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HOW MANY ARE EQUIAFFINE CONNECTIONS WITH TORSION

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Abstract. The question how many real analytic equiaffine connections with arbitrary torsion exist locally on a smooth manifold M of dimension n is studied. The families of general equiaffine connections and with skew-symmetric Ricci tensor, or with symmetric Ricci tensor, respectively, are described in terms of the number of arbitrary functions of n variables.

1. Introduction

When we consider an infinite family of well-determined geometric objects, it is natural to put the question about "how many" such objects there exist. In the real analytic case, the Cauchy-Kowalevski Theorem is the standard tool ([3], [7], [11]). Hence a natural way how to measure an infinite family of real analytic geometric objects is a finite family of arbitrary functions of k variables and (optionally) a family of arbitrary functions of $k-1$ variables, and, optionally, "modulo" another family of arbitrary functions of $k-1$ variables. The last (optional) family of functions corresponds to the family of automorphisms of any geometric object from the given family. A good example is the following question: How many there are real analytic Riemannian metrics in dimension 3? It is known (see [4], [8]) that every such metric can be put locally into a diagonal form and that all coordinate transformations preserving diagonal form of the given metric depend on 3 arbitrary functions of two variables. Hence all Riemannian metrics in dimension 3 can be locally described by 3 arbitrary functions of 3 variables modulo 3 arbitrary functions of 2 variables. An immediate question arise if we can "calculate the number" of more basic geometric objects, namely the affine connections, in an arbitrary dimension n. To the authors' knowledge, no attempts are known in this direction from the past. We shall be occupied with real analytic affine connections in arbitrary dimension n.

In the previous paper [2] the authors proved that the class of all real analytic affine connections can be described using $n(n^2-1)$ functions of n variables modulo $2n$ functions of $n-1$ variables. Further, it was proved that the class of all real analytic affine connections (with arbitrary torsion) with skew-symmetric Ricci form depends on $n(2n^2-n-3)/2$ functions of n variables and $n(n+1)/2$ functions

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of $n-1$ variables, modulo $2n$ functions of $n-1$ variables. The class of real analytic connections (with arbitrary torsion) with symmetric Ricci form depends on $n(2n^2-n-1)/2$ functions of n variables and $n(n-1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables. For these results, a direct approach using the Cauchy-Kowalevski Theorem can be used. The analogous results for affine connections without torsion were obtained in [1].

In the present paper, the equiaffine connections are studied in the analogous way. The affine connection is *equiaffine* if it admits a parallel volume form. It is well known (see e.g. [10]) that the connection with zero torsion is equiaffine if and only if the Ricci tensor is symmetric. Hence, for the case of a connection with zero torsion, the previous results obtained in [1] can be applied. Therefore, in the present paper, we characterize the class of equiaffine connections in dimension n with arbitrary torsion, and its natural subclasses, in terms of arbitrary functions of n variables and arbitrary functions of $n-1$ variables.

2. Preliminaries

For the aim of the next sections, and to remain self-contained, we shall formulate the important special case of order one of the Cauchy-Kowalevski Theorem.

Theorem 1. *Consider a system of partial differential equations for unknown functions $U^1(x^1,\dots,x^n),\dots,U^N(x^1,\dots,x^n)$ on an open domain in $\mathbb{R}^n$ and of the form*

$$\begin{aligned}
\frac{\partial U^1}{\partial x^1} &= H^1\Big(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\Big),\\
\frac{\partial U^2}{\partial x^1} &= H^2\Big(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\Big),\\
&\;\;\vdots\\
\frac{\partial U^N}{\partial x^1} &= H^N\Big(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\Big),
\end{aligned}$$

where H^i, $i=1,\dots,N$, are real analytic functions of all variables in a neighbourhood of $(x^1_0,\dots,x^n_0,a^1,\dots,a^N,a^1_2,\dots,a^1_n,\dots,a^N_2,\dots,a^N_n)$, where x^j_0, a^i, a^i_j are arbitrary constants.

Further, let the functions $\varphi^1(x^2,\dots,x^n),\dots,\varphi^N(x^2,\dots,x^n)$ be real analytic in a neighbourhood of $(x^2_0,\dots,x^n_0)$ and satisfy $\varphi^i(x^2_0,\dots,x^n_0)=a^i$ for $i=1,\dots,N$ and

$$\Big(\frac{\partial\varphi^1}{\partial x^2},\dots,\frac{\partial\varphi^1}{\partial x^n},\dots,\frac{\partial\varphi^N}{\partial x^2},\dots,\frac{\partial\varphi^N}{\partial x^n}\Big)(x^2_0,\dots,x^n_0)=(a^1_2,\dots,a^1_n,\dots,a^N_2,\dots,a^N_n)\,.$$

Then the system has a unique solution $(U^1(x^1,\dots,x^n),\dots,U^N(x^1,\dots,x^n))$ which is real analytic around $(x^1_0,\dots,x^n_0)$, and satisfies

$$U^i(x^1_0,x^2,\dots,x^n)=\varphi^i(x^2,\dots,x^n)\,,\qquad i=1,\dots,N\,.$$

We now recall the results from the previous paper [1], which will be used in further sections. We work locally with the spaces $\mathbb{R}[u^1,\dots,u^n]$, or $\mathbb{R}[x^1,\dots,x^n]$, respectively and we use the notation $\mathbf{u}=(u^1,\dots,u^n)$ and $\mathbf{x}=(x^1,\dots,x^n)$.

Lemma 2 ([1]). *For any affine connection determined by $\Gamma^h_{ij}(\mathbf{x})$, there exist a local transformation of coordinates determined by $\mathbf{x}=f(\mathbf{u})$ such that the connection in new coordinates satisfies $\bar\Gamma^h_{11}(\mathbf{bu})=0$, for $h=1,\dots,n$. All such transformations depend on $2n$ arbitrary functions of $n-1$ variables.*

The system of coordinates with the property from the above lemma is called *pre-semigeodesic* system of coordinates, see for example [9]. We finish this paragraph with the following existence theorem, which is a corollary of Lemma 2.

Theorem 3 ([2]). *All affine connections with torsion in dimension n depend locally on $n(n^2-1)$ arbitrary functions of n variables, modulo $2n$ arbitrary functions of $(n-1)$ variables.*

Proof. After the transformation into pre-semigeodesic coordinates, we obtain n Christoffel symbols equal to zero. We are left with $n^3-n=n(n^2-1)$ functions. The transformations into pre-semigeodesic coordinates is uniquely determined up to the choice of $2n$ functions $\varphi^i_0(u^2,\dots,u^n),\varphi^i_1(u^2,\dots,u^n)$ of $n-1$ variables. $\square$

We also recall the standard facts and formulas for the Ricci tensor. In the space $\mathbb{R}^n[x^i]$ with the coordinate vector fields $E_i=\frac{\partial}{\partial x^i}$, we denote derivatives with respect to x^i by the bottom index i. Using the standard definition

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z \tag{1}$$

we calculate the curvature operators

$$R(E_i,E_j)E_k=(\Gamma^\alpha_{jk})_iE_\alpha-(\Gamma^\beta_{ik})_jE_\beta+\Gamma^\alpha_{jk}\Gamma^\gamma_{i\alpha}E_\gamma-\Gamma^\beta_{ik}\Gamma^\delta_{j\beta}E_\delta\,.$$

For the Ricci form

$$\operatorname{Ric}(X,Y)=\operatorname{trace}\bigl[W\mapsto R(W,X)Y\bigr]\,,$$

we obtain

$$\operatorname{Ric}(E_i,E_j)=\sum_{k,l=1}^{n}\bigl[(\Gamma^k_{ij})_k-(\Gamma^k_{kj})_i+\Gamma^l_{ij}\Gamma^k_{kl}-\Gamma^l_{kj}\Gamma^k_{il}\bigr]\,. \tag{2}$$

3. Equiaffine connections with torsion

We work locally in the space $\mathbb{R}[x^1,\dots,x^n]$ with an affine connection ∇ with arbitrary torsion. The components of the connection ∇ are $\Gamma^1_{11},\dots,\Gamma^n_{nn}$ and we consider a volume element $\omega=f(x^1,\dots,x^n)\cdot dx^1\wedge\dots\wedge dx^n$. We want do determine connections ∇ for which

$$\nabla\omega=0\,.$$

This condition, with respect to coordinate vector fields $E_1, \dots, E_n$, gives the conditions

$$
\begin{aligned}
&(\nabla_{E_k}\omega)(E_1, \dots, E_n) \\
&\quad = \frac{\partial}{\partial x^k}\omega(E_1, \dots, E_n) - \omega(\nabla_{E_k}E_1, \dots, E_n) - \dots - \omega(E_1, \dots, \nabla_{E_k}E_n) = 0
\end{aligned}
$$

for $k = 1, \dots, n$. We obtain easily the following n equations

$$
f_{x^k} - f \cdot \sum_{i=1}^{n} \Gamma^i_{ki} = 0\,, \qquad k = 1, \dots, n\,. \tag{3}
$$

If we put $L(x^1, \dots, x^n) = \log(f(x^1, \dots, x^n))$, then these equations can be written in the form $f_{x^k} = f \cdot L_{x^k}$. We choose an arbitrary function $L(x^1, \dots, x^n)$ and we want the conditions

$$
L_{x^k} = \sum_{i=1}^{n} \Gamma^i_{ki}\,, \qquad k = 1, \dots, n \tag{4}
$$

to be satisfied. We can choose arbitrarily the Christoffel symbols Γ^i_{ki} for $k = 1, \dots, n$ and $i = 1, \dots, n-1$ and we calculate the Christoffel symbols Γ^n_{kn} from equations (4).

Theorem 4. *The family of equiaffine connections in dimension n depends on $n^3 - 2n + 1$ functions of n variables modulo a constant and modulo $2n$ functions of $n-1$ variables.*

Proof. The family of all connections depends on $n(n^2 - 1)$ Christoffel symbols. (The n Christoffel symbols are zero in pre-semigeodesic coordinates.) Out of them, n Christoffel symbols are determined from the n equations (4). Hence, we choose arbitrarily the function L and all Christoffel symbols except Γ^n_{kn}. Altogether, we choose arbitrarily the $n(n^2 - 1) - n + 1 = n^3 - 2n + 1$ functions. For any constant c, the function $L + c$ leads to the same equations (4) and the $2n$ functions of $n - 1$ variables appear because we have used pre-semigeodesic coordinates. □

4. Equiaffine connections with torsion and with skew-symmetric Ricci tensor

We use again formulas (4) and pre-semigeodesic coordinates. We choose arbitrarily the function L and Christoffel symbols Γ^i_{ki} for $i = 1, \dots, n-1$ and we determine Γ^n_{kn} from the formulas (4). We have fixed, so far, $n(n-1) + 1$ arbitrary functions of n variables. We continue with the formulas for the skew-symmetric Ricci form, which follow from the conditions

$$
\begin{aligned}
\mathrm{Ric}\,(E_1, E_1) &= 0\,, \\
\mathrm{Ric}\,(E_i, E_i) &= 0\,, \qquad && i > 1, \\
\mathrm{Ric}\,(E_1, E_i) + \mathrm{Ric}\,(E_i, E_1) &= 0\,, && i > 1, \\
\mathrm{Ric}\,(E_i, E_j) + \mathrm{Ric}\,(E_j, E_i) &= 0\,, && 1 < i < j \le n\,.
\end{aligned} \tag{5}
$$

Into these conditions, we substitute formulas (2) for the Ricci tensor. In each formula which follows, we denote by Λ'_{ij} the terms which involve first derivatives with respect to $x^2, \dots, x^n$ and by Λ_{ij} the terms which do not involve any differentiation (and which form a homogeneous polynomial of degree 2 in Γ^k_{ij}). We obtain the $n(n+1)/2$ conditions

$$\begin{aligned}
\sum_{k=2}^{n} (\Gamma^k_{k1})_1 &= \Lambda'_{11} + \Lambda_{11}\,, && \\
(\Gamma^1_{ii})_1 &= \Lambda'_{ii} + \Lambda_{ii}\,, && i > 1\,, \\
(\Gamma^1_{i1})_1 - \sum_{k=2}^{n} (\Gamma^k_{ki})_1 &= \Lambda'_{1i} + \Lambda_{1i}\,, && i > 1\,, \\
(\Gamma^1_{ij})_1 + (\Gamma^1_{ji})_1 &= \Lambda'_{ij} + \Lambda_{ij}\,, && 1 < i < j \le n\,.
\end{aligned} \tag{6}$$

Now, we keep, for example, the derivatives of the following Christoffel symbols on the left-hand sides of the respective equations (6): Γ^n_{n1}, Γ^1_{ii}, Γ^n_{ni}, Γ^1_{ij} (for $1 < i < j \le n$). We see that these Christoffel symbols have not been fixed yet by the procedure described in the previous section. We transport all other terms except the derivatives of the mentioned Christoffel symbols to the right-hand sides of the equations. Now, all Christoffel symbols whose derivatives are not on the left-hand sides and which are not fixed yet can be chosen as arbitrary functions. The Christoffel symbols whose derivatives are on the left-hand sides can be determined using the Cauchy-Kowalevski Theorem.

Theorem 5. *The family of equiaffine connections in dimension n which have skew-symmetric Ricci form depends on $\frac{2n^3-n^2-5n+2}{2}$ functions of n variables and $\frac{n(n+1)}{2}$ functions of $n-1$ variables modulo a constant and modulo $2n$ functions of $n-1$ variables.*

Proof. In the procedure above, we have started with the $n(n^2-1)$ Christoffel symbols in the pre-semigeodesic coordinates. Out of them, n were determined from the equations (4) and $n(n+1)/2$ of them were determined from the equations (6). Further, the function L was chosen arbitrarily. Altogether, the $n(n^2-1) - n - n(n+1)/2 + 1 = (2n^3 - n^2 - 5n + 2)/2$ functions were chosen arbitrarily. The $n(n+1)/2$ functions of less variables appear during solving the system (6) using the Cauchy-Kowalevski Theorem and the constant and $2n$ functions of less variables appear the same way as in Theorem 4. □

5. Equiaffine connections with torsion and with symmetric Ricci tensor

We do the same steps as in the previous section. First, we choose the function L and Christoffel symbols which appear in formulas (4) except Γ^n_{kn} arbitrarily. We determine the functions Γ^n_{kn} from the formulas (4). We continue with the formulas for the symmetric Ricci form, which are

$$\operatorname{Ric}(E_i, E_j) - \operatorname{Ric}(E_j, E_i) = 0\,, \qquad 1 \le i < j \le n\,. \tag{7}$$

After introducing the notation Λ_{ij} and Λ'_{ij} as in the previous section, we obtain formulas

$$
\begin{aligned}
(8)\qquad -\sum_{k=2}^{n}(\Gamma^k_{kj})_1 - (\Gamma^1_{j1})_1 &= \Lambda'_{1j} + \Lambda_{1j}\,, \quad 1 < j \le n,\\
(\Gamma^1_{ij})_1 - (\Gamma^1_{ji})_1 &= \Lambda'_{ij} + \Lambda_{ij}\,, \quad 1 < i < j \le n\,.
\end{aligned}
$$

We keep, for example, the derivatives of the following undetermined Christoffel symbols on the left-hand sides of the respective equations (8): Γ^n_{nj}, Γ^1_{ij} (for $1 < i < j \le n$). We see again that these Christoffel symbols were not fixed yet. We transport all other terms except the derivatives of the mentioned Christoffel symbols to the right-hand sides of the equations. Now, all Christoffel symbols whose derivatives are not on the left-hand sides and which are not fixed yet can be chosen as arbitrary functions. The Christoffel symbols whose derivatives are on the left-hand sides can be determined using the Cauchy-Kowalevski Theorem.

Theorem 6. *The family of equiaffine connections in dimension n which have symmetric Ricci form depends on $\frac{2n^3-n^2-3n+2}{2}$ functions of n variables and $\frac{n(n-1)}{2}$ functions of $n-1$ variables modulo a constant and modulo $2n$ functions of $n-1$ variables.*

Proof. In the procedure above, we have started with the $n(n^2-1)$ Christoffel symbols in the pre-semigeodesic coordinates. Out of them, n were determined from the equations (4) and $n(n-1)/2$ of them were determined from the equations (6). Further, the function L was chosed arbitrarily. Altogether, the $n(n^2-1)-n-n(n-1)/2+1 = (2n^3-n^2-3n+2)/2$ functions were chosen arbitrarily. The $n(n-1)/2$ functions of less variables appear during solving the system (6) using the Cauchy-Kowalevski Theorem and the constant and $2n$ functions of less variables appear the same way as in Theorem 4. □

6. Conclusions

Convention. Let $f(n)$ and $h(n)$ be two sequences depending on natural numbers and let $\lim_{n\to\infty}\frac{f(n)}{h(n)} = 1$. Then we say that $f(n)$ and $h(n)$ are *asymptotically equal at infinity.* Now, we can conclude with the following

Theorem 7. *The number of all equiaffine connections with torsion, or those with skew-symmetric Ricci tensor, or those with symmetric Ricci tensor, respectively, is asymptotically equal at infinity to the number of all affine connections with torsion.*

Proof. The result follows from the Theorems 3–6 because real analytic functions of $(n-1)$ variables are always of measure zero among real analytic functions of n variables (a result by Hilbert) and they need not be counted. □

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