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RELATIONS BETWEEN CONSTANTS OF MOTION AND CONSERVED FUNCTIONS

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Abstract. We study relations between functions on the cotangent bundle of a spacetime which are constants of motion for geodesics and functions on the odd-dimensional phase space conserved by the Reeb vector fields of geometrical structures generated by the metric and an electromagnetic field.

Introduction

We assume a classical spacetime $\boldsymbol{E}$ to be an oriented and time oriented 4-dimensional Loretzian manifold. In literature as phase space is usually considered the cotangent bundle $T^*\boldsymbol{E}$ and as infinitesimal symmetries are usually considered infinitesimal symmetries of the kinetic energy function. It is very well known, [13], that such infinitesimal symmetries are given as the Hamiltonian lifts of functions on $T^*\boldsymbol{E}$ which are *constants of motion* for geodesics. Constants of motion which are polynomial on fibres of the cotangent bundle are given by Killing k-vector fields, $k \geq 1$. For $k = 1$ the corresponding infinitesimal symmetries are the flow lifts of Killing vector fields and so they are projectable on infinitesimal symmetries of the spacetime. For $k \geq 2$ the corresponding infinitesimal symmetries are not projectable and they are called *hidden symmetries*. Moreover, if we consider coupling with an electromagnetic 2–form, constants of motion and the corresponding infinitesimal symmetries are generated by Killing-Maxwell multi-vector fields.

On the other hand the phase space of general relativistic test particle can be defined either as the *observer space*, [3], (a part of the unit pseudosphere bundle given by time–like future oriented vectors) or as the *1–jet space $\mathcal{J}_1\boldsymbol{E}$ of motions*, [11]. The metric and the electromagnetic fields then define geometrical structures given by a 1–form and a closed 2-form. As phase infinitesimal symmetries we define infinitesimal symmetries of these forms. Phase infinitesimal symmetries which are projectable on the spacetime were studied on the observer space by Iwai [3] and on 1-jet space of motions by Janyška and Vitolo [11]. In both situations projectable

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symmetries are given by the flow lifts of Killing vector fields (eventually Killing vector fields which are infinitesimal symmetries of the electromagnetic field).

In the paper [5] it was proved that nonprojectable (hidden) symmetries of the contact structure of the phase space generated by the metric are given by the Hamilton–Jacobi lifts of phase functions conserved by the Reeb vector field of the contact structure. Moreover, it was proved that such conserved functions are generated by Killing multi-vector fields. On the other hand if we assume the almost-cosymplectic-contact structure of the phase space given by the metric and the electromagnetic fields then in [7] it was proved that all infinitesimal symmetries are projectable and there are no hidden symmetries. In this case Killing-Maxwell multi-vector fields generate functions conserved by the Reeb vector field of the structure, but not infinitesimal symmetries.

In the paper we discus relations between functions on $T^*\boldsymbol{E}$ which are constants of motion and functions on $\mathcal{J}_1\boldsymbol{E}$ conserved by the the Reeb vector fields. We prove that conserved phase functions are obtained as a pull-back of constants of motion on $T^*\boldsymbol{E}$.

1. Infinitesimal symmetries of the kinetic energy function

A *classical spacetime* is assumed to be an oriented and time oriented 4-dimensional manifold $\boldsymbol{E}$ equipped with a Lorentzian metric g of signature $(1,3)$. We denote by (x^λ) local coordinates on $\boldsymbol{E}$ and by $(x^\lambda, \dot{x}_\lambda)$ the induced fibred coordinates on $T^*\boldsymbol{E}$. In what follows we shall use notation $d^\lambda = dx^\lambda$, $\dot{d}_\lambda = d\dot{x}_\lambda$, $\partial_\lambda = \frac{\partial}{\partial x^\lambda}$ and $\dot{\partial}^\lambda = \frac{\partial}{\partial \dot{x}_\lambda}$ The inverse metric will be denoted by $\bar{g}$.

1.1. **Canonical symplectic structure.** Suppose the phase space to be the cotangent bundle $T^*\boldsymbol{E}$. Then we have the canonical symplectic 2-form ω and the canonical Poisson 2-vector Λ given by

$$\omega = \dot{d}_\lambda \wedge d^\lambda\,, \qquad \Lambda = \dot{\partial}^\lambda \wedge \partial_\lambda\,.$$

Let us assume the *kinetic energy function*

$$H = \tfrac{1}{2} g^{\lambda\mu}\, \dot{x}_\lambda\, \dot{x}_\mu\,.$$

A function K on $T^*\boldsymbol{E}$ is said to be a *constant of motion* if

$$0 = \{H, K\} = L_{X_H} K = -L_{X_K} H = g^{\lambda\mu}\, \dot{x}_\lambda\, \partial_\mu K - \tfrac{1}{2}\, \dot{\partial}^\rho K\, \partial_\rho g^{\lambda\mu}\, \dot{x}_\lambda\, \dot{x}_\mu\,. \tag{1.1}$$

Remark 1.1. A phase function K is a constant of motion means that its Hamiltonian lift X_K is an infinitesimal symmetry of the kinetic energy function or that K is constant on geodesic curves since the Hamiltonian lift

$$X_H = g^{\lambda\rho}\, \dot{x}_\rho\, \partial_\lambda - \tfrac{1}{2}\, \partial_\lambda g^{\rho\sigma}\, \dot{x}_\rho\, \dot{x}_\sigma\, \dot{\partial}^\lambda\,. \tag{1.2}$$

is the tangent vector field of lifts of geodesics to $T^*\boldsymbol{E}$, [13]. □

Now, let us discuss functions satisfying the equation (1.1). If K is the pull-back of a spacetime function then K has to be a constant. Further suppose that $\overset{k}{K}$ is homogeneous of order k on fibres of $T^*\boldsymbol{E}$, i.e.

$$\overset{k}{K} = \overset{k}{K}{}^{\lambda_1\dots\lambda_k}\,\dot{x}_{\lambda_1}\dots\dot{x}_{\lambda_k}\,, \qquad \overset{k}{K}{}^{\lambda_1\dots\lambda_k} \in C^\infty(\boldsymbol{E})\,.$$

Then $\overset{k}{K}$ can be considered as a symmetric k-vector field $\overset{k}{K} = \overset{k}{K}{}^{\lambda_1\dots\lambda_k}\,\partial_{\lambda_1}\odot\dots\odot\partial_{\lambda_k}$. Then the equation (1.1) is satisfied if and only if $\overset{k}{K}$ satisfies the *Killing equation*

$$[\bar{g},\overset{k}{K}] = 0\,, \tag{1.3}$$

where $[\,,\,]$ is the Schouten bracket for symmetric multi-vector fields. So $\overset{k}{K}$, considered as a k-vector field, is a Killing tensor field.

Remark 1.2. For $k=1$ we obtain that a vector field $\overset{1}{K}$ admits a symmetry of the kinetic energy function if and only if it is a Killing vector field. Moreover, the Hamiltonian lift of the corresponding constant of motion is the flow lift $\mathcal{T}^*\overset{1}{K}$ of the vector field $\overset{1}{K}$ to the cotangent bundle. □

So functions of the type

$$K = \overset{0}{K} + \sum_{k\geq 1}\overset{k}{K}{}^{\lambda_1\dots\lambda_k}\,\dot{x}_{\lambda_1}\dots\dot{x}_{\lambda_k}\,, \quad \overset{0}{K},\ \overset{k}{K}{}^{\lambda_1\dots\lambda_k} \in C^\infty(\boldsymbol{E})\,, \tag{1.4}$$

are constants of motion if and only if $\overset{0}{K}$ is a constant and $\overset{k}{K}$, $k\geq 1$, are Killing multi-vector fields.

1.2. **(Souriau's) coupling with an electromagnetic field.** Let us consider a Maxwell (electromagnetic) field $F = F_{\lambda\mu}\,d^\lambda\wedge d^\mu$ satisfying the Maxwell equation $dF=0$. Then we consider the *total (joined) 2-form*, [2],

$$\omega^{\mathfrak{j}} = \omega + \tfrac{1}{2}F = \dot{d}_\lambda\wedge d^\lambda + \tfrac{1}{2}F_{\lambda\mu}\,d^\lambda\wedge d^\mu\,.$$

We obtain the corresponding *total (joined) Poisson 2-vector*

$$\Lambda^{\mathfrak{j}} = \Lambda + \Lambda^{\mathfrak{e}} = \dot{\partial}^\lambda\wedge\partial_\lambda + \tfrac{1}{2}F_{\lambda\mu}\,\dot{\partial}^\lambda\wedge\dot{\partial}^\mu\,.$$

Assume a function K on $T^*\boldsymbol{E}$ satisfying

$$0 = \{H,K\}^{\mathfrak{j}} = L_{X^{\mathfrak{j}}_H}K = g^{\lambda\rho}\,\dot{x}_\rho\,\partial_\lambda K - \big(\tfrac{1}{2}\,\partial_\lambda g^{\rho\sigma}\,\dot{x}_\rho\,\dot{x}_\sigma - F_{\rho\lambda}\,g^{\rho\sigma}\,\dot{x}_\sigma\big)\,\dot{\partial}^\lambda K\,, \tag{1.5}$$

where $\{,\}^{\mathfrak{j}}$ is the *total (joined) Poisson bracket*.

According to [13] functions of the type (1.4) satisfy the equation (1.5) if and only if

$$0 = \sum_{k\geq 1}\big(\tfrac{1}{2}\,[\bar{g},\overset{k-1}{K}]^{\sigma_1\dots\sigma_k} + k\,F_\rho{}^{\sigma_1}\,\overset{k}{K}{}^{\rho\sigma_2\dots\sigma_k}\big)\,\dot{x}_{\sigma_1}\dots\dot{x}_{\sigma_k}\,. \tag{1.6}$$

Corollary 1.1. *For a function $K = \overset{0}{K} + \overset{1}{K}{}^{\lambda}\, \dot{x}_{\lambda}$ two identities have to be satisfied*

$$0 = g^{\rho\sigma_1}\, \partial_{\rho}\overset{0}{K} + F_{\rho}{}^{\sigma_1}\, \overset{1}{K}{}^{\rho}\,, \qquad 0 = [\bar{g}\,, \overset{1}{K}]\,,$$

which implies that $\overset{1}{K}$ is a Killing vector field and the identity $d\overset{0}{K} + \overset{1}{K} \lrcorner F = 0$ is satisfied. Then $\overset{1}{K}$ is an infinitesimal symmetry of $\widehat{F}$, i.e. $L_{\overset{1}{K}} F = 0$. Moreover,

$$X^{\mathfrak{j}}_{K} = \overset{1}{K}{}^{\lambda}\, \partial_{\lambda} - (\partial_{\lambda}\overset{0}{K} + \partial_{\lambda}\overset{1}{K}{}^{\rho}\, \dot{x}_{\rho} + F_{\rho\lambda}\, \overset{1}{K}{}^{\rho})\, \dot{\partial}^{\lambda} = \overset{1}{K}{}^{\lambda}\, \partial_{\lambda} - \partial_{\lambda}\overset{1}{K}{}^{\rho}\, \dot{x}_{\rho}\, \dot{\partial}^{\lambda}$$

which is the flow lift $\mathcal{T}^{*}\overset{1}{K}$ of the vector field $\overset{1}{K}$. □

Corollary 1.2. *For a function function $\overset{k}{K} = \overset{k}{K}{}^{\lambda_1\dots\lambda_k}\, \dot{x}_{\lambda_1}\dots\dot{x}_{\lambda_k}$, $k \geq 2$, we get*

$$0 = [\bar{g}, \overset{k}{K}]\,, \qquad 0 = F_{\rho}{}^{(\sigma_1}\, \overset{k}{K}{}^{\sigma_2\dots\sigma_k)\rho}\,,$$

i.e. $\overset{k}{K}$ is a Killing-Maxwell k-vector field, [2]. *Moreover, the corresponding vector field $X^{\mathfrak{j}}_{\overset{k}{K}}$ is not projectable on spacetime and the infinitesimal symmetry of H is hidden.* □

2. Infinitesimal symmetries of the gravitational contact phase structure

In what follows we shall consider a phase space of a general relativistic test particle considered as the 1-jet space of motions. Our theory is explicitly independent of scales, so we introduce the spaces of scales in the sense of [10]. Any tensor field carries explicit information on its scale dimension. We assume the following basic spaces of scales: the space of *time intervals* $\mathbb{T}$, the space of *lengths* $\mathbb{L}$ and the space of *mass* $\mathbb{M}$. We assume the *speed of light* $c \in \mathbb{T}^* \otimes \mathbb{L}$ and the *Planck constant* $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$ as the *universal scales.* We denote as $u^0 \in \mathbb{T}^*$ a base.

2.1. **Classical phase space.** Now we assume the metric to be scaled, i.e. $g\colon \boldsymbol{E} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{E} \odot T^*\boldsymbol{E})$. A *spacetime chart* is defined to be a chart $(x^{\lambda}) \equiv (x^0, x^i) \in C^{\infty}(\boldsymbol{U}, \mathrm{I\!R} \times \mathrm{I\!R}^3)$, $\boldsymbol{U} \subset \boldsymbol{E}$ is open, of $\boldsymbol{E}$, which fits the orientation of spacetime and such that the vector field ∂_0 is timelike and time oriented and the vector fields $\partial_1, \partial_2, \partial_3$ are spacelike. Greek indices $\lambda, \mu, \dots$ will span spacetime coordinates, while Latin indices $i, j, \dots$ will span spacelike coordinates.

For a particle with mass m it is very convenient to use the *re-scaled metric* $G = \frac{m}{\hbar} g\colon \boldsymbol{E} \to \mathbb{T} \otimes (T^*\boldsymbol{E} \odot T^*\boldsymbol{E})$, $G^0_{\lambda\mu} = \frac{m}{\hbar_0}\, g_{\lambda\mu}$, and the associated contravariant re-scaled metric $\bar{G} = \frac{\hbar}{m}\bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \odot T\boldsymbol{E})$, $G_0^{\lambda\mu} = \frac{\hbar_0}{m}\, g^{\lambda\mu}$.

We assume *time* to be a one-dimensional affine space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} = \mathbb{T} \otimes \mathbb{R}$. A *motion* is defined to be a 1-dimensional timelike submanifold $s\colon \boldsymbol{T} \hookrightarrow \boldsymbol{E}$. The *1st differential* of the motion s is defined to be the tangent map $ds\colon T\boldsymbol{T} = \boldsymbol{T} \times \bar{\mathbb{T}} \to T\boldsymbol{E}$.

We assume as *phase space* the open subspace $\mathcal{J}_1\boldsymbol{E} \subset J_1(\boldsymbol{E}, 1)$ consisting of all 1-jets of motions. So elements of $\mathcal{J}_{1x}\boldsymbol{E}$ are classes of non-parametrized curves which have in a point $x \in \boldsymbol{E}$ the same tangent line lying inside the light cone, [8]. $\pi^1_0 \colon \mathcal{J}_1\boldsymbol{E} \to \boldsymbol{E}$ is a fibred manifold but NOT an affine bundle! The *velocity* of a motion s is defined to be its 1–jet $j_1 s \colon \boldsymbol{T} \to \mathcal{J}_1(\boldsymbol{E},1)$. For each 1-dimensional submanifold $s : \boldsymbol{T} \hookrightarrow \boldsymbol{E}$ and for each $x \in \boldsymbol{T}$, we have $j_1 s(x) \in \mathcal{J}_1\boldsymbol{E}$ if and only if $ds(x)(u) \in T_{s(x)}\boldsymbol{E}$ is timelike, where $u \in \mathbb{T}$.

Any spacetime chart (x^0, x^i) is related to each motion s which means that s can be locally expressed by $(x^0, x^i = s^i(x^0))$. Then we obtain the induced fibred coordinate chart (x^0, x^i, x^i_0) on $\mathcal{J}_1\boldsymbol{E}$ such that $x^i_0 \circ s = \partial_0 s^i$. Moreover, there exists a time unit function $\boldsymbol{T} \to \mathbb{T}$ such that the 1st differential of s, considered as the map $ds \colon \boldsymbol{T} \to \bar{\mathbb{T}}^* \otimes T\boldsymbol{E}$, is normalized by $g(ds, ds) = -c^2$, for details see [8].

We shall always refer to the above fibred charts.

We define the *contact map* to be the unique fibred morphism $д \colon \mathcal{J}_1\boldsymbol{E} \to \bar{\mathbb{T}}^* \otimes T\boldsymbol{E}$ over $\boldsymbol{E}$, such that $д \circ j_1 s = ds$, for each motion s. We have $g\,(д, д) = -c^2$. The coordinate expression of $д$ is

$$(2.1)\quad д = c\,\alpha^0\,(\partial_0 + x^i_0\,\partial_i)\,, \qquad \text{where} \qquad \alpha^0 := 1/\sqrt{|g_{00} + 2\,g_{0j}\,x^j_0 + g_{ij}\,x^i_0\,x^j_0|}\,.$$

We define the *time form* to be the fibred morphism $\tau = -\frac{1}{c^2}\,g^\flat(д) \colon \mathcal{J}_1\boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$, considered as the scaled horizontal 1-form of $\mathcal{J}_1\boldsymbol{E}$. We have the coordinate expression

$$(2.2)\qquad \tau = \tau_\lambda\,d^\lambda = -\tfrac{\alpha^0}{c}\,(g_{0\lambda} + g_{i\lambda}\,x^i_0)\,d^\lambda\,.$$

Note 2.1. In what follows it is very convenient to use the following notation $\breve{\delta}^i_\lambda = \delta^i_\lambda - x^i_0\,\delta^0_\lambda$ and $\breve{\delta}^\mu_0 = \delta^\mu_0 + \delta^\mu_p\,x^p_0$. Then $д = c\,\alpha^0\,\breve{\delta}^\mu_0\,\partial_\mu$ and $\tau = -\frac{\alpha^0}{c}\,\breve{g}_{0\lambda}\,d^\lambda$, where $\breve{g}_{0\lambda} = g_{\mu\lambda}\,\breve{\delta}^\mu_0$. □

Let $V\mathcal{J}_1\boldsymbol{E} \subset T\mathcal{J}_1\boldsymbol{E}$ be the vertical tangent subbundle over $\boldsymbol{E}$. The vertical prolongation of the contact map yields the mutually inverse linear fibred isomorphisms

$$\nu_\tau \colon \mathcal{J}_1\boldsymbol{E} \to \mathbb{T} \otimes V^*_\tau\boldsymbol{E} \otimes V\mathcal{J}_1\boldsymbol{E} \qquad \text{and} \qquad \nu^{-1}_\tau \colon \mathcal{J}_1\boldsymbol{E} \to V^*\mathcal{J}_1\boldsymbol{E} \otimes \mathbb{T}^* \otimes V_\tau\boldsymbol{E}\,,$$

where $V_\tau\boldsymbol{E} = \ker\tau \subset T\boldsymbol{E}$, with coordinate expressions

$$\nu_\tau = \frac{1}{c\,\alpha^0}\,\breve{\delta}^i_\lambda\,d^\lambda \otimes \partial^0_i\,, \quad \nu^{-1}_\tau = c\,\alpha^0\,d^i_0 \otimes \big(\partial_i - c\,\alpha^0\tau_i\,\breve{\delta}^\lambda_0\,\partial_\lambda\big)\,.$$

2.2. **Infinitesimal symmetries of the gravitational contact phase structure.** For a particle with mass m we can unscale the time 1-form and obtain a contact 1-form $\widehat{\tau} = \frac{m\,c^2}{\hbar}\,\tau = \widehat{\tau}_\lambda\,d^\lambda$, where $\widehat{\tau}_\lambda = -\frac{m\,c\,\alpha^0}{\hbar}\,\breve{g}_{0\lambda}$. So the metric g defines on the phase space $\mathcal{J}_1\boldsymbol{E}$ the *gravitational contact structure* $(-\widehat{\tau}, \Omega^{\mathfrak{g}})$, where $\Omega^{\mathfrak{g}} = -d\widehat{\tau}$. Then we have the dual Jacobi pair $(-\widehat{\gamma}^{\mathfrak{g}}, \Lambda^{\mathfrak{g}})$ given by the Reeb vector field $-\widehat{\gamma}^{\mathfrak{g}}$ and the 2-vector field $\Lambda^{\mathfrak{g}}$, [8].

We define an *infinitesimal symmetry of the gravitational contact phase structure* to be a phase vector field X on $\mathcal{J}_1\boldsymbol{E}$ which is a symmetry of $\widehat{\tau}$, i.e. $L_X\widehat{\tau} = 0$. By naturality we have $L_X\Omega^{\mathfrak{g}} = 0$, $L_X\widehat{\gamma}^{\mathfrak{g}} = [X, \widehat{\gamma}^{\mathfrak{g}}] = 0$ and $L_X\Lambda^{\mathfrak{g}} = [X, \Lambda^{\mathfrak{g}}] = 0$. According to [5] any infinitesimal symmetry of the pair $(-\widehat{\tau}, \Omega^{\mathfrak{g}})$ is the Hamilton-Jacobi lift

$$(2.3)\qquad X = d(\widehat{\tau}(\underline{X}))^{\sharp\mathfrak{g}} + \widehat{\tau}(\underline{X})\,\widehat{\gamma}^{\mathfrak{g}}$$

of the phase function $\widehat{\tau}(\underline{X})$, where $\underline{X} = T\pi^1_0(X) \colon \mathcal{J}_1\boldsymbol{E} \to T\boldsymbol{E}$ is a *generalized vector field* in the sense of [12] such that $\widehat{\gamma}^{\mathfrak{g}} \cdot (\widehat{\tau}(\underline{X})) = 0$. So, a generalized vector field $\underline{X}$ has to satisfy the following conditions:

1. (Projectability condition) The Hamilton-Jacobi lift (2.3) of the phase function $\widehat{\tau}(\underline{X})$ projects on $\underline{X}$.

2. (Conservation condition) The phase function $\widehat{\tau}(\underline{X})$ is *conserved*, i.e.

$$\widehat{\gamma}^{\mathfrak{g}} \cdot (\widehat{\tau}(\underline{X})) = 0\,.$$

The following results were proved in [5].

Theorem 2.1. *Let $\underline{X} = \underline{X}^\lambda\,\partial_\lambda \colon \mathcal{J}_1\boldsymbol{E} \to T\boldsymbol{E}$, $\underline{X}^\lambda \in C^\infty(\mathcal{J}_1\boldsymbol{E})$, be a generalized vector field, then the following assertions are equivalent:*

1. *The Hamilton-Jacobi lift* (2.3) *projects on $\underline{X}$.*

2. *The vertical prolongation*

$$V\underline{X} \colon V\mathcal{J}_1\boldsymbol{E} \to VT\boldsymbol{E} = T\boldsymbol{E} \oplus T\boldsymbol{E}$$

has values in the kernel of $\widehat{\tau}$.

3. *In coordinates*

$$\breve{g}_{0\rho}\,\partial^0_j \underline{X}^\rho = 0\,. \tag{2.4}$$ □

Lemma 2.2. *For generalized vector fields $\underline{X}$ and $\underline{Y}$ satisfying the projectability condition we have*

$$\{\widehat{\tau}(\underline{X}), \widehat{\tau}(\underline{Y})\}^{\mathfrak{g}} + \widehat{\tau}(\underline{X})\,\widehat{\gamma}^{\mathfrak{g}} \cdot (\widehat{\tau}(\underline{Y})) - \widehat{\tau}(\underline{Y})\,\widehat{\gamma}^{\mathfrak{g}} \cdot (\widehat{\tau}(\underline{X})) = \widehat{\tau}(\underline{[X,Y]})\,. \tag{2.5}$$ □

Remark 2.1. Let us remark that on the left hand side of (2.5) there is the Jacobi bracket of functions $\widehat{\tau}(\underline{X})$ and $\widehat{\tau}(\underline{Y})$. □

Theorem 2.3. *Let $\underline{X}$ be a generalized vector field satisfying the projectability condition. Then the following assertions are equivalent:*

1. *The Hamilton-Jacobi lift* (2.3) *is an infinitesimal symmetry of the gravitational contact phase structure.*
2. *The phase function $\widehat{\tau}(\underline{X})$ is conserved, i.e. $\widehat{\gamma}^{\mathfrak{g}} \cdot \widehat{\tau}(\underline{X}) = 0$.*
3. *The vector field $[\widehat{\gamma}^{\mathfrak{g}}, X]$ is in* $\ker \widehat{\tau}$.
4. *In coordinates*

$$0 = \breve{\delta}^\rho_0\,\breve{\delta}^\sigma_0\,(g_{\rho\omega}\,\partial_\sigma \underline{X}^\omega + \tfrac{1}{2}\,\underline{X}^\omega\,\partial_\omega g_{\rho\sigma})\,. \tag{2.6}$$ □

Corollary 2.4. *For generalized vector fields $\underline{X}$ and $\underline{Y}$ satisfying the projectability and the conservation conditions we have*

$$\{\widehat{\tau}(\underline{X}), \widehat{\tau}(\underline{Y})\} = \widehat{\tau}(\underline{[X,Y]})\,. \tag{2.7}$$

Moreover, the phase function $\widehat{\tau}(\underline{[X,Y]})$ is conserved. □

Theorem 2.5. *The Lie algebra of infinitesimal symmetries of the gravitational contact phase structure is formed by the Hamilton-Jacobi lifts of phase functions $\widehat{\tau}(\underline{X})$, where generalized vector fields $\underline{X}$ satisfy the projectability and the conservation conditions* (2.4) *and* (2.6).

Moreover, if $\underline{X}$ factorises through a spacetime vector field, then the corresponding infinitesimal symmetry is projectable and it is the jet flow lift $\mathcal{J}_1\underline{X}$. If $\underline{X}$ is a generalized vector field which is not factorisable through a spacetime vector field, then the corresponding infinitesimal symmetry is hidden. □

2.3. **Infinitesimal symmetries of the gravitational contact phase structure generated by Killing multi-vector fields.** In [5] it was proved that a symmetric k-vector field $\overset{k}{K}$, $k \geq 1$, admits generalized vector field satisfying the projectability condition. Such generalized vector fields are given by

$$\underline{X}[\overset{k}{K}] = k\,\underbrace{\widehat{\tau}\,\lrcorner \ldots \lrcorner\,\widehat{\tau}\,\lrcorner}_{(k-1)-\text{times}}\overset{k}{K} - (k-1)\overset{k}{K}(\widehat{\tau}, \ldots, \widehat{\tau})\,\widehat{Д} \colon \mathcal{J}_1\boldsymbol{E} \to T\boldsymbol{E}\,, \tag{2.8}$$

where $\widehat{Д} = \frac{\hbar}{m\,c^2}Д$. Then we obtain the induced phase function

$$\widehat{\tau}\big(\underline{X}[\overset{k}{K}]\big) = \overset{k}{K}(\widehat{\tau}) := \overset{k}{K}(\widehat{\tau}, \ldots, \widehat{\tau}) = \overset{k}{K}{}^{\lambda_1 \ldots \lambda_k}\,\widehat{\tau}_{\lambda_1} \ldots \widehat{\tau}_{\lambda_k}\,.$$

Theorem 2.6. *The phase function $\overset{k}{K}(\widehat{\tau})$ is conserved with respect to the gravitational Reeb vector field, i.e. $\widehat{\gamma}^{\mathfrak{g}} \cdot \overset{k}{K}(\widehat{\tau}) = 0$, if and only if $\overset{k}{K}$ is a Killing k-vector field.* □

Remark 2.2. Let $\overset{0}{K}$ be a spacetime function. Then $\widehat{\gamma}^{\mathfrak{g}}.\overset{0}{K} = 0$ if and only if $\overset{0}{K}$ is a constant. □

Theorem 2.7. *The Hamilton-Jacobi lift of a phase function*

$$K = \overset{0}{K} + \sum_{k\geq 1} \overset{k}{K}(\widehat{\tau}), \tag{2.9}$$

is an infinitesimal symmetry of the gravitational contact phase structure $(-\widehat{\tau}, \Omega^{\mathfrak{g}})$ if and only if $\overset{0}{K}$ is a constant and $\overset{k}{K}$, $k \geq 1$, are Killing k-vector fields. □

Remark 2.3. For a Killing vector field $\overset{1}{K}$ the Hamilton–Jacobi lift of $\widehat{\tau}(\overset{1}{K})$ coincides with the jet flow lift $\mathcal{J}_1\overset{1}{K}$ and the corresponding infinitesimal symmetry is projectable on spacetime. For $k \geq 2$ the corresponding infinitesimal symmetry is hidden. □

Remark 2.4. For a constant $\overset{0}{K}$ and Killing k-vector fields $\overset{k}{K}$, $k \geq 1$, the conserved phase function of the type (2.9) admits the infinitesimal symmetry

$$X[K] = \overset{0}{K}\,\widehat{\gamma} + \sum_{k\geq 1} X[\overset{k}{K}]$$

which projects on the generalized vector field

$$\underline{X}[K] = \big(\overset{0}{K} - \sum_{k\geq 2}(k-1)\,\overset{k}{K}(\widehat{\tau})\big)\,\widehat{Д} + \overset{1}{K} + \sum_{k\geq 2} k\,\underbrace{\widehat{\tau}\,\lrcorner \ldots \lrcorner\,\widehat{\tau}\,\lrcorner}_{(k-1)-\text{times}}\overset{k}{K}$$

satisfying the projectability and the conservation conditions. $\square$

2.4. **Comparison with infinitesimal symmetries of the kinetic energy function.** Suppose the morphism

$$-\widehat{\tau}\colon \mathcal{J}_1\boldsymbol{E} \longrightarrow T^*\boldsymbol{E}\,.$$

over $\boldsymbol{E}$. In coordinates we have

$$x^\lambda = x^\lambda\,, \quad \dot{x}_\lambda = -\widehat{\tau}_\lambda = \tfrac{m\,c\,\alpha^0}{\hbar}\,\breve{g}_{0\lambda}\,.$$

Remark 2.5. Let us note that the image of the mapping $-\widehat{\tau}$ is the subset of $T^*\boldsymbol{E}$ given by elements satisfying the condition

$$\widehat{G}^{\lambda\mu}\,\dot{x}_\lambda\,\dot{x}_\mu = -1\,,$$

where $\widehat{G} = \frac{\hbar^2}{m^2\,c^2}\,\bar{g}$ is the unscaled metric. $\square$

Lemma 2.8. *Let K be a constant of motion on $T^*\boldsymbol{E}$, i.e. $X_H\cdot K = 0$, then its pull–back $-\widehat{\tau}^*(K)$ is a conserved function, i.e. $\widehat{\gamma}^{\mathfrak{g}}\cdot\widehat{\tau}^*(K) = 0$.*

Proof. First, it is easy to see that we have

$$-T_e\widehat{\tau}(\widehat{\gamma}^{\mathfrak{g}}) = X_H(\widehat{\tau}(e))\,,$$

where $e\in\mathcal{J}_1\boldsymbol{E}$ and X_H is the vector field (1.2).

Now,

$$\begin{aligned}\widehat{\gamma}^{\mathfrak{g}}\cdot\widehat{\tau}^*(K) &= i_{\widehat{\gamma}^{\mathfrak{g}}}d\widehat{\tau}^*(K)\\ &= i_{\widehat{\gamma}^{\mathfrak{g}}}\widehat{\tau}^*(dK) = dK(T\widehat{\tau}(\widehat{\gamma}^{\mathfrak{g}})) = -dK(X_H) = -X_H\cdot K\,.\ \square\end{aligned}$$

Now, let us assume a function (1.4) on $T^*\boldsymbol{E}$. The above function is a constant of motion if and only if $\overset{0}{K}(x)$ is a constant and $\overset{k}{K}(x)$, $k\geq 1$, are Killing k-vector fields. The pull-back of the function (1.4) is the phase function (2.9) which is a function conserved by the gravitational Reeb vector field. Let us note that in [5] the conserved functions of the type (2.9) were obtained in a different way by using generalized vector fields (2.8).

By Lemma 2.8 we get the following diagram

$$\begin{tikzcd}
(T^*\boldsymbol{E},\omega) & \text{constants of motion } K,\ \{H,K\}=0 \arrow[r, "\text{Hamiltonian}", "\text{lift}"'] \arrow[dd, "-\widehat{\tau}^*"] & \text{ISs of } H \\
\text{Killing multi-vectors} \arrow[ur] \arrow[dr] & & \\
(\mathcal{J}_1\boldsymbol{E}, -\widehat{\tau}, \Omega^{\mathfrak{g}}) & \text{conserved functions, } \widehat{\gamma}^{\mathfrak{g}}\cdot(-\widehat{\tau}^*(K))=0 \arrow[r, "\text{Hamilton-Jacobi}", "\text{lift}"'] & \text{ISs of } (-\widehat{\tau},\Omega^{\mathfrak{g}})
\end{tikzcd}$$

For Killing vector fields we obtain in both cases projectable infinitesimal symmetries which are obtained by the flow lifts. For Killing k-vector fields, $k\geq 2$, the corresponding infinitesimal symmetries are hidden.

3. Infinitesimal symmetries of the total almost-cosymplectic-contact phase structure

We assume the *total (joined) almost-cosymplectic-contact structure* $(-\widehat{\tau}, \Omega^{\mathfrak{j}})$ on the phase space given naturally by the metric and an electromagnetic field.

3.1. **Total almost-cosymplectic-contact phase structure.** We assume an *electromagnetic field* to be a closed scaled 2-form on $\boldsymbol{E}$

$$F\colon \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \bigwedge^2 T^*\boldsymbol{E}\,.$$

Given a particle with *charge* $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$ the rescaled electromagnetic field $\widehat{F} = \frac{q}{\hbar} F$ can be incorporated into the geometrical structure of the phase space, i.e. the gravitational form. Namely, we define the *total (joined) phase 2-form*

$$\Omega^{\mathfrak{j}} := \Omega^{\mathfrak{g}} + \Omega^{\mathfrak{e}} = \Omega^{\mathfrak{g}} + \tfrac{1}{2}\widehat{F}\colon \mathcal{J}_1\boldsymbol{E} \to \bigwedge^2 T^*\mathcal{J}_1\boldsymbol{E}. \tag{3.1}$$

The pair $(-\widehat{\tau}, \Omega^{\mathfrak{j}})$ is almost-cosymplectic-contact, i.e. it is regular and and the 2–form $\Omega^{\mathfrak{j}}$ is closed, [9]. Then the dual almost-coPoisson-Jacobi pair is $(-\widehat{\gamma}^{\mathfrak{j}}, \Lambda^{\mathfrak{j}})$. Here $\widehat{\gamma}^{\mathfrak{j}} = \frac{\hbar}{m\,c^2}(\gamma^{\mathfrak{g}} + \gamma^{\mathfrak{e}})$, where

$$\gamma^e\colon \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\mathcal{J}_1\boldsymbol{E}$$

with the coordinate expression

$$\gamma^e = -\breve{G}^{i\lambda}_0\, \breve{\delta}^{\rho}_0\, \widehat{F}_{\rho\lambda}\, u^0 \otimes \partial^0_i\,, \tag{3.2}$$

$\breve{G}^{i\lambda}_0 = \breve{\delta}^i_\mu\, G^{\mu\lambda}_0$. Further, $\Lambda^{\mathfrak{j}} = \Lambda^{\mathfrak{g}} + \Lambda^{\mathfrak{e}}$, where

$$\Lambda^{\mathfrak{e}} = \frac{1}{2\,(c_0\alpha^0)^2}\breve{G}^{i\lambda}_0\, \breve{G}^{j\mu}_0\, \widehat{F}_{\lambda\mu}\, \partial^0_i \wedge \partial^0_j\,. \tag{3.3}$$

3.2. **Infinitesimal symmetries of the total almost-cosymplectic-contact phase structure.** We define a *phase infinitesimal symmetry* of the total almost-cosymplectic-contact phase structure to be a vector field X on $\mathcal{J}_1\boldsymbol{E}$ such that: (1) $L_X\widehat{\tau} = 0$; (2) $L_X\Omega^{\mathfrak{j}} = 0$.

Remark 3.1. The conditions (1) and (2) are equivalent to $[X, \widehat{\gamma}^{\mathfrak{j}}] = 0$ and $[X, \Lambda^{\mathfrak{j}}] = 0$. □

Lemma 3.1. *A phase vector field X is an infinitesimal symmetry of $\Omega^{\mathfrak{j}}$ if and only if it is of the form*

$$X = df^{\sharp\mathfrak{j}} + h\,\widehat{\gamma}^{\mathfrak{j}}\,, \tag{3.4}$$

where f is a conserved phase function, i.e. $\widehat{\gamma}^{\mathfrak{j}} \cdot f = 0$, and $h = \widehat{\tau}(\underline{X})$, $\underline{X} = T\pi^1_0(X)$.

Proof. We have the splitting $T\mathcal{J}_1\boldsymbol{E} = \ker\widehat{\tau} \oplus \langle\widehat{\gamma}^{\mathfrak{j}}\rangle$, i.e. $X = \widetilde{X} + h\,\widehat{\gamma}^{\mathfrak{j}}$, where $\widehat{\tau}(\widetilde{X}) = 0$ and h is a phase function. Then from $\widehat{\tau}(\widehat{\gamma}^{\mathfrak{j}}) = 1$ we have $h = \widehat{\tau}(\underline{X})$.

Further the phase 2-form $\Omega^{\mathfrak{j}}$ is closed, then from $i_{\widehat{\gamma}^{\mathfrak{j}}}\Omega^{\mathfrak{j}} = 0$ we obtain $0 = L_X\Omega^{\mathfrak{j}} = d i_{\widetilde{X}}\Omega^{\mathfrak{j}}$, which implies locally that $i_{\widetilde{X}}\Omega^{\mathfrak{j}} = df$ for a phase function f, i.e. $\widetilde{X} = df^{\sharp\mathfrak{j}}$. Moreover, $\widehat{\gamma}^{\mathfrak{j}} \cdot f = i_{\widehat{\gamma}^{\mathfrak{j}}} df = i_{\widehat{\gamma}^{\mathfrak{j}}} i_{\widetilde{X}}\Omega^{\mathfrak{j}} = -i_{\widetilde{X}} i_{\widehat{\gamma}^{\mathfrak{j}}}\Omega^{\mathfrak{j}} = 0$. □

Theorem 3.2. *A phase vector field* (3.4)*, where where f is a conserved phase function, i.e. $\widehat{\gamma}^{\mathfrak{j}} \cdot f = 0$, and $h = \widehat{\tau}(\underline{X})$, $\underline{X} = T\pi^1_0(X)$, is an infinitesimal symmetry of $\widehat{\tau}$ if and only if f is of the form*

$$f = \widehat{\tau}(\underline{X}) + \breve{f} \tag{3.5}$$

for a generalized vector field $\underline{X}$ and a spacetime function $\breve{f} \in C^\infty(\boldsymbol{E})$ such that

$$d\breve{f} = \underline{X} \lrcorner \widehat{F}\,. \tag{3.6}$$

Proof. By Lemma 3.1 infinitesimal symmetries of $(-\widehat{\tau}, \Omega^{\mathfrak{j}})$ are of the form (3.4), where $\widehat{\gamma}^{\mathfrak{j}}.f = \widehat{\gamma}^{\mathfrak{g}} \cdot f + \widehat{\gamma}^{\mathfrak{e}}.f = 0$, and $h = \widehat{\tau}(\underline{X})$. If X is an infinitesimal symmetry of $\widehat{\tau}$ then by [4]

$$- i_{df^{\sharp\mathfrak{j}}} \Omega^{\mathfrak{g}} - h\, i_{\widehat{\gamma}^{\mathfrak{e}}} \Omega^{\mathfrak{g}} + dh = 0\,. \tag{3.7}$$

But

$$i_{\widehat{\gamma}^{\mathfrak{e}}} \Omega^{\mathfrak{g}} = -\widehat{Д} \lrcorner \widehat{F}$$

and

$$i_{df^{\sharp\mathfrak{j}}} \Omega^{\mathfrak{g}} = df + \tfrac{1}{c_0\,\alpha^0} \breve{G}^{i\lambda}_0\, \partial^0_i f \widehat{F}_{\lambda\mu}\, d^\mu = df + (G^\sharp \circ \nu_\tau)(d_V f) \lrcorner \widehat{F}\,.$$

which follows from

$$\begin{aligned}
(\Omega^{\flat\mathfrak{g}} \circ \Lambda^{\sharp\mathfrak{g}})(df) &= df - (\widehat{\gamma}^{\mathfrak{g}} \cdot f)\, \widehat{\tau} = df + (\widehat{\gamma}^{\mathfrak{e}} \cdot f)\, \widehat{\tau}\,,\\
(\Omega^{\flat\mathfrak{g}} \circ \Lambda^{\sharp\mathfrak{e}})(df) &= -(\widehat{\gamma}^{\mathfrak{e}} \cdot f)\, \widehat{\tau} + \tfrac{1}{c_0\,\alpha^0} \breve{G}^{i\lambda}_0\, \partial^0_i f \widehat{F}_{\lambda\mu}\, d^\mu\,.
\end{aligned}$$

Then (3.7) reads as

$$d(h - f) = -h\, \widehat{Д} \lrcorner \widehat{F} + (G^\sharp \circ \nu_\tau)(d_V f) \lrcorner \widehat{F}\,. \tag{3.8}$$

Now, if we put $\widehat{\tau}(X) = \widehat{\tau}(\underline{X}) = h$, then we can rewrite (3.8) as

$$\begin{aligned}
df &= d(\widehat{\tau}(\underline{X})) + \widehat{\tau}(\underline{X})\, \widehat{Д} \lrcorner \widehat{F} - (G^\sharp \circ \nu_\tau)(d_V f) \lrcorner \widehat{F}\\
&= d(\widehat{\tau}(\underline{X})) - \big((\alpha^0)^2 \breve{g}_{0\rho}\, \underline{X}^\rho\, \breve{\delta}^\lambda_0 + \tfrac{1}{c_0\,\alpha^0} \breve{G}^{i\lambda}_0\, \partial^0_i f\big)\, \widehat{F}_{\lambda\mu}\, d^\mu
\end{aligned} \tag{3.9}$$

which implies that

$$\partial^0_i f = \partial^0_i \widehat{\tau}(\underline{X}) = -c_0\, \alpha^0 (\breve{G}^0_{i\rho}\, \underline{X}^\rho + \breve{G}^0_{0\rho}\, \partial^0_i \underline{X}^\rho)\,, \tag{3.10}$$

where $\breve{G}^0_{i\rho} = G^0_{i\rho} + (\alpha^0)^2\, \breve{g}_{0i}\, \breve{G}^0_{0\rho}$, and we can rewrite (3.9), by using the identity $\breve{G}^{i\lambda}_0\, \breve{G}^0_{i\rho} = \delta^\lambda_\rho + (\alpha^0)^2\, \breve{g}_{0\rho}\, \breve{\delta}^\lambda_0$, as

$$df = d(\widehat{\tau}(\underline{X})) + \big(\underline{X}^\lambda + \breve{G}^{i\lambda}_0\, \breve{G}^0_{0\rho}\, \partial^0_i \underline{X}^\rho\big)\, \widehat{F}_{\lambda\mu}\, d^\mu\,. \tag{3.11}$$

If we consider the condition that the vector field (3.4), where df is given by (3.11), projects on $\underline{X}$ which is equivalent with (2.4) we get

$$df = d(\widehat{\tau}(\underline{X})) + \underline{X}^\lambda\, \widehat{F}_{\lambda\mu}\, d^\mu = d(\widehat{\tau}(\underline{X})) + \underline{X} \lrcorner \widehat{F}\,. \tag{3.12}$$

So we get (3.5), where $\breve{f} \in C^\infty(\boldsymbol{E})$ such that (3.6) is satisfied. □

Theorem 3.3. *All phase infinitesimal symmetries of the total phase structure are vector fields of the type*

$$X = d(\widehat{\tau}(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}} + \widehat{\tau}(\underline{X})\,\widehat{\gamma}^{\mathfrak{j}} \tag{3.13}$$

where $\underline{X}$ *is a generalized vector field and* $\breve{f} \in C^\infty(\boldsymbol{E})$ *satisfying the following conditions:*

1) $d\breve{f} = \underline{X} \lrcorner \widehat{F}$.

2) (*Projectability condition*) *The vector field* (3.13) *projects on* $\underline{X}$.

3) (*Conservation condition*) *The phase function* $\widehat{\tau}(\underline{X}) + \breve{f}$ *is conserved, i.e.* $\widehat{\gamma}^{\mathfrak{j}} \cdot (\widehat{\tau}(\underline{X}) + \breve{f}) = 0$.

Proof. It follows from Lemma 3.1 and Theorem 3.2. □

Remark 3.2. Let us note that to find a pair $(\underline{X}, \breve{f})$ satisfying the projectability condition 2) of the above Theorem 3.3, it is sufficient to find a generalized vector field satisfying the projectability condition (2.4) of Theorem 2.1. It follows from the fact that $d\breve{f}^{\sharp\mathfrak{j}}$ is a vertical vector field.

Further, if the condition 1) and 2) are satisfied, then the conservation condition $\widehat{\gamma}^{\mathfrak{j}} \cdot (\widehat{\tau}(\underline{X}) + \breve{f}) = 0$ is equivalent with the conservation condition given by (2.6) in Theorem 2.3 which follows from $\widehat{\gamma}^{\mathfrak{g}} \cdot \breve{f} = -\widehat{\gamma}^{\mathfrak{e}} \cdot \widehat{\tau}(\underline{X})$. □

Lemma 3.4. *Let* $(\underline{X}, \breve{f})$ *and* $(\underline{Y}, \breve{h})$ *be pairs of generalized vector fields and spacetime functions such that the projectability condition of Theorem 3.3 is satisfied. Let* X *and* Y *are phase vector fields given by* (3.13). *Then*

$$\widehat{\tau}([X,Y]) = \widehat{\tau}(\underline{[X,Y]}) = c_0\,\alpha^0\,\breve{G}^0_{0\lambda}\,(\underline{X}^\rho\,\partial_\rho \underline{Y}^\lambda - \underline{Y}^\rho\,\partial_\rho \underline{X}^\lambda)\,. \tag{3.14}$$

Proof. For a pair $(\underline{X}, \breve{f})$ satisfying the projectability condition (2.4) we obtain

$$\begin{aligned} X = \underline{X}^\lambda\,\partial_\lambda - \breve{G}^{i\rho}_0\,\big[-\frac{1}{c_0\,\alpha^0}\partial_\rho \breve{f} \\ + \underline{X}^\sigma\,\partial_\sigma \breve{G}^0_{0\rho} + \breve{G}^0_{0\sigma}\,\partial_\rho \underline{X}^\sigma + \frac{1}{c_0\,\alpha^0}\underline{X}^\sigma\,\widehat{F}_{\sigma\rho}\big]\,\partial^0_i\,. \end{aligned} \tag{3.15}$$

The same expression we have for a pair $(\underline{Y}, \breve{h})$ which implies Lemma 3.4. □

Remark 3.3. Let us note that if, moreover, the condition (3.6) is satisfied, then the vector field (3.15) is reduced to

$$X = \underline{X}^\lambda\,\partial_\lambda - \breve{G}^{i\rho}_0\,\big[\underline{X}^\sigma\,\partial_\sigma \breve{G}^0_{0\rho} + \breve{G}^0_{0\sigma}\,\partial_\rho \underline{X}^\sigma\big]\,\partial^0_i\,. \tag{3.16}$$ □

Lemma 3.5. *Let* $(\underline{X}, \breve{f})$ *and* $(\underline{Y}, \breve{h})$ *be pairs of generalized vector fields and spacetime functions satisfying the projectability condition of Theorem 3.3. Then*

$$\begin{aligned} \{\widehat{\tau}(\underline{X}), \widehat{\tau}(\underline{Y})\}^{\mathfrak{j}} &= \widehat{\tau}\big(\underline{[X,Y]}\big) - \widehat{\tau}(\underline{X})\,\widehat{\gamma}^{\mathfrak{g}} \cdot \widehat{\tau}(\underline{Y}) + \widehat{\tau}(\underline{Y})\,\widehat{\gamma}^{\mathfrak{g}} \cdot \widehat{\tau}(\underline{X}) \\ &\quad + \tfrac{1}{2}\,\widehat{F}(\underline{X},\underline{Y}) - \tfrac{1}{2}\,\widehat{\tau}(\underline{X})\,\widehat{F}(\widehat{Д},\underline{Y}) + \tfrac{1}{2}\,\widehat{\tau}(\underline{Y})\,\widehat{F}(\widehat{Д},\underline{X})\,, \end{aligned} \tag{3.17}$$

$$\{\widehat{\tau}(\underline{X})\,, \breve{h}\}^{\mathfrak{j}} = \underline{X} \cdot \breve{h} - \widehat{\tau}(\underline{X})\,\widehat{Д} \cdot \breve{h}\,, \tag{3.18}$$

$$\{\breve{f}, \breve{h}\}^{\mathfrak{j}} = 0\,. \tag{3.19}$$

Proof. We have for any phase functions f, h

$$\begin{aligned}\{f,h\}^{\mathfrak{j}} = \tfrac{1}{c_0\,\alpha^0}\,\big[\breve{G}^{p\rho}_0\,(\partial_\rho f\,\partial^0_p h - \partial^0_p f\,\partial_\rho h) \\ + \breve{G}^{i\rho}_0\,\breve{G}^{j\sigma}_0\,(\partial_\sigma \breve{G}^0_{0\rho} - \partial_\rho \breve{G}^0_{0\sigma} + \tfrac{1}{c_0\,\alpha^0}\,\widehat{F}_{\rho\sigma})\,\partial^0_i h\,\partial^0_j f\big]\,,\end{aligned}$$

and, for generalized vector fields $\underline{X}$, $\underline{Y}$ and spacetime functions $\breve{f}$ and $\breve{h}$ satisfying the projectability condition (2.4), we obtain

$$\begin{aligned}\{\widehat{\tau}(\underline{X}),\widehat{\tau}(\underline{Y})\}^{\mathfrak{j}} &= c_0\,\alpha^0\,\big[\breve{G}^0_{0\lambda}\,(\underline{Y}^\rho\,\partial_\rho \underline{X}^\lambda - \underline{X}^\rho\,\partial_\rho \underline{Y}^\lambda) \\ &\quad + (\alpha^0)^2\,\breve{G}^0_{0\mu}\,\underline{Y}^\mu\,(\breve{g}_{0\lambda}\,\breve{\delta}^\rho_0\,\partial_\rho \underline{X}^\lambda + \tfrac12\,\underline{X}^\lambda\,\partial_\lambda \widehat{g}_{00}) \\ &\quad - (\alpha^0)^2\,\breve{G}^0_{0\lambda}\,\underline{X}^\lambda\,(\breve{g}_{0\mu}\,\breve{\delta}^\rho_0\,\partial_\rho \underline{Y}^\mu + \tfrac12\,\underline{Y}^\mu\,\partial_\mu \widehat{g}_{00}) \\ &\quad + \tfrac{1}{c_0\,\alpha^0}\,\widehat{F}_{\lambda\mu}\,\underline{X}^\lambda\,\underline{Y}^\mu - \tfrac{\alpha^0}{c_0}\,\breve{\delta}^\rho_0\,(\breve{g}_{0\lambda}\,\widehat{F}_{\rho\mu} - \breve{g}_{0\mu}\,\widehat{F}_{\rho\lambda})\,\underline{X}^\lambda\,\underline{Y}^\mu\big] \\ &= \widehat{\tau}\big(\underline{[X,Y]}\big) - \widehat{\tau}(\underline{X})\,\widehat{\gamma}^{\mathfrak{g}}\cdot\widehat{\tau}(\underline{Y}) + \widehat{\tau}(\underline{Y})\,\widehat{\gamma}^{\mathfrak{g}}\cdot\widehat{\tau}(\underline{X}) \\ &\quad \tfrac12\,\widehat{F}(\underline{X},\underline{Y}) - \tfrac12\,\widehat{\tau}(\underline{X})\,\widehat{F}(\widehat{Д},\underline{Y}) + \tfrac12\,\widehat{\tau}(\underline{Y})\,\widehat{F}(\widehat{Д},\underline{X})\,, \\ \{\widehat{\tau}(\underline{X})\,,\breve{h}\}^{\mathfrak{j}} &= \underline{X}^\rho\,\partial_\rho \breve{h} + (\alpha^0)^2\,\breve{g}_{0\sigma}\,\underline{X}^\sigma\,\breve{\delta}^\rho_0\,\partial_\rho \breve{h} \\ &= \underline{X}.\breve{h} - \widehat{\tau}(\underline{X})\,\widehat{Д}\cdot\breve{h}\,, \\ \{\breve{f},\breve{h}\}^{\mathfrak{j}} &= 0\,,\end{aligned}$$

where $\widehat{g}_{00} = \breve{\delta}^\lambda_0\,\breve{\delta}^\mu_0\,g_{\lambda\mu}$, which follows from Lemma 3.4 and

$$\widehat{\tau}(\underline{Y})\,(\widehat{\gamma}^{\mathfrak{g}}\cdot\widehat{\tau}(\underline{X})) = c_0\,(\alpha^0)^3\,\breve{G}^0_{0\mu}\,\underline{Y}^\mu\,\big(\breve{g}_{0\lambda}\,\breve{\delta}^\rho_0\,\partial_\rho \underline{X}^\lambda + \tfrac12\,\underline{X}^\rho\,\partial_\rho \widehat{g}_{00}\big)\,. \qquad \square$$

Theorem 3.6. *All infinitesimal symmetries of the total almost-cosymplectic-contact phase structure* $(-\widehat{\tau},\Omega^{\mathfrak{j}})$ *are projectable.*

Proof. Let us consider two infinitesimal symmetries of the almost-cosymplectic-contact phase structure $(-\widehat{\tau},\Omega^{\mathfrak{j}})$

$$X = d(\widehat{\tau}(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}} + \widehat{\tau}(\underline{X})\,\widehat{\gamma}^{\mathfrak{j}}\,,\quad Y = d(\widehat{\tau}(\underline{Y}) + \breve{h})^{\sharp\mathfrak{j}} + \widehat{\tau}(\underline{Y})\,\widehat{\gamma}^{\mathfrak{j}}\,,$$

where the pairs $(\underline{X},\breve{f})$ and $(\underline{Y},\breve{h})$ satisfy the conditions 1), 2) and 3) of Theorem 3.3. Then the Lie bracket $[X,Y]$ is also an infinitesimal symmetry of $(-\widehat{\tau},\Omega^{\mathfrak{j}})$. By [5] (Lemma 2.5) we have

$$\begin{aligned}[X,Y] = d\{\widehat{\tau}(\underline{X}) + \breve{f},\widehat{\tau}(\underline{Y}) + \breve{h}\}^{\sharp\mathfrak{j}} + \big(\{\widehat{\tau}(\underline{X}) + \breve{f},\widehat{\tau}(\underline{Y})\}^{\mathfrak{j}} \\ - \{\widehat{\tau}(\underline{Y}) + \breve{h},\widehat{\tau}(\underline{X})\}^{\mathfrak{j}} + \Omega^{\mathfrak{g}}(d(\widehat{\tau}(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}}, d(\widehat{\tau}(\underline{Y}) + \breve{h})^{\sharp\mathfrak{j}})\big)\,\widehat{\gamma}^{\mathfrak{j}}\,.\end{aligned}$$

But $[X,Y]$ is an infinitesimal symmetry of $(-\widehat{\tau},\Omega^{\mathfrak{j}})$ if and only if the difference

$$\begin{aligned}\{\widehat{\tau}(\underline{X}) + \breve{f},\widehat{\tau}(\underline{Y}) + \breve{h}\}^{\mathfrak{j}} - \big(\{\widehat{\tau}(\underline{X}) + \breve{f},\widehat{\tau}(\underline{Y})\}^{\mathfrak{j}} - \{\widehat{\tau}(\underline{Y}) + \breve{h},\widehat{\tau}(\underline{X})\}^{\mathfrak{j}} \\ + \Omega^{\mathfrak{g}}(d(\widehat{\tau}(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}}, d(\widehat{\tau}(\underline{Y}) + \breve{h})^{\sharp\mathfrak{j}})\big)\end{aligned}$$

is a spacetime function. The above difference is

$$-\{\widehat{\tau}(\underline{X}),\widehat{\tau}(\underline{Y})\}^{\mathfrak{j}} - \Omega^{\mathfrak{g}}(d(\widehat{\tau}(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}}, d(\widehat{\tau}(\underline{Y}) + \breve{h})^{\sharp\mathfrak{j}})\,. \tag{3.20}$$

But $\Omega^{\mathfrak{g}} = \Omega^{\mathfrak{j}} - \Omega^{\mathfrak{e}}$. Then from the duality and

$$T\pi^1_0(d(\widehat{\tau}(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}}) = \underline{X} - \widehat{\tau}(\underline{X})\,\widehat{\text{Д}}$$

we get

$$\begin{aligned}
&\Omega^{\mathfrak{j}}(d(\widehat{\tau}(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}}, d(\widehat{\tau}(\underline{Y}) + \breve{h})^{\sharp\mathfrak{j}}) = -\{\widehat{\tau}(\underline{X}) + \breve{f}, \widehat{\tau}(\underline{Y}) + \breve{h}\}^{\mathfrak{j}}\,,\\
&\Omega^{\mathfrak{e}}(d(\widehat{\tau}(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}}, d(\widehat{\tau}(\underline{Y}) + \breve{h})^{\sharp\mathfrak{j}}) = \tfrac{1}{2}\widehat{F}(\underline{X} - \widehat{\tau}(\underline{X})\,\widehat{\text{Д}}, \underline{Y} - \widehat{\tau}(\underline{Y})\,\widehat{\text{Д}})
\end{aligned}$$

which implies

$$\begin{aligned}
\Omega^{\mathfrak{g}}&(d(\widehat{\tau}(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}}, d(\widehat{\tau}(\underline{Y}) + \breve{h})^{\sharp\mathfrak{j}})\\
&\qquad = -\,\{\widehat{\tau}(\underline{X}) + \breve{f}, \widehat{\tau}(\underline{Y}) + \breve{h}\}^{\mathfrak{j}} - \tfrac{1}{2}\widehat{F}(\underline{X}, \underline{Y}) + \tfrac{1}{2}\widehat{\tau}(\underline{X})\,\widehat{F}(\widehat{\text{Д}}, \underline{Y})\\
&\qquad\quad + \tfrac{1}{2}\widehat{\tau}(\underline{Y})\,\widehat{F}(\underline{X}, \widehat{\text{Д}})\,.
\end{aligned}$$

Then (3.20) is

$$\{\widehat{\tau}(\underline{X})\,, \breve{h}\}^{\mathfrak{j}} + \{\breve{f}\,, \widehat{\tau}(\underline{Y})\}^{\mathfrak{j}} + \tfrac{1}{2}\widehat{F}(\underline{X}, \underline{Y}) - \tfrac{1}{2}\widehat{\tau}(\underline{X})\,\widehat{F}(\widehat{\text{Д}}, \underline{Y}) - \tfrac{1}{2}\widehat{\tau}(\underline{Y})\,\widehat{F}(\underline{X}, \widehat{\text{Д}})$$

and, from (3.18), it can be rewritten as

$$\begin{aligned}
&\underline{X}\cdot\breve{h} - \widehat{\tau}(\underline{X})\,\widehat{\text{Д}}\cdot\breve{h} - \underline{Y}\cdot\breve{f} + \widehat{\tau}(\underline{Y})\,\widehat{\text{Д}}\cdot\breve{f} + \tfrac{1}{2}\widehat{F}(\underline{X}, \underline{Y})\\
&\qquad - \tfrac{1}{2}\widehat{\tau}(\underline{X})\,\widehat{F}(\widehat{\text{Д}}, \underline{Y}) - \tfrac{1}{2}\widehat{\tau}(\underline{Y})\,\widehat{F}(\underline{X}, \widehat{\text{Д}})\,.
\end{aligned}$$

Finally, from (3.6) and $\underline{Y} \lrcorner \underline{X} \lrcorner \widehat{F} = \frac{1}{2}\widehat{F}(\underline{X}, \underline{Y})$, we get that the difference is equal to $-\frac{1}{2}\widehat{F}(\underline{X}, \underline{Y})$ which is a spacetime function if and only if $\underline{X}$ and $\underline{Y}$ are spacetime vector fields. So all infinitesimal symmetries of the almost-cosymplectic-contact structure $(-\widehat{\tau}, \Omega^{\mathfrak{j}})$ are projectable and there are no nonprojectable (hidden) symmetries. □

Remark 3.4. Let us note that projectable infinitesimal symmetries of of the almost-cosymplectic-contact phase structures were classified in [11]. It was proved that all projectable infinitesimal symmetries are vector fields of the type (3.13) where $\underline{X}$ is a spacetime Killing vector field and $\breve{f}$ is a spacetime function such that the condition (3.6) is satisfied. In this case the vector field (3.13) reduces to (3.16). But if $\underline{X}$ is a Killing vector field, then $L_{\underline{X}}\bar{G} = 0$ which in coordinates reads as

$$\underline{X}^{\sigma}\,\partial_{\sigma}G^0_{\lambda\mu} + G^0_{\lambda\sigma}\,\partial_{\mu}\underline{X}^{\sigma} + G^0_{\mu\sigma}\,\partial_{\lambda}\underline{X}^{\sigma} = 0\,,$$

i.e.

$$\underline{X}^{\sigma}\,\partial_{\sigma}\breve{G}^0_{0\rho} + \breve{G}^0_{0\sigma}\,\partial_{\rho}\underline{X}^{\sigma} = -G^0_{\rho\sigma}\,\breve{\delta}^{\omega}_0\,\partial_{\omega}\underline{X}^{\sigma}$$

which implies that the vector field (3.16) can be rewritten as

$$\begin{aligned}
X &= \underline{X}^{\lambda}\,\partial_{\lambda} + \breve{G}^{i\rho}_0\,G^0_{\rho\sigma}\,\breve{\delta}^{\omega}_0\,\partial_{\omega}\underline{X}^{\sigma}\,\partial^0_i = \underline{X}^{\lambda}\,\partial_{\lambda} + \breve{\delta}^i_{\sigma}\,\breve{\delta}^{\omega}_0\,\partial_{\omega}\underline{X}^{\sigma}\,\partial^0_i\\
&= \underline{X}^{\lambda}\,\partial_{\lambda} + \big(\partial_0\underline{X}^i + x^p_0\,\partial_p\underline{X}^i - x^i_0\,\partial_0\underline{X}^0 - x^i_0\,x^p_0\,\partial_p\,\underline{X}^0\big)\,\partial^0_i
\end{aligned}\tag{3.21}$$

which is the 1-jet flow lift of $\underline{X}$ to $\mathcal{J}_1\boldsymbol{E}$. □

3.3. **Conserved functions and Killing-Maxwell multi-vector fields.** Now, let us consider a phase function (2.9) given by a spacetime function $\breve{f} = \overset{0}{K}$ and symmetric multi-vector fields $\overset{k}{K}$, $k \geq 1$. If we consider the phase vector field

$$X = dK^{\sharp\mathfrak{j}} + \sum_{k\geq 1} \overset{k}{K}(\widehat{\tau})\, \widehat{\gamma}^{\mathfrak{j}}$$

then this vector field coincides with the vector field (3.13) for the generalized vector field

$$\begin{aligned} \underline{X}[K] &= T\pi^1_0\big(dK^{\sharp\mathfrak{j}} + \sum_{k\geq 1} \overset{k}{K}(\widehat{\tau})\, \widehat{\gamma}^{\mathfrak{j}}\big) \\ &= \overset{1}{K} - \sum_{k\geq 2}(k-1)\, \overset{k}{K}(\widehat{\tau})\, \widehat{\text{Д}} + \sum_{k\geq 2} k\, \underbrace{\widehat{\tau}\,\lrcorner \ldots \lrcorner\, \widehat{\tau}\,\lrcorner}_{(k-1)-\text{times}} \overset{k}{K}\,. \end{aligned} \tag{3.22}$$

Really, we have $\widehat{\tau}(\underline{X}[K]) = \sum_{k\geq 1} \overset{k}{K}(\widehat{\tau})$. Such generalized vector field satisfies the projectability condition and we have to find conditions for the function (2.9) to be conserved by the joined Reeb vector field, i.e. $\widehat{\gamma}^{\mathfrak{j}}.K = 0$.

Lemma 3.7. *We have*

$$\begin{aligned} &\widehat{\gamma}^{\mathfrak{e}} \cdot \overset{0}{K} = 0\,, \\ &\widehat{\gamma}^{\mathfrak{e}}.\overset{k}{K}(\widehat{\tau}) = -k\, \frac{\hbar^2}{m^2\, c^2}\, \overset{k}{K}{}^{\rho\lambda_1\ldots\lambda_{k-1}}\, \widehat{F}^{\lambda_k}{}_{\rho}\, \widehat{\tau}_{\lambda_1} \ldots \widehat{\tau}_{\lambda_k}\,, \quad k \geq 1\,, \end{aligned}$$

where $\widehat{F}^{\lambda_k}{}_{\rho} = g^{\sigma\lambda_k}\, \widehat{F}_{\sigma\rho}$.

Proof. We have

$$\widehat{\gamma}^{\mathfrak{e}}.\widehat{\tau}_{\rho} = \frac{\alpha^0\, \hbar}{m\, c}\, \breve{\delta}^{\sigma}_0\, \widehat{F}_{\sigma\rho}$$

which implies, with $\frac{\alpha^0\,\hbar}{m\,c}\, \breve{\delta}^{\sigma}_0 = -\frac{\hbar^2}{m^2\,c^2}\, g^{\sigma\omega}\, \widehat{\tau}_{\omega}$,

$$\begin{aligned} \widehat{\gamma}^{\mathfrak{e}} \cdot \overset{k}{K}(\widehat{\tau}) &= k\, \overset{k}{K}{}^{\rho\lambda_1\ldots\lambda_{k-1}}\, (\widehat{\gamma}^{\mathfrak{e}}.\widehat{\tau}_{\rho})\, \widehat{\tau}_{\lambda_1} \ldots \widehat{\tau}_{\lambda_{k-1}} \\ &= -k\, \frac{\hbar^2}{m^2\, c^2}\, \overset{k}{K}{}^{\rho\lambda_1\ldots\lambda_{k-1}}\, \widehat{F}_{\sigma\rho}\, g^{\sigma\lambda_k}\, \widehat{\tau}_{\lambda_1} \ldots \widehat{\tau}_{\lambda_k}\,. \end{aligned}$$

□

Lemma 3.8. *Let us suppose a phase function* $B = B^{\lambda_1\ldots\lambda_k}\, \widehat{\tau}_{\lambda_1} \ldots \widehat{\tau}_{\lambda_k}$, $k \geq 1$, $B^{\lambda_1\ldots\lambda_k} \in C^{\infty}(\boldsymbol{E})$. *Then* $B = 0$ *if and only if* $B^{\lambda_1\ldots\lambda_k} = 0$ *for all* $\lambda_1, \ldots, \lambda_k$.

Proof. $B = 0$ if and only if $B^{\lambda_1\ldots\lambda_k}\, g_{\lambda_1\rho_1} \ldots g_{\lambda_k\rho_k}\, \breve{\delta}^{\rho_1}_0 \ldots \breve{\delta}^{\rho_k}_0 = 0$ which is a polynomial function on fibres of $\mathcal{J}_1\boldsymbol{E}$. Then $B^{\lambda_1\ldots\lambda_k}\, g_{\lambda_1\rho_1} \ldots g_{\lambda_k\rho_k} = 0$ for all indices $\rho_1, \ldots, \rho_k$ and from regularity of the metric we get Lemma 3.8. □

Theorem 3.9. *A phase function* (2.9) *is conserved by the joined Reeb vector field, i.e.* $\widehat{\gamma}^{\mathfrak{j}} \cdot K = 0$*, if and only if*

$$g^{\rho\lambda}\,\partial_\rho \overset{0}{K} + \overset{1}{K}{}^{\rho}\,\widehat{F}^{\lambda}{}_{\rho} = 0\,, \tag{3.23}$$

$$\nabla^{(\lambda_1}\overset{k}{K}{}^{\lambda_2\dots\lambda_{k+1})} + (k+1)\,\overset{k+1}{K}{}^{\rho(\lambda_1\dots\lambda_k}\,\widehat{F}^{\lambda_{k+1})}{}_{\rho} = 0\,, \tag{3.24}$$

for $k = 1, 2, \dots$.

Proof. From Lemma 3.7 we have

$$\begin{aligned}\widehat{\gamma}^{\mathfrak{j}} \cdot K = \widehat{\gamma}^{\mathfrak{g}} \cdot K + \widehat{\gamma}^{\mathfrak{e}} \cdot K = -\frac{\hbar^2}{m^2\,c^2}\,&\big[\big(g^{\rho\lambda}\,\partial_\rho \overset{0}{K} + \overset{1}{K}{}^{\rho}\,\widehat{F}^{\lambda}{}_{\rho}\big)\,\widehat{\tau}_{\lambda_1}\\ &+\sum_{k\geq 1}\big(\tfrac{1}{2}[\bar{g},\overset{k}{K}]^{\lambda_1\dots\lambda_{k+1}} + (k+1)\,\overset{k+1}{K}{}^{\rho\lambda_1\dots\lambda_k}\,\widehat{F}^{\lambda_{k+1}}{}_{\rho}\big)\,\widehat{\tau}_{\lambda_1}\dots\widehat{\tau}_{\lambda_{k+1}}\big]\,.\end{aligned}$$

From

$$\nabla^{(\lambda_1}\overset{k}{K}{}^{\lambda_2\dots\lambda_{k+1})} = \tfrac{1}{2}[\bar{g},\overset{k}{K}]^{\lambda_1\dots\lambda_{k+1}}$$

and Lemma (3.8) we obtain Theorem 3.9. □

Corollary 3.10. *The vector field* $X[K] = dK^{\sharp\mathfrak{j}} + h\,\widehat{\gamma}^{\mathfrak{j}}$*, where the phase function* K *is given by* (2.9) *and* $h = K - \overset{0}{K}$*, is an infinitesimal symmetry of the almost–cosymplectic–contact pair* $(-\widehat{\tau}, \Omega^{\mathfrak{j}})$ *if and only if the conditions* (3.23) *and* (3.24) *are satisfied and* $d\overset{0}{K} = \underline{X}[K] \lrcorner \widehat{F}$. □

Remark 3.5. Let us assume a (special) phase function $K = \overset{0}{K} + \overset{1}{K}(\widehat{\tau})$. Then the conditions (3.23) and (3.24) are reduced to

$$\partial_\rho \overset{0}{K} - \overset{1}{K}{}^{\sigma}\,\widehat{F}_{\sigma\rho} = 0\,, \qquad \nabla^{(\lambda_1}\overset{1}{K}{}^{\lambda_2)} = 0$$

and we obtain the result of [11], i.e. $\overset{1}{K}$ is a Killing vector field and $\overset{0}{K}$ and $\overset{1}{K}$ are related by the formula $d\overset{0}{K} = \overset{1}{K} \lrcorner \widehat{F}$ which implies that $\overset{1}{K}$ is an infinitesimal symmetry of $\widehat{F}$. Moreover, the corresponding infinitesimal symmetry is the flow lift $\mathfrak{J}_1\overset{1}{K}$ which projects on $\overset{1}{K}$, see Remark 3.4. Let us note that in this case the condition (3.6) coincides with the condition (3.23). □

Remark 3.6. Let us assume a phase function $K = \overset{k}{K}(\widehat{\tau})$, $k \geq 2$. Then the conditions (3.23) and (3.24) are reduced to

$$\nabla^{(\lambda_1}\overset{k}{K}{}^{\lambda_2\dots\lambda_{k+1})} = 0\,, \qquad \overset{k}{K}{}^{\rho(\lambda_1\dots\lambda_{k-1}}\,\widehat{F}^{\lambda_k)}{}_{\rho} = 0 \tag{3.25}$$

and we obtain that $\overset{k}{K}$ is a Killing-Maxwell k-vector field, [2]. The condition (3.6) has the form $\underline{X}[K] \lrcorner \widehat{F} = 0$ which from (3.22) is equivalent with

$$0 = \sum_{k\geq 2}\big((k-1)\overset{k}{K}(\widehat{\tau})\,\widehat{\mathfrak{H}} \lrcorner \widehat{F} - k\,(\underbrace{\widehat{\tau} \lrcorner \dots \lrcorner \widehat{\tau} \lrcorner}_{(k-1)-\text{times}} \overset{k}{K}) \lrcorner \widehat{F}\big)\,. \tag{3.26}$$

Now, from $\widehat{\text{Д}} = \widehat{G}^\sharp \widehat{\tau}$ and Lemma 3.8 we obtain the coordinate expression of (3.26) in the form

$$\overset{k}{K}{}^{(\lambda_1\dots\lambda_k}\,\widehat{F}^{\lambda_{k+1})}{}_\mu = 0\,, \qquad \overset{k}{K}{}^{\rho\lambda_1\dots\lambda_{k-1}}\,\widehat{F}_{\rho\mu} = 0\,. \tag{3.27}$$

For a nonvanishing k-multi vector field the equations (3.27) are satisfied only for $\widehat{F} \equiv 0$ but in this case the geometrical phase structure is contact. For a nonvanishing electromagnetic field the equations (3.27) are satisfied only for vanishing k-multi vector field and we have no induced infinitesimal hidden symmetry. □

Remark 3.7. As an example let us assume the canonical Killing-Maxwell 2-vector field $\overset{2}{K}{}^{\lambda\mu} = \widehat{G}^{\lambda\mu}$ for the unscaled metric, [13]. Then the above conditions (3.27) are in the form

$$\widehat{G}^{\rho\lambda}\widehat{F}_{\rho\mu} = 0\,, \quad \widehat{G}^{(\lambda_1\lambda_2}\widehat{F}^{\lambda_3)}{}_\mu = 0\,.$$

which implies $F \equiv 0$. □

3.4. **Comparison with infinitesimal symmetries of the kinetic energy function.**

Lemma 3.11. *Let K be a function on $T^*\boldsymbol{E}$ constant of motion, i.e. $X_H^{\mathfrak{j}} \cdot K = 0$, then its pull-back $-\widehat{\tau}^*(K)$ is a conserved function, i.e. $\widehat{\gamma}^{\mathfrak{j}} \cdot \widehat{\tau}^*(K) = 0$.*

Proof. The proof is the same as the proof of Lemma 2.8 by observing that

$$-T_e\widehat{\tau}(\widehat{\gamma}^{\mathfrak{e}}) = \widehat{F}^\rho{}_\lambda\,\widehat{\tau}_\rho(e)\,\dot{\partial}^\lambda\,, \quad e \in \mathcal{J}_1\boldsymbol{E}\,,$$

and by using equation (1.5). □

If we consider the electromagnetic field, then in both approaches we get the same results for projectable infinitesimal symmetries, see Corollary 1.1 and Remark 3.5. On the other hand, Killing-Maxwell multi-vector fields of rank ≥ 2 admits hidden infinitesimal symmetries of H on $T^*\boldsymbol{E}$. On $\mathcal{J}_1\boldsymbol{E}$ Killing-Maxwell multi-vector fields admit functions conserved by the Reeb vector field of the joined structure but to obtain infinitesimal symmetries of the joined almost-cosymplectic-contact structure we need a further strong condition (3.26) which implies either $F \equiv 0$ and the structure is reduced to the gravitational one or there are no hidden infinitesimal symmetries.

We can summarize the results in the following diagram

$$\begin{tikzcd}
(T^*\boldsymbol{E}, \omega^{\mathfrak{j}}) & \text{constants of motion } K,\ \{H,K\}^{\mathfrak{j}} = 0 \arrow[r] \arrow[dd, "-\widehat{\tau}^*"] & \text{ISs of } H \\
\text{Killing–Maxwell multi-v.f.s} \arrow[ur] \arrow[dr] & & \text{projectable ISs of } (-\widehat{\tau}, \Omega^{\mathfrak{j}}) \\
(\mathcal{J}_1\boldsymbol{E}\,, -\widehat{\tau}\,, \Omega^{\mathfrak{j}}) & \text{conserved functions, } \widehat{\gamma}^{\mathfrak{j}}.(-\widehat{\tau}^*K) = 0 \arrow[ur, "k=1"] \arrow[dr, "k\geq 2"'] & \\
& & \text{no hidden ISs of } (-\widehat{\tau}, \Omega^{\mathfrak{j}})
\end{tikzcd}$$

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