta^n$, we see that

$$m\lambda\theta^{n+d} = a_{d-1}m\lambda\theta^{n+d-1} + \cdots + a_0 m\lambda\theta^n$$

and

$$t_{n+d} := a_{d-1}t_{n+d-1} + \cdots + a_0 t_n\,, \quad \forall n \geq 0\,;$$

thus $T := (t_n)_{n\geq 0} \in \pounds_\theta$. Moreover, as

$$\mathrm{Trace}\,(m\lambda\theta^n) = \sum_{i=1}^{d} \sigma_i(m\lambda\theta^n) = m\sum_{i=1}^{d} \sigma_i(\lambda)\theta_i^n\,,$$

we have

$$\{\lambda\theta^n\} = \frac{t_n}{m} - [\lambda\theta^n] - \sum_{i=2}^{d} \theta_i^n \sigma_i(\lambda)\,, \tag{1}$$

where $[\,\cdot\,]$ is the integer part function.

Recall also that for every $((g_n)_{n\geq 0}, k) \in \pounds_\theta \times \mathbb{N}$, the sequence $(g_n \bmod k)_{n\geq 0}$ is eventually periodic (there are k possible values $\bmod\, k$, for each component of the vectors $(g_n, g_{n+1}, \ldots, g_{n+d-1})$) and so there are two rational integers $p = p((g_n)_{n\geq 0}, k) \geq 1$ and $q \geq 0$ such that $g_{n+p} \equiv g_n \bmod k$, for all $n \geq q$.
Setting $\overline{b_1 b_2 \ldots b_p}$ for the period of $(g_n \bmod k)_{n\geq 0}$, we deduce that the set $R = R((g_n)_{n\geq 0}, k) := \{r_1, \ldots, r_{s((g_n)_{n\geq 0}, k)}\}$ of distinct terms of the sequence $(g_n \bmod k)_{n\geq 0}$, occurring infinitely often, satisfies $R = \{b_1, \ldots, b_p\} \subseteq \{0, 1, \ldots, k-1\}$.

Finally, notice that in a similar manner as in the proof of Theorem 2 of [4], which uses a result of Smyth [6], saying that the conjugates of a Pisot number with the same modulus are complex conjugates, we easily obtain from (1) the following assertion.

Proposition 2. *With the above notation, where $R = R(T, m)$ and $R/m := \{r_1/m, \ldots, r_{s(T,m)}/m\}$, the relations below are true.*

(i) *If $r \in R\diagdown\{0\}$, then $r/m \in R/m$, and so $0 \notin R \Rightarrow L = R/m$.*

(ii) *$\theta \in \mathbb{S}_{\mathrm{st}} \Rightarrow L = R/m$ or $L = R/m \cup \{1\}\diagdown\{0\}$.*

(iii) *Suppose $\theta \notin \mathbb{S}_{\mathrm{st}}$, $0 \in R$ and p is even with $r_i = r_j = 0 \Rightarrow i \equiv j \bmod 2$, $\forall (i,j) \in \{1, \ldots, p\}^2$ (resp. and p is odd, or p is even with $r_i = r_j = 0$ for some $(i,j) \in \{1, \ldots, p\}^2$ satisfying $i \not\equiv j \bmod 2$). Then, $L = R/m$ or $L = R/m \cup \{1\}\diagdown\{0\}$ (resp. Then, $L = R/m \cup \{1\}$).*

It follows, in particular, that L equals R/m or $R/m \cup \{1\}$ or $R/m \cup \{1\} \diagdown \{0\}$, and so

$$1 \leq s(T,m) \leq \mathrm{Card}\,\big(L(\theta,\lambda)\big) \leq s(T,m) + 1 \leq \min\{m+1,\ p(T,m)+1\}\,. \tag{2}$$

The result below yields that the converse of the above defined correspondence between the elements of the field $\mathbb{Q}(\theta)$ and the pairs of the set $\pounds_\theta \times \mathbb{N}$ holds too.

Theorem 1. *If $(G,k) \in \pounds_\theta \times \mathbb{N}$, then there is $\lambda \in \mathbb{Q}(\theta)$ such that $G = (\mathrm{Trace}\,(k\lambda\theta^n))_{n\geq 0}$ and $k\lambda \in \mathbb{Z}[\theta]/M'_\theta(\theta)$. Moreover, we have*

$$\mathrm{Card}\,\big(L(\theta,\lambda)\big) \in \{s(G,k),\ s(G,k)+1\}\,.$$

Proof. It is clear that $\mathbb{Z}[\theta]/M'_\theta(\theta)$ and $\mathcal{L}_\theta$ are, respectively, subgroups of the additive groups $\mathbb{Q}(\theta)$ and $\mathbb{Z}^{\mathbb{N}}$. Let φ be the mapping defined from $\mathbb{Z}[\theta]/M'_\theta(\theta)$ to $\mathcal{L}_\theta$, by the equality

$$\varphi(\gamma) = (\text{Trace } \big(\gamma\theta^n)\big)_{n\geq 0} \quad \forall \gamma \in \mathbb{Z}[\theta]/M'_\theta(\theta)\,.$$

Then, we claim that the isomorphic image of $\mathbb{Z}[\theta]/M'_\theta(\theta)$ by φ is $\mathcal{L}_\theta$. Indeed, by the above, the mapping φ is well defined and is a group homomorphism, since the function Trace is linear. Moreover, for each $(g_n)_{n\geq 0} \in \mathcal{L}_\theta$ there are complex numbers $\gamma_1, \dots, \gamma_d$ such that

$$g_n = \sum_{i=1}^{d} \gamma_i \theta_i^n \quad \forall n \geq 0\,,$$

and again by Lemma 2 of [7] we have that $\gamma_i = \sigma_i(\gamma)$, $\forall i \in \{1, \dots, d\}$, where $\gamma_1 = \gamma \in \mathbb{Z}[\theta]/M'_\theta(\theta)$; thus $(g_n)_{n\geq 0} = \varphi(\gamma)$ and φ is onto. Finally, if $\gamma \in \ker\varphi$, then $(\text{Trace}\,(\gamma\theta^n))_{n\geq 0}$ is the zero sequence, and the equalities $\text{Trace}\,(\gamma\theta^n) = 0$, where $n \in \{0, 1, \dots, d-1\}$, , yield to an homogenous linear system whose determinant, namely $\det[{\theta_i}^{j-1}]_{1\leq i,j\leq d}$, is non-zero, as the quantity $\Delta_\theta := (\det[{\theta_i}^{j-1}]_{1\leq i,j\leq d})^2$ is the discriminant of the polynomial M_θ; thus $\gamma = 0$ and the claim is proved. It follows when $(G, k) \in \mathcal{L}_\theta \times \mathbb{N}$, that $G = (\text{Trace}\,(\gamma\theta^n))_{n\geq 0}$ for some $\gamma \in \mathbb{Z}[\theta]/M'_\theta(\theta)$, and so

$$G = \big(\,\text{Trace}\,(k\lambda\theta^n)\big)_{n\geq 0}\,,$$

where $\lambda = \gamma/k \in \mathbb{Q}(\theta)$ and $k\lambda \in \mathbb{Z}[\theta]/M'_\theta(\theta)$. The second assertion in Theorem 1 is an immediate corollary of Proposition 2 (or the relation (2)). □

Remark. Since each element of the subgroup $\mathbb{Z}[\theta]$ of $\mathbb{Z}[\theta]/M'_\theta(\theta)$, can be written $k_0 + \cdots + k_{d-1}\theta^{d-1}$, for some $(k_0, \dots, k_{d-1}) \in \mathbb{Z}^d$, and $\varphi(k_0 + \cdots + k_{d-1}\theta^{d-1}) = (k_0 s_n + \cdots + k_{d-1}s_{n+d-1})_{n\geq 0}$, where $s_n := \text{Trace}\,(\theta^n)$, we deduce that the isomorphic image of $\mathbb{Z}[\theta]$ by φ, is a subgroup D_θ of $\mathcal{L}_\theta$, whose elements are those $(g_n)_{n\geq 0} \in \mathcal{L}_\theta$ satisfying the following equalities:

$$(3) \qquad \begin{cases} k_0 s_0 + k_1 s_1 + \cdots + k_{d-1}s_{d-1} = g_0 \\ \quad \vdots \\ k_0 s_{d-1} + k_1 s_d + \cdots + k_{d-1}s_{2d-2} = g_{d-1}\,, \end{cases}$$

for some $(k_0, \dots, k_{d-1}) \in \mathbb{Z}^d$. Since the determinant of the the system (3) satisfies $\det([s_{i+j-2}]_{1\leq i,j\leq d}) = \Delta_\theta > 1$, [5], we see immediately that $D_\theta \subsetneq \mathcal{L}_\theta$. However, it is easy to see that $\{(g_n \bmod m)_{n\geq 0} \mid (g_n)_{n\geq 0} \in D_\theta\} = \{(g_n \bmod m)_{n\geq 0} \mid (g_n)_{n\geq 0} \in \mathcal{L}_\theta\}$ for any $m \in \mathbb{N}$ satisfying $\gcd(\Delta_\theta, m) = 1$.

3. Proof of the corollaries

Proof of Corollary 1. First we claim that there exists $(g_n)_{n\geq 0} \in \mathcal{L}_\theta$ such that $0 < g_n < g_{n+1},\ \ \forall n \geq 0$. Indeed, consider the element $(h_n)_{n\geq 0}$ of $\mathcal{L}_\theta$ which is defined by the equation $(h_0,\dots,h_{d-2},h_{d-1})=(0,\dots,0,\varepsilon)$, where $\varepsilon\in\{-1,1\}$ and satisfies

$$\varepsilon(-1)^{d-1}(\det[{\theta_i}^{j-1}]_{1\le i,j\le d})\prod_{2\le i<j\le d}(\theta_j-\theta_i)>0\,.$$

Similarly as in the proof of Theorem 1, we have that there is $\gamma\in\mathbb{Q}(\theta)$ such that

$$h_n=\sum_{i=1}^{d}\sigma_i(\gamma)\theta_i^n \quad \forall n\ge 0. \tag{4}$$

Since $\det[\theta_i^{j-1}]_{1\le i,j\le d}$ is the determinant of the linear system

$$h_n=\sum_{i=1}^{d}\sigma_i(\gamma)\theta_i^n \qquad n\in\{0,1,\dots,d-1\},$$

we see that the product $\gamma(\det[{\theta_i}^{j-1}]_{1\le i,j\le d})$ equals

$$\det\begin{bmatrix} h_0 & 1 & 1 & \dots & 1\\ h_1 & \theta_2 & \theta_3 & \dots & \theta_d\\ \vdots & \vdots & \vdots & & \vdots\\ h_{d-1} & \theta_2^{d-1} & \theta_3^{d-1} & \dots & \theta_d^{d-1}\end{bmatrix}=(-1)^{d-1}\varepsilon\det\begin{bmatrix} 1 & 1 & \dots & 1\\ \theta_2 & \theta_3 & \dots & \theta_d\\ \vdots & \vdots & & \vdots\\ \theta_2^{d-2} & \theta_3^{d-2} & \dots & \theta_d^{d-2}\end{bmatrix}$$

$$=(-1)^{d-1}\varepsilon\prod_{2\le i<j\le d}(\theta_j-\theta_i)\,,$$

and so $\gamma>0$. Hence, there is $N_1\in\mathbb{N}$ such that $h_n>0$ for all $n\ge N_1$, because the relation (4) yields $\lim_{n\to\infty}h_n/\theta^n=\gamma$. Also, if $N_2\in\mathbb{N}$ and satisfies $\theta^{N_2}>2(\sum_{i=2}^{d}|\sigma_i(\gamma)|)/\gamma(\theta-1)$, then we have, by (4),

$$\begin{aligned} h_{n+1}-h_n&=\gamma\theta^n(\theta-1)+\sum_{i=2}^{d}\sigma_i(\gamma)(\theta_i^{n+1}-\theta_i^n)\\ &\ge\gamma\theta^{N_2}(\theta-1)+\sum_{i=2}^{d}\sigma_i(\gamma)(\theta_i^{n+1}-\theta_i^n)\quad\forall n\ge N_2\,,\end{aligned}$$

and so

$$h_{n+1}-h_n>2\Big(\sum_{i=2}^{d}|\sigma_i(\gamma)|\Big)+\sum_{i=2}^{d}\sigma_i(\gamma)(\theta_i^{n+1}-\theta_i^n)>0\,;$$

thus $(h_n)_{n\ge N_2}$ is strictly increasing, the sequence $G=(g_n)_{n\ge 0}:=(h_n)_{n\ge\max(N_1,N_2)}$ satisfies the desired property, and the claim is proved.

Now define the numbers k_n inductively by $k_1:=g_1$, and $k_{n+1}:=g_{k_n}$, for all $n\ge 1$. It is clear that $2\le s(G,k_1)\le k_1$,

$$k_n+1\le s(G,k_{n+1})\le k_{n+1}$$

and so

$$s(G,k_n) \leq k_n < k_n + 1 \leq s(G,k_{n+1})\,.$$

It follows by Theorem 1, that there are some elements $\lambda_1, \lambda_2, \lambda_3, \ldots$ of $\mathbb{Q}(\theta)$ such that $\mathrm{Card}\,(L(\theta,\lambda_n)) \in \{s(G,k_n),\, s(G,k_n)+1\}$, for all $n \geq 1$, and the result follows immediately by considering (for instance) the sequence $(\mathrm{Card}\,(L(\theta,\lambda_{2n+1})))_{n\geq 0}$, since the sequence $(s(G,k_n))_{n\geq 1}$ is strictly increasing. □

Proof of Corollary 2. Let $\theta \in \mathbb{S}_{\mathrm{st}}$ and $c \in \mathbb{N}$. Suppose that there is $\lambda \in \mathbb{Q}(\theta)$ such that $\mathrm{Card}\,(L(\theta,\lambda)) = c$. We have already seen that there is an element $(T,m) \in \pounds_\theta \times \mathbb{N}$, satisfying $m\lambda \in \mathbb{Z}[\theta]/M'_\theta(\theta)$ and $T = (\mathrm{Trace}\,(m\lambda\theta^n))_{n\geq 0}$. Moreover, by Proposition 2 (ii), we have $L(\theta,\lambda) = R(T,m)/m$ or $L(\theta,\lambda) = R(T,m)/m \cup \{1\} \diagdown \{0\}$ and so $c = s(T,m)$. The converse follows trivially from the first assertion in Theorem 1. □

Proof of Corollary 3. It is clear, by Proposition 2, that $\mathrm{Card}\,(L(\theta,\lambda)) = 1 \Rightarrow$ $(s = 1$ and $\theta \in \mathbb{S}_{\mathrm{st}})$ or $(s = 1$ and $R \neq \{0\})$, where $s = s(\mathrm{Trace}\,(m\lambda\theta^n), m)$ and $m\lambda \in \mathbb{Z}[\theta]/M'_\theta(\theta)$, because $s = 1 \Rightarrow p = 1 \Rightarrow p$ odd and so $\mathrm{Card}(L) = 2 > 1$ when $R = \{0\}$ and $\theta \notin \mathbb{S}_{\mathrm{st}}$. Since, the condition $R = \{r\} \neq \{0\}$, implies that $m \geq 2$ and $(t_n \bmod m)_{n\geq q} = \overline{r}$, for some $q \geq 0$, it follows by the relation $r \equiv a_{d-1}r + \cdots + a_0 r \bmod m$, or equivalently $rM_\theta(1) \equiv 0 \bmod m$, that $|M_\theta(1)| \neq 1$, as $0 < r < m$. Conversely, a simple calculation gives $L(\theta,-1) = \{0\}$ when $\theta \in \mathbb{S}_{\mathrm{st}}$. Suppose $|M_\theta(1)| \neq 1$, set $m = |M_\theta(1)|$ and consider the element $(g_n)_{n\geq 0}$ of $\pounds_\theta$ defined by the equalities: $g_0 = \cdots = g_{d-1} = 1$. Then, $g_d = a_{d-1} + \cdots + a_0 = 1 - M_\theta(1) \equiv 1 \bmod m$, and a simple induction gives that $g_n \equiv 1 \bmod m$ for all $n \geq 0$. Now, by the first assertion in Theorem 1 and Proposition 2 (i), we immediately see that there is $\lambda \in \mathbb{Q}(\theta)$ such that $L(\theta,\lambda) = \{1/\,|M_\theta(1)|\}$, as $(g_n \bmod m)_{n\geq 0} = \overline{1}$, and $1 \neq 0$. □

Proof of Corollary 4. It is clear that Corollary 4 is true when $d = 1$. Suppose $d \geq 2$, set $m = |M_\theta(-1)|$ and consider the element $(g_n)_{n\geq 0}$ of $\pounds_\theta$ defined by the relations $g_{2k} = 1$, (resp. $g_{2k+1} = m - 1$) where $k \in \{0,\ldots,(d-1)/2\}$ (resp. $k \in \{1,\ldots,(d-2)/2\}$). Then, a simple calculation shows that the sequence $(g_n \bmod m)_{n\geq 0}$ is purely periodic, and takes alternatively the values 1 and $m - 1$. It follows by the first assertion in Theorem 1 and Proposition 2 (i), that there is $\lambda \in \mathbb{Q}(\theta)$ such that $L(\theta,\lambda) = \{1/\,|M_\theta(-1)|\,, 1 - 1/\,|M_\theta(-1)|\}$, and so Corollary 4 holds. □

4. Proof of Theorem 2

Consider the polynomial

$$M(x) = x^d - (bn+1)x^{d-1} + n(b-a)x^{d-2} + 1\,,$$

where the rational integers a, b, d and n satisfy the inequalities $1 \leq a < b$, $3 \leq d$ and $2 \leq n$. Then, the relations

$$\left|(bn+1)z^{d-1}\right| = bn + 1 > 2 + n(b-a) \geq \left|z^d + 1 + n(b-a)z^{d-2}\right|\,,$$

where z is a complex number with modulus one, yield, by Rouché's theorem, that the roots of the polynomial M are all of modulus less than one, except one, say θ. A short computation gives, for n sufficiently large, $M(nb+\frac{a}{b}) = \frac{a}{b}(nb+\frac{a}{b})^{d-2}(\frac{a}{b}-1)+1 < 0$,

$$M(nb+\frac{a}{b}+\frac{1}{n}) = (nb+\frac{a}{b}+\frac{1}{n})^{d-2}(b+\frac{a}{b}(\frac{2}{n}-1+\frac{a}{b})+\frac{1}{n}(\frac{1}{n}-1))+1 > 0$$

and so

$$nb+\frac{a}{b} < \theta < nb+\frac{a}{b}+\frac{1}{n}\,; \tag{5}$$

thus, the algebraic integer θ is a Pisot number and the minimal polynomial of θ is M. Furthermore, we see by (5) that we may, again, choose n sufficiently large, in a way to make the fractional part of θ arbitrarily close to a/b, and the result follows immediately, when the fixed degree d is greater than 2, since the rational numbers a/b are dense in the unit interval. To complete the proof of Theorem 2, it remains to consider the quadratic case. In fact it is easy to show that the following proposition is true: *If K is a real quadratic extension of $\mathbb{Q}$, then the fractional parts of quadratic Pisot numbers, belonging to K, are dense in the unit interval.* Indeed, set $K := \mathbb{Q}(\sqrt{D})$, where D is a square-free positive rational integer. Then, for each $y \in \mathbb{N}$, there is $x \in \mathbb{Z}$ such that $x + y\sqrt{D} > 1$ and $-1 < x - y\sqrt{D} < 1$ (choose for instance $x = [y\sqrt{D}]$), and so the element $\theta_y := x + y\sqrt{D}$ of K, is a quadratic Pisot number such that $\{\theta_y\} = \{y\sqrt{D}\}$. Since the fractional parts of the numbers of the form $y\sqrt{D}$, when y runs through $\mathbb{N}$, are dense in the unit interval, the above cited assertion follows immediately.

References

[1] Bertin, M.J., Decomps-Guilloux, A., Grandet-Hugo, M., Pathiaux-Delefosse, M., Schreiber, J.P., *Pisot and Salem numbers*, Birkhäuser Verlag Basel, 1992.

[2] Boyd, D.W., *Linear recurrence relations for some generalized Pisot sequences*, Advances in Number Theory, Proc. of the 1991 CNTA Conference, Oxford University Press, 1993, pp. 333–340.

[3] Bugeaud, Y., *An introduction to diophantine approximation*, Cambridge University Press, Cambridge, 2012.

[4] Dubickas, A., *On the limit points of the fractional parts of powers of Pisot numbers*, Arch. Math. (Brno) **42** (2006), 151–158.

[5] Marcus, D., *Number fields*, Springer, Berlin, 1977, 3rd edition.

[6] Smyth, C.J., *The conjugates of algebraic integers*, Amer. Math. Monthly **82** (86) (1975).

[7] Zaïmi, T., *Comments on the distribution modulo one of powers of Pisot and Salem numbers*, Publ. Math. Debrecen **80** (2012), 417–426.

[8] Zaïmi, T., *On the spectra of Pisot numbers*, Glasgow Math. J. **54** (2012), 127–132.

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