ARCHIVUM MATHEMATICUM (BRNO)
Tomus 51 (2015), 315–329

ON THE COMPOSITION STRUCTURE OF THE TWISTED VERMA MODULES FOR $\mathfrak{sl}(3,\mathbb{C})$

Libor Křižka and Petr Somberg

Abstract. We discuss some aspects of the composition structure of twisted Verma modules for the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$, including the explicit structure of singular vectors for both $\mathfrak{sl}(3,\mathbb{C})$ and one of its Lie subalgebras $\mathfrak{sl}(2,\mathbb{C})$, and also of their generators. Our analysis is based on the use of partial Fourier tranform applied to the realization of twisted Verma modules as $\mathcal{D}$-modules on the Schubert cells in the full flag manifold for $\mathrm{SL}(3,\mathbb{C})$.

Introduction

The objects of central interest in the representation theory of complex simple Lie algebras are the Harish-Chandra modules. It is well known that there is a categorical equivalence between principal series Harish-Chandra modules and twisted Verma modules as objects of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. The twisted Verma modules are studied from various perspectives including the Lie algebra (co)homology of the twisted nilradical, the Schubert cell decomposition of full flag manifolds and algebraic techniques of twisting functors applied to Verma modules, in [4], [2], [14] and references therein.

Combinatorial conditions for the existence of homomorphisms between twisted Verma modules were studied in [1], but there is basically no information on precise positions and properties of elements responsible for a non-trivial composition structure of twisted Verma modules. The modest aim of the present article is the study of some aspects related to the composition structure of twisted Verma modules for the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ by geometrical methods, through their realization as $\mathcal{D}$-modules supported on Schubert cells, cf. [7], [3]. Namely, we discuss in the case of $\mathfrak{sl}(3,\mathbb{C})$ a few results parallel to the development for (untwisted) generalized Verma modules in [10], [11].

Let us briefly describe the content of our article. First of all, in Section 1 we briefly review various characterizing properties of twisted Verma modules compared to the untwisted Verma modules. Based on the action of a simple Lie algebra on its full flag manifold, see e.g. [7], [12] for rather explicit description, in Section 2 we write down explicit realizations of the highest weight twisted $\mathfrak{sl}(3,\mathbb{C})$-Verma modules.

2010 *Mathematics Subject Classification*: primary 53A30; secondary 22E47, 33C45, 58J70.
Key words and phrases: Lie algebra $\mathfrak{sl}(3,\mathbb{C})$, twisted Verma modules, composition structure, $\mathcal{D}$-modules.
DOI: 10.5817/AM2015-5-315

For all twistings realized by elements w of the Weyl group W, the isomorphism given by a partial Fourier transform allows us to analyze several basic questions on twisted Verma modules not accessible in the literature. We shall carry out this procedure for the structure of singular vectors and the generators of $\mathfrak{sl}(3,\mathbb{C})$-Verma modules twisted by $w = s_1$. Another our result concerns the application of ideas on the decomposition of twisted Verma modules with respect to a reductive Lie subalgebra $\mathfrak{sl}(3,\mathbb{C})$, thereby generalizing the results analogous to [11] towards the twisted Verma modules. Here we consider the simplest example of an embedded $\mathfrak{sl}(2,\mathbb{C}) \subset \mathfrak{sl}(3,\mathbb{C})$ and produce a complete list of singular vectors responsible for the branching problem of twisted $\mathfrak{sl}(3,\mathbb{C})$-Verma module. In our situation we also observe that the s_1-twisted $\mathfrak{sl}(3,\mathbb{C})$-Verma modules are generated by single vector (which is not of highest weight), a property analogous to the case of (untwisted) Verma modules. In the last Section 3 we highlight our results in the framework of (un)known properties of the objects of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ (see e.g. [8]).

1. Twisted Verma modules and their characterizations

Let G be a connected complex semisimple Lie group, $H \subset G$ a maximal torus of G, $B \subset G$ a Borel subgroup of G containing H, and $W = N_G(H)/H$ the Weyl group of G. Furthermore, let $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{b}$ be the Lie algebras of G, H and B, respectively. Finally, let $\mathfrak{n}$ be the positive nilradical of the Borel subalgebra $\mathfrak{b}$ and $\bar{\mathfrak{n}}$ the opposite (negative) nilradical. We denote by N and $\overline{N}$ the Lie subgroups of G corresponding to the Lie subalgebras $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively.

The objects of our interest are the twisted Verma modules $M^w_{\mathfrak{g}}(\lambda)$, parametrized by $\lambda \in \mathfrak{h}^*$ and the twisting $w \in W$. The twisted Verma modules $M^w_{\mathfrak{g}}(\lambda)$ have for all $w \in W$ the same character as the Verma module $M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ induced from the 1-dimensional $\mathfrak{b}$-module $\mathbb{C}_\lambda$,

$$M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda) \equiv M^e_{\mathfrak{g}}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \,,$$

with highest weight $\lambda \in \mathfrak{h}^*$ and $e \in W$. However, the extensions of simple sub-quotients in twisted Verma modules differ from extensions in Verma modules. As $U(\mathfrak{g})$-modules they are objects of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$, i.e. finitely generated $U(\mathfrak{g})$-modules, $\mathfrak{h}$-semisimple and locally $\mathfrak{n}$-finite.

Let us denote by e and w_0 the identity and the longest element of W, respectively, and let $\ell\colon W \to \mathbb{N}_0$ be the length function on W. The Weyl group W acts by ρ-affine action on $\mathfrak{h}^*$, $w \cdot \lambda = w(\lambda + \rho) - \rho$, and gives four Lie subalgebras

1) $\bar{\mathfrak{n}} = \mathfrak{n}^-_w \oplus \bar{\mathfrak{n}}^-_w$: $\quad \mathfrak{n}^-_w = \bar{\mathfrak{n}} \cap \mathrm{Ad}(\dot{w})(\mathfrak{n})$, $\bar{\mathfrak{n}}^-_w = \bar{\mathfrak{n}} \cap \mathrm{Ad}(\dot{w})(\bar{\mathfrak{n}})$,
2) $\mathfrak{n} = \mathfrak{n}^+_w \oplus \bar{\mathfrak{n}}^+_w$: $\quad \mathfrak{n}^+_w = \mathfrak{n} \cap \mathrm{Ad}(\dot{w})(\mathfrak{n})$, $\bar{\mathfrak{n}}^+_w = \mathfrak{n} \cap \mathrm{Ad}(\dot{w})(\bar{\mathfrak{n}})$.

The universal enveloping algebra $U(\mathfrak{n}^-_w)$ is a graded subalgebra of $U(\bar{\mathfrak{n}})$, determined by $U(\mathfrak{n}^-_w)_0 = \mathbb{C}$, $U(\mathfrak{n}^-_w)_{-1} = \mathfrak{n}^-_w$ for all $w \in W$, and $U(\mathfrak{n}^-_e) = \mathbb{C}$, $U(\mathfrak{n}^-_{w_0}) = U(\bar{\mathfrak{n}})$. The graded dual of $U(\mathfrak{n}^-_w)$ is defined by $(U(\mathfrak{n}^-_w))^*_n = \mathrm{Hom}_{\mathbb{C}}((U(\mathfrak{n}^-_w))_{-n}, \mathbb{C})$ for all $n \in \mathbb{Z}$.

There are several equivalent characterizing properties of twisted Verma modules $M^w_{\mathfrak{g}}(\lambda)$ for $\lambda \in \mathfrak{h}^*$ and $w \in W$, see [2], [8, Chapter 12] for detailed discussion.

1) The Lie algebra cohomology of the twisted opposite nilradical $\overline{\mathfrak{n}}^+_w \oplus \overline{\mathfrak{n}}^-_w = \mathrm{Ad}(\dot{w})(\overline{\mathfrak{n}})$ with coefficients in $M^w_{\mathfrak{g}}(\lambda)$ is

$$H^i(\overline{\mathfrak{n}}^+_w \oplus \overline{\mathfrak{n}}^-_w, M^w_{\mathfrak{g}}(\lambda)) \simeq \begin{cases} \mathbb{C}_{\lambda+w(\rho)+\rho} & \text{if} \quad i = \dim \mathfrak{n} - \ell(w)\,, \\ 0 & \text{if} \quad i \neq \dim \mathfrak{n} - \ell(w) \end{cases} \tag{1.1}$$

as $\mathfrak{h}$-modules. In particular, $M^w_{\mathfrak{g}}(\lambda)$ is a free $U(\mathrm{Ad}(\dot{w})(\overline{\mathfrak{n}})\cap\overline{\mathfrak{n}})$-module, while its graded dual $(M^w_{\mathfrak{g}}(\lambda))^*$ is a free $U(\mathrm{Ad}(\dot{w})(\overline{\mathfrak{n}}) \cap \mathfrak{n})$-module.

2) Let us consider the full flag manifold G/B and the Schubert cell X_w for $w \in W$ defined as the N-orbit $X_w = NwB/B \subset G/B$, where $\dim X_w = \ell(w)$. Then there is an isomorphism of $U(\mathfrak{g})$-modules for the local, relative to X_w, sheaf cohomology of a homogeneous vector bundle $\mathcal{L}(\lambda)$,

$$H^i_{X_w}(G/B, \mathcal{L}(\lambda)) \simeq \begin{cases} M^w_{\mathfrak{g}}(ww_0 \cdot \lambda) & \text{if} \quad i = \dim \mathfrak{n} - \ell(w)\,, \\ 0 & \text{if} \quad i \neq \dim \mathfrak{n} - \ell(w)\,. \end{cases} \tag{1.2}$$

In particular, the Verma modules are supported on the closed Schubert cell while the contragradient Verma modules are supported on the open (dense) Schubert cell.

3) For $w \in W$, the $U(\mathfrak{g})$-bimodule $S_w = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}^-_w)} (U(\mathfrak{n}^-_w))^*$ allows to define a functor $T_w \colon \mathcal{O} \to \mathcal{O}$ (called twisting functor) by

$$T_w \colon M \mapsto \varphi_w(S_w \otimes_{U(\mathfrak{g})} M). \tag{1.3}$$

Here $\varphi_w = \mathrm{Ad}(\dot{w}^{-1}) \colon \mathfrak{g} \to \mathrm{Aut}(\mathfrak{g})$ indicates the conjugation of the action by $\mathfrak{g}$ on the twisted module. In particular, we have $M^w_{\mathfrak{g}}(\lambda) = T_w(M^{\mathfrak{g}}_{\mathfrak{b}}(w \cdot \lambda))$.

Twisted Verma modules $M^w_{\mathfrak{g}}(\lambda)$ for $\lambda \in \mathfrak{h}^*$ and $w \in W$ can be realized in the framework of $\mathcal{D}$-modules on the flag manifold $X = G/B$. There is a G-equivariant sheaf of rings of twisted differential operators $\mathcal{D}^\lambda_X$ on X, see [3], [9], which is for an integral dominant weight $\lambda + \rho$ a sheaf of rings of differential operators acting on $\mathcal{L}(\lambda + \rho)$. The G-equivariance of $\mathcal{D}^\lambda_X$ ensures the existence of a Lie algebra morphism

$$\alpha_\lambda \colon \mathfrak{g} \to \Gamma(X, \mathcal{D}^\lambda_X) \tag{1.4}$$

and a localization functor

$$\Delta\colon \mathrm{Mod}(\mathfrak{g}) \to \mathrm{Mod}(\mathcal{D}^\lambda_X)\,. \tag{1.5}$$

Then the $\mathcal{D}^\lambda_X$-module $\Delta(M^w_{\mathfrak{g}}(w \cdot (\lambda - \rho)))$ is realized in the vector space of distributions supported on the Schubert cell X_w of X, see [6, Chapter 11].

2. Twisted Verma modules for $\mathfrak{sl}(3,\mathbb{C})$

We shall consider the complex semisimple Lie group $G = \mathrm{SL}(3,\mathbb{C})$ given by 3×3 complex matrices of unit determinant and its Lie algebra $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by diagonal matrices $\mathfrak{h} = \{\mathrm{diag}(a_1, a_2, a_3);\ a_1, a_2, a_3 \in \mathbb{C},\ a_1 + a_2 + a_3 = 0\}$. For $i = 1, 2, 3$, we define $\varepsilon_i \in \mathfrak{h}^*$ by $\varepsilon_i(\mathrm{diag}(a_1, a_2, a_3)) = a_i$. Then the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is $\Delta = \{\varepsilon_i - \varepsilon_j;\ 1 \leq i \neq j \leq 3\}$, the positive root system is $\Delta^+ = \{\varepsilon_i - \varepsilon_j;\ 1 \leq i < j \leq 3\}$ and the set of simple roots is $\Pi = \{\alpha_1, \alpha_2\}$, $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$. The fundamental weights are $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$, and the smallest regular integral dominant weight is $\rho = \omega_1 + \omega_2$. The notation $\lambda = (\lambda_1, \lambda_2)$ means $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$.

We choose the basis of root spaces of $\mathfrak{g}$ as

$$f_1 = f_{\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = f_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_{12} = f_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$e_1 = e_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = e_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{12} = e_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the basis of the Cartan subalgebra $\mathfrak{h}$ is given by coroots

$$h_1 = h_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h_2 = h_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These matrices fulfill, among others, the commutation relations $[f_{\alpha_1}, f_{\alpha_2}] = -f_{\alpha_1+\alpha_2}$ and $[e_{\alpha_1}, e_{\alpha_2}] = e_{\alpha_1+\alpha_2}$.

The Weyl group W of G is generated by simple reflections $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$, where the action of W on $\mathfrak{h}^*$ is given by

$$s_1(\alpha_1) = -\alpha_1, \quad s_1(\alpha_2) = \alpha_1 + \alpha_2\,, \quad s_2(\alpha_1) = \alpha_1 + \alpha_2, \quad s_2(\alpha_2) = -\alpha_2\,,$$

and $|W| = 6$ with $W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 = s_2s_1s_2\}$. Consequently, there are six Schubert cells X_w in G/B isomorphic to $X_w \simeq \mathbb{C}^{\ell(w)}$:

$$\dim(X_e) = 0\,, \quad \dim(X_{s_1}) = \dim(X_{s_2}) = 1\,, \quad \dim(X_{s_1s_2}) = \dim(X_{s_2s_1}) = 2\,,$$

$$\dim(X_{s_1s_2s_1}) = 3\,.$$

For the representatives of the elements of W in G we take the matrices

$$\dot{e} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \dot{s}_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \dot{s}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\dot{s}_1\dot{s}_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \dot{s}_2\dot{s}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \dot{s}_1\dot{s}_2\dot{s}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We denote by (x, y, z) the linear coordinate functions on $\bar{\mathfrak{n}}$ with respect to the basis (f_1, f_2, f_{12}) of the opposite nilradical $\bar{\mathfrak{n}}$, and by (ξ_x, ξ_y, ξ_z) the dual linear coordinate functions on $\bar{\mathfrak{n}}^*$.

Let us consider the partial dual space $\bar{\mathfrak{n}}^{*,w}$ of $\bar{\mathfrak{n}}$ defined by

$$\bar{\mathfrak{n}}^{*,w} = (\bar{\mathfrak{n}}^-_{w^{-1}})^* \oplus \mathfrak{n}^-_{w^{-1}}\,, \tag{2.1}$$

so that

$$(\xi_{x_\alpha},\, \alpha \in w^{-1}(\Delta^+) \cap \Delta^+,\ x_\alpha,\, \alpha \in w^{-1}(-\Delta^+) \cap \Delta^+) \tag{2.2}$$

with $x_{\alpha_1} = x$, $x_{\alpha_2} = y$, $x_{\alpha_1+\alpha_2} = z$ are linear coordinate functions on $\bar{\mathfrak{n}}^{*,w}$. Moreover, the Weyl algebra $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}$ of $\bar{\mathfrak{n}}$ is generated by $\{x, \partial_x, y, \partial_y, z, \partial_z\}$, and the Weyl algebra $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}^{*,w}}$ of $\bar{\mathfrak{n}}^{*,w}$ is generated by

$$\{\xi_{x_\alpha}, \partial_{\xi_{x_\alpha}}, \alpha \in w^{-1}(\Delta^+) \cap \Delta^+,\ x_\alpha, \partial_{x_\alpha}, \alpha \in w^{-1}(-\Delta^+) \cap \Delta^+\}\,. \tag{2.3}$$

There is a canonical isomorphism $\mathcal{F}^w \colon \mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}} \to \mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}^{*,w}}$ of associative $\mathbb{C}$-algebras called the partial Fourier transform, defined with respect to the generators (2.3) by

$$\begin{aligned} &x_\alpha \mapsto -\partial_{\xi_{x_\alpha}}, \quad \partial_{x_\alpha} \mapsto \xi_{x_\alpha}, \quad \text{for} \quad \alpha \in w^{-1}(\Delta^+) \cap \Delta^+\,, \\ &x_\alpha \mapsto x_\alpha, \qquad \partial_{x_\alpha} \mapsto \partial_{x_\alpha}, \quad \text{for} \quad \alpha \in w^{-1}(-\Delta^+) \cap \Delta^+. \end{aligned} \tag{2.4}$$

The partial Fourier transform is independent of the choice of linear coordinates on $\bar{\mathfrak{n}}$.

The Verma modules $M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda - \rho)$, $\lambda \in \mathfrak{h}^*$, can be realized as $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_e$ for I_e the left ideal of $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}$ defined by $I_e = (x, y, z)$, see e.g. [12]. The structure of $\mathfrak{g}$-module on $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_e$ is realized through the embedding $\pi_\lambda \colon \mathfrak{g} \to \mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}$ given by

$$\begin{aligned} \pi_\lambda(X) = &-\sum_{\alpha \in \Delta^+} \left[\frac{\operatorname{ad}(u(x))e^{\operatorname{ad}(u(x))}}{e^{\operatorname{ad}(u(x))} - \operatorname{id}_{\bar{\mathfrak{n}}}}\,(e^{-\operatorname{ad}(u(x))}X)_{\bar{\mathfrak{n}}}\right]_\alpha \partial_{x_\alpha} \\ &+ (\lambda + \rho)((e^{-\operatorname{ad}(u(x))}X)_{\mathfrak{b}}) \end{aligned} \tag{2.5}$$

for all $X \in \mathfrak{g}$, where $[Y]_\alpha$ denotes the α-th coordinate of $Y \in \bar{\mathfrak{n}}$ with respect to the basis $(f_\alpha;\, \alpha \in \Delta^+)$ of $\bar{\mathfrak{n}}$ and $u(x) = \sum_{\alpha \in \Delta^+} x_\alpha f_\alpha$. The twisted Verma modules are realized by $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_w$, where I_w is the left ideal of $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}$ defined by

$$I_w = (x_\alpha,\, \alpha \in w^{-1}(\Delta^+) \cap \Delta^+\,, \quad \partial_{x_\alpha},\, \alpha \in w^{-1}(-\Delta^+) \cap \Delta^+) \tag{2.6}$$

with $x_{\alpha_1} = x$, $x_{\alpha_2} = y$, $x_{\alpha_1+\alpha_2} = z$. The list of all possibilities looks as follows:

1) $w = e$, $I_e = (x, y, z)$, $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_e \simeq \mathbb{C}[\partial_x, \partial_y, \partial_z]$;
2) $w = s_1$, $I_{s_1} = (\partial_x, y, z)$, $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_{s_1} \simeq \mathbb{C}[x, \partial_y, \partial_z]$;
3) $w = s_2$, $I_{s_2} = (x, \partial_y, z)$, $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_{s_2} \simeq \mathbb{C}[\partial_x, y, \partial_z]$;
4) $w = s_1 s_2$, $I_{s_1 s_2} = (x, \partial_y, \partial_z)$, $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_{s_1 s_2} \simeq \mathbb{C}[\partial_x, y, z]$;
5) $w = s_2 s_1$, $I_{s_2 s_1} = (\partial_x, y, \partial_z)$, $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_{s_2 s_1} \simeq \mathbb{C}[x, \partial_y, z]$;
6) $w = s_1 s_2 s_1$, $I_{s_1 s_2 s_1} = (\partial_x, \partial_y, \partial_z)$, $\mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_{s_1 s_2 s_1} \simeq \mathbb{C}[x, y, z]$.

In particular, the twisted Verma modules are realized as $M^w_{\mathfrak{g}}(\lambda) \simeq \mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_w$, where $\pi^w_\lambda \colon \mathfrak{g} \to \mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}}/I_w$ is defined by

$$\pi^w_\lambda = \pi_{w^{-1}(\lambda+\rho)} \circ \operatorname{Ad}(\dot{w}^{-1}) \tag{2.7}$$

with w^{-1} acting in the standard and not the ρ-shifted manner. Since $\mathcal{F}^w \colon \mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}} \to \mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}^{*,w}}$ is an isomorphism of associative $\mathbb{C}$-algebras, the composition

$$\hat{\pi}^w_\lambda = \mathcal{F}^w \circ \pi^w_\lambda \tag{2.8}$$

gives the homomorphism $\hat{\pi}^w_\lambda \colon U(\mathfrak{g}) \to \mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}^{*,w}}$ of associative $\mathbb{C}$-algebras and the twisted Verma modules are realized as $M^w_{\mathfrak{g}}(\lambda) \simeq \mathcal{A}^{\mathfrak{g}}_{\bar{\mathfrak{n}}^{*,w}}/\mathcal{F}^w(I_w)$.

2.1. **(Untwisted) Verma modules.** Let us first consider the case of Verma modules. The untwisted Verma module $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda) \equiv M_{\mathfrak{g}}^{e}(\lambda)$ for $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ is isomorphic to

$$M_{\mathfrak{g}}^{e}(\lambda) \simeq \mathcal{A}_{\mathfrak{n}}^{\mathfrak{g}}/I_e \simeq \mathbb{C}[\partial_x, \partial_y, \partial_z], \quad I_e = (x, y, z), \tag{2.9}$$

where the embedding $\pi_\lambda^e \colon \mathfrak{g} \to \mathcal{A}_{\mathfrak{n}}^{\mathfrak{g}}$ and so the $\mathfrak{g}$-module structure on $\mathcal{A}_{\mathfrak{n}}^{\mathfrak{g}}/I_e$ are given by

$$\begin{aligned}
\pi_\lambda^e(f_1) &= -\partial_x + \tfrac{1}{2} y\partial_z, \\
\pi_\lambda^e(f_2) &= -\partial_y - \tfrac{1}{2} x\partial_z, \\
\pi_\lambda^e(f_{12}) &= -\partial_z, \\
\pi_\lambda^e(e_1) &= x^2\partial_x + (z - \tfrac{1}{2}xy)\partial_y + (\tfrac{1}{4}x^2y + \tfrac{1}{2}xz)\partial_z + (\lambda_1 + 2)x, \\
\pi_\lambda^e(e_2) &= y^2\partial_y - (z + \tfrac{1}{2}xy)\partial_x - (\tfrac{1}{4}xy^2 - \tfrac{1}{2}yz)\partial_z + (\lambda_2 + 2)y, \\
\pi_\lambda^e(e_{12}) &= (xz + \tfrac{1}{2}x^2y)\partial_x + (yz - \tfrac{1}{2}xy^2)\partial_y + (z^2 + \tfrac{1}{4}x^2y^2)\partial_z \\
&\quad + (\lambda_1 + \lambda_2 + 4)z + \tfrac{1}{2}(\lambda_1 - \lambda_2)xy, \\
\pi_\lambda^e(h_1) &= 2x\partial_x - y\partial_y + z\partial_z + \lambda_1 + 2, \\
\pi_\lambda^e(h_2) &= -x\partial_x + 2y\partial_y + z\partial_z + \lambda_2 + 2.
\end{aligned}$$

This representation corresponds to the Verma module with the highest weight $\lambda = (\lambda_1, \lambda_2)$, and the Fourier dual representation acts in the Fourier dual variables on $\mathbb{C}[\xi_x, \xi_y, \xi_z]$ by

$$\begin{aligned}
\hat{\pi}_\lambda^e(f_1) &= -\xi_x - \tfrac{1}{2}\xi_z\partial_{\xi_y}, \\
\hat{\pi}_\lambda^e(f_2) &= -\xi_y + \tfrac{1}{2}\xi_z\partial_{\xi_x}, \\
\hat{\pi}_\lambda^e(f_{12}) &= -\xi_z, \\
\hat{\pi}_\lambda^e(e_1) &= -\xi_y\partial_{\xi_z} + (\xi_x\partial_{\xi_x} + \tfrac{1}{2}\xi_z\partial_{\xi_z} - \lambda_1)\partial_{\xi_x} - \tfrac{1}{2}(\xi_y + \tfrac{1}{2}\xi_z\partial_{\xi_x})\partial_{\xi_x}\partial_{\xi_y}, \\
\hat{\pi}_\lambda^e(e_2) &= \xi_x\partial_{\xi_z} + (\xi_y\partial_{\xi_y} + \tfrac{1}{2}\xi_z\partial_{\xi_z} - \lambda_2)\partial_{\xi_y} - \tfrac{1}{2}(\xi_x - \tfrac{1}{2}\xi_z\partial_{\xi_y})\partial_{\xi_x}\partial_{\xi_y}, \\
\hat{\pi}_\lambda^e(e_{12}) &= (\xi_x\partial_{\xi_x} + \xi_y\partial_{\xi_y} + \xi_z\partial_{\xi_z} - \lambda_1 - \lambda_2)\partial_{\xi_z} \\
&\quad - \tfrac{1}{2}(\xi_x\partial_{\xi_x} - \xi_y\partial_{\xi_y} - \lambda_1 + \lambda_2 - \tfrac{1}{2}\xi_z\partial_{\xi_x}\partial_{\xi_y})\partial_{\xi_x}\partial_{\xi_y}, \\
\hat{\pi}_\lambda^e(h_1) &= -2\xi_x\partial_{\xi_x} + \xi_y\partial_{\xi_y} - \xi_z\partial_{\xi_z} + \lambda_1, \\
\hat{\pi}_\lambda^e(h_2) &= \xi_x\partial_{\xi_x} - 2\xi_y\partial_{\xi_y} - \xi_z\partial_{\xi_z} + \lambda_2.
\end{aligned}$$

The next result, which is easy to verify, determines the singular vectors responsible for the composition series of $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$. It can be regarded as a degenerate case in the series of parabolic subalgebras with Heisenberg type nilradicals discussed in [12]. We write the statement for general highest weight $\lambda \in \mathfrak{h}^*$, so that the number of singular vectors is reduced for the weights which are not dominant and regular. In particular, in the case λ is a regular dominant integral weight there are six singular vectors.

Lemma 1. *Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ be the highest weight for the Verma module $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda) \simeq \mathbb{C}[\partial_x, \partial_y, \partial_z]$. Then the singular vectors and their weights are*

1) $v_\lambda = 1$, (λ_1, λ_2)
2) $v_{s_1\cdot\lambda} = \partial_x^{\lambda_1+1}$, $(-\lambda_1 - 2, \lambda_1 + \lambda_2 + 1)$,
3) $v_{s_2\cdot\lambda} = \partial_y^{\lambda_2+1}$, $(\lambda_1 + \lambda_2 + 1, -\lambda_2 - 2)$,
4) $v_{s_2s_1\cdot\lambda} = \sum_{k=0}^{\lambda_2+1} \frac{k!}{2^k}\binom{\lambda_2+1}{k}\binom{\lambda_1+\lambda_2+2}{k}\partial_z^k\partial_y^{\lambda_2-k+1}\partial_x^{\lambda_1+\lambda_2-k+2}$, $(-\lambda_1 - \lambda_2 - 3, \lambda_1)$,
5) $v_{s_1s_2\cdot\lambda} = \sum_{k=0}^{\lambda_1+1} \frac{(-1)^k k!}{2^k}\binom{\lambda_1+1}{k}\binom{\lambda_1+\lambda_2+2}{k}\partial_z^k\partial_x^{\lambda_1-k+1}\partial_y^{\lambda_1+\lambda_2-k+2}$, $(\lambda_2, -\lambda_1 - \lambda_2 - 3)$,
6) $v_{s_1s_2s_1\cdot\lambda} = \sum_{k=0}^{\lambda_1+\lambda_2+2} \frac{k!}{2^k}\binom{\lambda_1+\lambda_2+2}{k}\sum_{\ell=0}^{k}\binom{\lambda_2+1}{\ell}\binom{\lambda_1+1}{k-\ell}(-1)^\ell$ $\partial_z^k\partial_x^{\lambda_1+\lambda_2-k+2}\partial_y^{\lambda_1+\lambda_2-k+2}$, $(-\lambda_2 - 2, -\lambda_1 - 2)$.

The Hasse diagram corresponding to the affine orbit of the Weyl group W for a regular dominant integral weight λ of $\mathfrak{sl}(3,\mathbb{C})$ is drawn on Figure 1a. The nodes of the overall graph correspond to Verma modules and the arrows are their homomorphisms, and the dots and arrows in each node (corresponding to a Verma module) represent the singular vectors and the Verma submodules they generate, respectively.

2.2. **Twisted Verma modules for $w = s_1$.** For $w = s_1$, we have $I_w = (\partial_x, y, z)$, $M_\mathfrak{g}^w(\lambda) \simeq \mathbb{C}[x, \partial_y, \partial_z]$, and

$$\pi^w_{(\lambda_1,\lambda_2)} = \pi_{(-\lambda_1-1,\lambda_1+\lambda_2+2)} \circ \mathrm{Ad}(\dot{w}^{-1}) \tag{2.10}$$

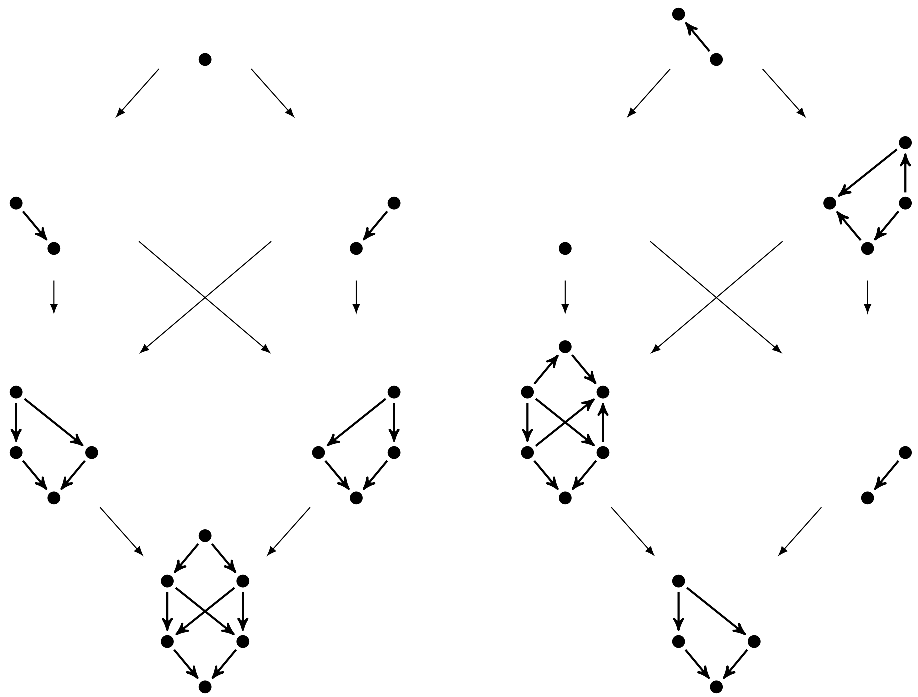
since $w^{-1}(\lambda+\rho) = w^{-1}(\lambda_1+1, \lambda_2+1) = (-\lambda_1-1, \lambda_1+\lambda_2+2)$ for $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$. Because

$$\begin{aligned} &\mathrm{Ad}(\dot{w}^{-1})(e_1) = -f_1, \qquad \mathrm{Ad}(\dot{w}^{-1})(e_{12}) = e_2, \qquad \mathrm{Ad}(\dot{w}^{-1})(e_2) = -e_{12}, \\ &\mathrm{Ad}(\dot{w}^{-1})(f_1) = -e_1, \qquad \mathrm{Ad}(\dot{w}^{-1})(f_{12}) = f_2, \qquad \mathrm{Ad}(\dot{w}^{-1})(f_2) = -f_{12}, \\ &\qquad\qquad \mathrm{Ad}(\dot{w}^{-1})(h_1) = -h_1, \qquad \mathrm{Ad}(\dot{w}^{-1})(h_2) = h_1 + h_2, \end{aligned} \tag{2.11}$$

we obtain

$$\begin{aligned} \pi^w_\lambda(f_1) &= -x^2\partial_x - (z - \tfrac{1}{2}xy)\partial_y - (\tfrac{1}{4}x^2y + \tfrac{1}{2}xz)\partial_z + \lambda_1 x, \\ \pi^w_\lambda(f_2) &= \partial_z, \\ \pi^w_\lambda(f_{12}) &= -\partial_y - \tfrac{1}{2}x\partial_z, \\ \pi^w_\lambda(e_1) &= \partial_x - \tfrac{1}{2}y\partial_z, \\ \pi^w_\lambda(e_2) &= -(xz + \tfrac{1}{2}x^2y)\partial_x - (yz - \tfrac{1}{2}xy^2)\partial_y - (z^2 + \tfrac{1}{4}x^2y^2)\partial_z \\ &\quad - (\lambda_2 + 3)z + \tfrac{1}{2}(2\lambda_1 + \lambda_2 + 3)xy, \\ \pi^w_\lambda(e_{12}) &= y^2\partial_y - (z + \tfrac{1}{2}xy)\partial_x - (\tfrac{1}{4}xy^2 - \tfrac{1}{2}yz)\partial_z + (\lambda_1 + \lambda_2 + 3)y, \\ \pi^w_\lambda(h_1) &= -2x\partial_x + y\partial_y - z\partial_z + \lambda_1, \\ \pi^w_\lambda(h_2) &= x\partial_x + y\partial_y + 2z\partial_z + \lambda_2 + 3\,. \end{aligned}$$

The vector $1 \in \mathbb{C}[x, \partial_y, \partial_z]$ has the weight $\lambda = (\lambda_1, \lambda_2)$. In the partial Fourier dual picture of the representation, the Lie algebra $\mathfrak{g}$ acts on the polynomial algebra



(A) $w = e$ (B) $w = s_1$

FIG. 1. Generalized weak BGG resolution for $w = e$ and $w = s_1$

$\mathbb{C}[x, \xi_y, \xi_z]$ by

$$
\begin{aligned}
\hat{\pi}^w_\lambda(f_1) &= \xi_y \partial_{\xi_z} - x(x\partial_x - \tfrac{1}{2}\xi_z\partial_{\xi_z} - \lambda_1) - \tfrac{1}{2}x(\xi_y - \tfrac{1}{2}x\xi_z)\partial_{\xi_y}, \\
\hat{\pi}^w_\lambda(f_2) &= \xi_z, \\
\hat{\pi}^w_\lambda(f_{12}) &= -\xi_y - \tfrac{1}{2}x\xi_z, \\
\hat{\pi}^w_\lambda(e_1) &= \partial_x + \tfrac{1}{2}\xi_z\partial_{\xi_y}, \\
\hat{\pi}^w_\lambda(e_2) &= (x\partial_x - \xi_y\partial_{\xi_y} - \xi_z\partial_{\xi_z} + \lambda_2)\partial_{\xi_z} \\
&\quad + \tfrac{1}{2}x(x\partial_x + \xi_y\partial_{\xi_y} - \tfrac{1}{2}x\xi_z\partial_{\xi_y} - 2\lambda_1 - \lambda_2 - 1)\partial_{\xi_y}, \\
\hat{\pi}^w_\lambda(e_{12}) &= \partial_x\partial_{\xi_z} + (\xi_y\partial_{\xi_y} + \tfrac{1}{2}\xi_z\partial_{\xi_z} - \lambda_1 - \lambda_2 - \tfrac{1}{2})\partial_{\xi_y} + \tfrac{1}{2}x(\partial_x - \tfrac{1}{2}\xi_z\partial_{\xi_y})\partial_{\xi_y}, \\
\hat{\pi}^w_\lambda(h_1) &= -2x\partial_x - \xi_y\partial_{\xi_y} + \xi_z\partial_{\xi_z} + \lambda_1, \\
\hat{\pi}^w_\lambda(h_2) &= x\partial_x - \xi_y\partial_{\xi_y} - 2\xi_z\partial_{\xi_z} + \lambda_2.
\end{aligned}
$$

Lemma 2. *The twisted Verma module $M^w_{\mathfrak{g}}(\lambda)$ for $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ is generated by one vector v_λ. For $\lambda_1 \notin \mathbb{N}_0$ this generator is $v_\lambda = 1 \in \mathbb{C}[x, \partial_y, \partial_z]$, while for $\lambda_1 \in \mathbb{N}_0$ the generator is equal to $v_\lambda = x^{\lambda_1+1} \in \mathbb{C}[x, \partial_y, \partial_z]$.*

Proof. We observe $\pi^w_\lambda(f_1^k)1 = k!\binom{\lambda_1}{k}x^k$ for $k \in \mathbb{N}_0$. It follows that the vectors $\{\pi^w_\lambda(f_1^k)1;\ k \in \mathbb{N}_0\}$ generate the subspace $\mathbb{C}[x] \subset \mathbb{C}[x,\partial_y,\partial_z]$ for $\lambda_1 \notin \mathbb{N}_0$. Further, for $\lambda_1 \in \mathbb{N}_0$ we have $\pi^w_\lambda(e_1^k)x^{\lambda_1+1} = k!\binom{\lambda_1+1}{k}x^{\lambda_1+1-k}$ and $\pi^w_\lambda(f_1^k)x^{\lambda_1+1} = (-1)^k k! x^{\lambda_1+1+k}$ for $k \in \mathbb{N}_0$, and therefore the vectors $\{\pi^w_\lambda(f_1^k)x^{\lambda_1+1}, \pi^w_\lambda(e_1^k)x^{\lambda_1+1};\ k \in \mathbb{N}_0\}$ generate again the subspace $\mathbb{C}[x] \subset \mathbb{C}[x,\partial_y,\partial_z]$. Now, from the form of elements $\pi^w_\lambda(f_2) = \partial_z$ and $\pi^w_\lambda(f_{12}) = -\partial_y - \frac{1}{2}x\partial_z$ the rest of the proof easily follows. $\square$

Lemma 3. *Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ and let I^w_λ be the left ideal of $U(\mathfrak{g})$ defined by*

$$(2.12)\quad I^w_\lambda = \begin{cases} (e_1, e_2, e_{12}, h_1 - \lambda_1, h_2 - \lambda_2) & \text{if} \quad \lambda_1 \notin \mathbb{N}_0, \\ (e_2, e_{12}, e_1^{\lambda_1+2}, f_1e_1, h_1 + \lambda_1 + 2, h_2 - \lambda_1 - \lambda_2 - 1) & \text{if} \quad \lambda_1 \in \mathbb{N}_0. \end{cases}$$

Then we have $U(\mathfrak{g})/I^w_\lambda \simeq M^w_\mathfrak{g}(\lambda)$ as $\mathfrak{g}$-modules.

Proof. Let us consider $w_\lambda = 1 \bmod I^w_\lambda$ as an element in $U(\mathfrak{g})/I^w_\lambda$. The generator v_λ of $M^w_\mathfrak{g}(\lambda)$ constructed in Lemma 2 allows to define a homomorphism $\varphi\colon U(\mathfrak{g}) \to M^w_\mathfrak{g}(\lambda)$ of $U(\mathfrak{g})$-modules by $\varphi(1) = v_\lambda$. Since the generator v_λ is annihilated by the left ideal I^w_λ of $U(\mathfrak{g})$, we get the surjective homomorphism $\tilde{\varphi}\colon U(\mathfrak{g})/I^w_\lambda \to M^w_\mathfrak{g}(\lambda)$ of $U(\mathfrak{g})$-modules. Then $\tilde{\varphi}$ is an isomorphism once we prove that the modules have the same formal characters. There are clearly two complementary cases to be considered:

i) Let us assume first $\lambda_1 \notin \mathbb{N}_0$. Then $h_1w_\lambda = \lambda_1w_\lambda$ and $h_2w_\lambda = \lambda_2w_\lambda$, therefore $U(\mathfrak{h})w_\lambda = \mathbb{C}w_\lambda$. By $e_1w_\lambda = 0$, $e_2w_\lambda = 0$ and $e_{12}w_\lambda = 0$ it follows $U(\mathfrak{n})w_\lambda = \mathbb{C}w_\lambda$. The PBW theorem applied to $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ is equivalent to $U(\mathfrak{g}) = U(\bar{\mathfrak{n}})U(\mathfrak{n})U(\mathfrak{h})$, hence we get $U(\mathfrak{g})/I^w_\lambda = U(\mathfrak{g})w_\lambda = U(\bar{\mathfrak{n}})w_\lambda$. But the characters of $U(\mathfrak{g})/I^w_\lambda$ and $M^w_\mathfrak{g}(\lambda)$ are equal, because $U(\mathfrak{g})/I^w_\lambda$ is free $U(\bar{\mathfrak{n}})$-module generated by w_λ and hence a Verma module with the highest weight λ.

ii) Let us now assume $\lambda_1 \in \mathbb{N}_0$. Then $h_1w_\lambda = (-\lambda_1 - 2)w_\lambda$ and $h_2w_\lambda = (\lambda_1+\lambda_2+1)w_\lambda$, therefore $U(\mathfrak{h})w_\lambda = \mathbb{C}w_\lambda$. By $e_2w_\lambda = 0$ and $e_{12}w_\lambda = 0$ it follows that $U(\mathfrak{n}_w \cap \mathfrak{n})w_\lambda = \mathbb{C}w_\lambda$. We have $e_1^{\lambda_1+2}w_\lambda = 0$, and so $U(\bar{\mathfrak{n}}_w \cap \mathfrak{n})w_\lambda = \bigoplus_{k=0}^{\lambda_1+1}\mathbb{C}e_1^kw_\lambda$. Finally, the condition $f_1e_1w_\lambda = 0$ implies an equality of vector spaces

$$U(\mathfrak{n}_w \cap \bar{\mathfrak{n}})U(\bar{\mathfrak{n}}_w \cap \mathfrak{n})w_\lambda = \bigoplus_{k=0}^{\lambda_1}\mathbb{C}e_1^{\lambda_1+1-k}w_\lambda \oplus \bigoplus_{k\in\mathbb{N}_0}\mathbb{C}f_1^kw_\lambda.$$

By PBW theorem applied to the vector space decomposition $\mathfrak{g} = (\bar{\mathfrak{n}}_w \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{n}_w \cap \bar{\mathfrak{n}}) \oplus (\bar{\mathfrak{n}}_w \cap \mathfrak{n}) \oplus (\mathfrak{n}_w \cap \mathfrak{n}) \oplus \mathfrak{h}$ we get

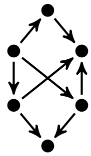
$$U(\mathfrak{g})/I^w_\lambda = U(\mathfrak{g})w_\lambda = U(\bar{\mathfrak{n}})w_\lambda \oplus \bigoplus_{k=0}^{\lambda_1} U(\mathfrak{n}_w \cap \bar{\mathfrak{n}})\,\mathbb{C}e_1^{\lambda_1+1-k}w_\lambda,$$

where we used the fact that $U(\mathfrak{g})/I^w_\lambda$ is a free $U(\bar{\mathfrak{n}}_w \cap \bar{\mathfrak{n}})$-module. Hence the characters of $U(\mathfrak{g})/I^w_\lambda$ and $M^w_\mathfrak{g}(\lambda)$ coincide.
Hence the proof is complete. $\square$

The structure of s_1-twisted Verma modules $M^w_\mathfrak{g}(\lambda)$ for the highest weights $\lambda \in \mathfrak{h}^*$ lying on the affine orbit of W for an integral dominant weight is shown on the list below. Namely, the points on the figures denote singular vectors in $M^w_\mathfrak{g}(\lambda)$ or some of its quotients, where the vectors which were not singular vectors in the former twisted Verma module are singular vectors in a quotient Verma module.

The composition structure among these vectors is expressed by the arrows: there is a directed arrow from one vector to another vector if and only if for any choice of vectors projecting under a quotient homomorphism onto singular vectors is the latter vector generated by the former vector.

The structure of $M^w_{\mathfrak{g}}(\lambda)$ for a specific highest weight $\lambda \in \mathfrak{h}^*$ follows immediately from Lemma 3 and its proof. If $\lambda_1 \notin \mathbb{N}_0$, then $M^w_{\mathfrak{g}}(\lambda)$ is isomorphic to the Verma module $M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$. On the other hand, if $\lambda_1 \in \mathbb{N}_0$, then $M^w_{\mathfrak{g}}(\lambda)/N^w_{\mathfrak{g}}(\lambda)$ is isomorphic to the Verma module $M^{\mathfrak{g}}_{\mathfrak{b}}(s_1 \cdot \lambda)$, where $N^w_{\mathfrak{g}}(\lambda)$ is the $\mathfrak{g}$-submodule generated by $1 \in \mathbb{C}[x, \partial_y, \partial_z]$.

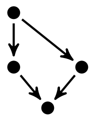


1

x^{λ_1+1} $\partial_z^{\lambda_2+1}$

$x^{\lambda_1+1}\partial_z^{\lambda_1+\lambda_2+1}$ $x^{\lambda_1+1}\left(\sum_{k=0}^{\lambda_2+1} \frac{1}{2^k}\binom{\lambda_2+1}{k} a_k (x\partial_z)^k \partial_y^{\lambda_2+1-k}\right)$, $(\lambda_1+k+3)a_{k+2} = \lambda_1 a_{k+1} + (k+1)a_k$

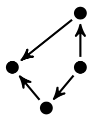
$(x\partial_z - 2\partial_y)^{\lambda_1+\lambda_2+2}$



1

$\partial_z^{\lambda_1+\lambda_2+2}$ $(x\partial_z - 2\partial_y)^{\lambda_2+1}$

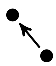
$(x\partial_z - 2\partial_y)^{\lambda_2+1}\partial_z^{\lambda_1+1}$



1

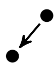
$(x\partial_z - 2\partial_y)^{\lambda_1+1}$ $x^{\lambda_1+\lambda_2+2}$

$x^{\lambda_1+\lambda_2+2}\partial_z^{\lambda_1+1}$



1

x^{λ_2+1}



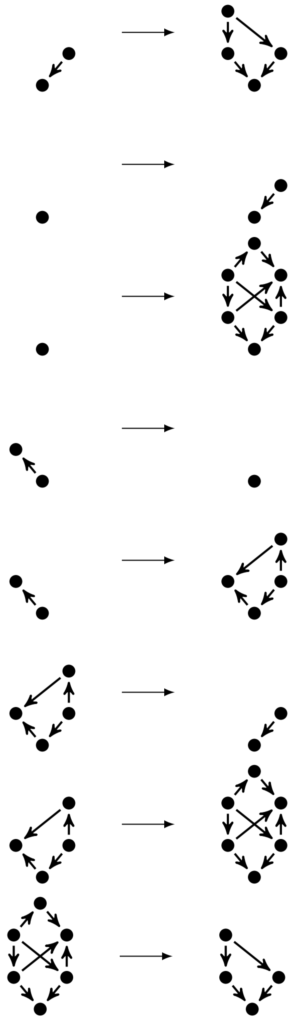
1

$\partial_z^{\lambda_1+1}$

1

Let us recall that we have the twisting functor $T_w\colon \mathcal{O} \to \mathcal{O}$. If we apply this functor to the standard BGG resolution for Verma modules shown on Figure 1a, we obtain twisted BGG resolution shown on Figure 1b.

Let us describe the homomorphisms between twisted Verma modules $M^w_{\mathfrak{g}}(\lambda)$ drawn on Figure 1b explicitly. Because $M^w_{\mathfrak{g}}(\lambda)$ is generated by one vector $v_\lambda \in M^w_{\mathfrak{g}}(\lambda)$, a homomorphism $\varphi\colon M^w_{\mathfrak{g}}(\lambda) \to M^w_{\mathfrak{g}}(\mu)$ is uniquely determined by $\varphi(v_\lambda) \in M^w_{\mathfrak{g}}(\mu)$.



$$1 \mapsto (x\partial_z - 2\partial_y)^{\lambda_2+1}$$

$$1 \mapsto \partial_z^{\lambda_1+1}$$

$$1 \mapsto (x\partial_z - 2\partial_y)^{\lambda_1+\lambda_2+2}$$

$$x^{\lambda_2+1} \mapsto 1$$

$$x^{\lambda_2+1} \mapsto x^{\lambda_2+1}(x\partial_z - 2\partial_y)^{\lambda_1+1}$$

$$x^{\lambda_1+\lambda_2+2} \mapsto 1$$

$$x^{\lambda_1+\lambda_2+2} \mapsto x^{\lambda_1+1}\Big(\sum_{k=0}^{\lambda_2+1} \frac{1}{2^k}\binom{\lambda_2+1}{k} a_k (x\partial_z)^k \partial_y^{\lambda_2+1-k}\Big),$$
$$(\lambda_1+k+3)a_{k+2} = \lambda_1 a_{k+1} + (k+1)a_k,\ a_{-1}=0,\ a_0=1$$

$$x^{\lambda_1+1} \mapsto 1$$

There is another aspect of the composition structure of twisted Verma modules, related to a choice of Lie subalgebras of $\mathfrak{sl}(3,\mathbb{C})$. As for the parallel results for (untwisted) Verma modules, we refer to [12], [11]. For concreteness, we shall stick to the case of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ embedded on the first simple root of $\mathfrak{sl}(3,\mathbb{C})$.

Lemma 4. *Let us consider the Lie subalgebra* $\mathfrak{sl}(2,\mathbb{C})$ *of the Lie algebra* $\mathfrak{sl}(3,\mathbb{C})$ *generated by the elements* $\{e_1, f_1, h_1\}$*. Then the set of singular vectors for this Lie subalgebra, i.e.* $v \in M^w_{\mathfrak{g}}(\lambda)$ *such that* $\pi^w_\lambda(e_1)v = 0$*, is given by* $\mathbb{C}[\partial_z, x\partial_z - 2\partial_y]$*, and the corresponding weight spaces with respect to the Cartan subalgebra* $\mathbb{C}h_1$ *of* $\mathfrak{sl}(2,\mathbb{C})$ *are*

$$\bigoplus_{b\in\mathbb{N}_0} \mathbb{C}\partial_z^{a+2b}(x\partial_z - 2\partial_y)^b \tag{2.13}$$

with the weight $(\lambda_1 + a)\omega_1$*.*

Proof. First of all, we claim that there is an isomorphism of graded $\mathbb{C}$-algebras

$$\mathbb{C}[x, \partial_y, \partial_z] \xrightarrow{\simeq} \mathbb{C}[\partial_z, x\partial_z - 2\partial_y] \otimes_{\mathbb{C}} \mathbb{C}[x] \tag{2.14}$$

with $\deg(x) = 1$, $\deg(\partial_y) = 2$ and $\deg(\partial_z) = 1$. The mapping is clearly surjective, and the fact that $x, \partial_y, \partial_z$ are algebraically independent implies by induction on the degree of ∂_y the algebraic independence of $x, x\partial_z - 2\partial_y, \partial_z$, hence the claim follows.

An elementary calculation shows that any element in $\mathbb{C}[\partial_z, x\partial_z - 2\partial_y]$ is in the kernel of $\pi^w_\lambda(e_1)$, hence the action of $\pi^w_\lambda(e_1)$ on $\mathbb{C}[\partial_z, x\partial_z - 2\partial_y] \otimes_{\mathbb{C}} \mathbb{C}[x]$ reduces to the action of $1 \otimes \partial_x$. Therefore, the kernel of $\pi^w_\lambda(e_1)$ on $\mathbb{C}[\partial_z, x\partial_z - 2\partial_y] \otimes_{\mathbb{C}} \mathbb{C}[x]$ is equal to $\mathbb{C}[\partial_z, x\partial_z - 2\partial_y] \otimes_{\mathbb{C}} \mathbb{C}$, and thus to $\mathbb{C}[\partial_z, x\partial_z - 2\partial_y]$ by the isomorphism (2.14). $\square$

After the application of partial Fourier transform, the isomorphism (2.14) *can be interpreted as a graded version of the Fischer tensor product decomposition for the graded algebra* $\mathbb{C}[x, \xi_y, \xi_z]$ *with respect to the differential operator* $\partial_x + \frac{1}{2}\xi_z\partial_{\xi_y}$*, cf.* [5]*. Though we were not able to find an explicit result on the branching rules for twisted Verma modules in the available literature, the coincidence of their characters with characters of (untwisted) Verma modules suggests branching rules in* $K(\mathcal{O})$ *parallel to those derived in* [10].

2.3. **Twisted Verma modules for** $w = s_1s_2$**.** *For* $w = s_1s_2$*, we have* $I_w = (x, \partial_y, \partial_z)$*,* $M^w_{\mathfrak{g}}(\lambda) \simeq \mathbb{C}[\partial_x, y, z]$*, and*

$$\pi^w_{(\lambda_1,\lambda_2)} = \pi_{(\lambda_2+1,-\lambda_1-\lambda_2-2)} \circ \mathrm{Ad}(\dot{w}^{-1}) \tag{2.15}$$

since $w^{-1}(\lambda+\rho) = w^{-1}(\lambda_1+1, \lambda_2+1) = (\lambda_2+1, -\lambda_1-\lambda_2-2)$ *for* $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$*. Due to*

$$\begin{gathered} \mathrm{Ad}(\dot{w}^{-1})(e_1) = -f_{12}, \qquad \mathrm{Ad}(\dot{w}^{-1})(e_{12}) = -f_2, \qquad \mathrm{Ad}(\dot{w}^{-1})(e_2) = e_1, \\ \mathrm{Ad}(\dot{w}^{-1})(f_1) = -e_{12}, \qquad \mathrm{Ad}(\dot{w}^{-1})(f_{12}) = -e_2, \qquad \mathrm{Ad}(\dot{w}^{-1})(f_2) = f_1, \\ \mathrm{Ad}(\dot{w}^{-1})(h_1) = -h_1 - h_2, \qquad \mathrm{Ad}(\dot{w}^{-1})(h_2) = h_1, \end{gathered} \tag{2.16}$$

we obtain

$$\begin{aligned} \pi^w_\lambda(f_1) = {} & -(xz + \tfrac{1}{2}x^2y)\partial_x - (yz - \tfrac{1}{2}xy^2)\partial_y - (z^2 + \tfrac{1}{4}x^2y^2)\partial_z + (\lambda_1 - 1)z \\ & - \tfrac{1}{2}(\lambda_1 + 2\lambda_2 + 3)xy\,, \end{aligned}$$

$$\begin{aligned}
\pi^w_\lambda(f_2) &= -\partial_x + \tfrac{1}{2}y\partial_z\,,\\
\pi^w_\lambda(f_{12}) &= -y^2\partial_y + (z+\tfrac{1}{2}xy)\partial_x + (\tfrac{1}{4}xy^2 - \tfrac{1}{2}yz)\partial_z + (\lambda_1+\lambda_2+1)y\,,\\
\pi^w_\lambda(e_1) &= \partial_z\,,\\
\pi^w_\lambda(e_2) &= x^2\partial_x + (z-\tfrac{1}{2}xy)\partial_y + (\tfrac{1}{4}x^2y + \tfrac{1}{2}xz)\partial_z + (\lambda_2+2)x\,,\\
\pi^w_\lambda(e_{12}) &= \partial_y + \tfrac{1}{2}x\partial_z\,,\\
\pi^w_\lambda(h_1) &= -x\partial_x - y\partial_y - 2z\partial_z + \lambda_1 - 1\,,\\
\pi^w_\lambda(h_2) &= 2x\partial_x - y\partial_y + z\partial_z + \lambda_2 + 2\,.
\end{aligned}$$

The vector $1 \in \mathbb{C}[\partial_x, y, z]$ has the weight $\lambda = (\lambda_1, \lambda_2)$. In the partial Fourier dual picture of the representation, $\mathfrak{g}$ acts on $\mathbb{C}[\xi_x, y, z]$ by

$$\begin{aligned}
\hat{\pi}^w_\lambda(f_1) &= -z(-\xi_x\partial_{\xi_x} + y\partial_y + z\partial_z - \lambda_1)\\
&\qquad + \tfrac{1}{2}y\big(-\xi_x\partial_{\xi_x} - y\partial_y - \tfrac{1}{2}y\partial_{\xi_x}\partial_z + 2\lambda_2 + \lambda_1 + 1\big)\partial_{\xi_x},\\
\hat{\pi}^w_\lambda(f_2) &= -\xi_x + \tfrac{1}{2}y\partial_z,\\
\hat{\pi}^w_\lambda(f_{12}) &= \xi_x z - y(y\partial_y + \tfrac{1}{2}z\partial_z - \lambda_1 - \lambda_2 - \tfrac{1}{2}) - \tfrac{1}{2}y(\xi_x + \tfrac{1}{2}y\partial_z)\partial_{\xi_x},\\
\hat{\pi}^w_\lambda(e_1) &= \partial_z,\\
\hat{\pi}^w_\lambda(e_2) &= z\partial_y + (\xi_x\partial_{\xi_x} - \tfrac{1}{2}z\partial_z - \lambda_2)\partial_{\xi_x} + \tfrac{1}{2}y(\partial_y + \tfrac{1}{2}\partial_z\partial_{\xi_x})\partial_{\xi_x},\\
\hat{\pi}^w_\lambda(e_{12}) &= \partial_y - \tfrac{1}{2}\partial_{\xi_x}\partial_z,\\
\hat{\pi}^w_\lambda(h_1) &= \xi_x\partial_{\xi_x} - y\partial_y - 2z\partial_z + \lambda_1,\\
\hat{\pi}^w_\lambda(h_2) &= -2\xi_x\partial_{\xi_x} - y\partial_y + z\partial_z + \lambda_2.
\end{aligned}$$

2.4. **Twisted Verma modules for $w = s_1s_2s_1$.** *For $w = s_1s_2s_1$, we have $I_w = (\partial_x, \partial_y, \partial_z)$, $M^w_{\mathfrak{g}}(\lambda) \simeq \mathbb{C}[x,y,z]$, and*

$$\pi^w_{(\lambda_1,\lambda_2)} = \pi_{(-\lambda_2-1,-\lambda_1-1)} \circ \mathrm{Ad}(\dot{w}^{-1}) \tag{2.17}$$

since $w^{-1}(\lambda+\rho) = w^{-1}(\lambda_1+1, \lambda_2+1) = (-\lambda_2-1, -\lambda_1-1)$ for $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$. Due to

$$\begin{gathered}
\mathrm{Ad}(\dot{w}^{-1})(e_1) = -f_2, \qquad \mathrm{Ad}(\dot{w}^{-1}) = f_{12}, \qquad \mathrm{Ad}(\dot{w}^{-1})(e_2) = -f_1,\\
\mathrm{Ad}(\dot{w}^{-1})(f_1) = -e_2, \qquad \mathrm{Ad}(\dot{w}^{-1})(f_{12}) = e_{12}, \qquad \mathrm{Ad}(\dot{w}^{-1})(f_2) = -e_1,\\
\mathrm{Ad}(\dot{w}^{-1})(h_1) = -h_2, \qquad \mathrm{Ad}(\dot{w}^{-1})(h_2) = -h_1,
\end{gathered} \tag{2.18}$$

we obtain

$$\begin{aligned}
\pi^w_\lambda(f_1) &= -y^2\partial_y + (z+\tfrac{1}{2}xy)\partial_x + (\tfrac{1}{4}xy^2 - \tfrac{1}{2}yz)\partial_z + \lambda_1 y\,,\\
\pi^w_\lambda(f_2) &= -x^2\partial_x - (z-\tfrac{1}{2}xy)\partial_y - (\tfrac{1}{4}x^2y + \tfrac{1}{2}xz)\partial_z + \lambda_2 x\,,
\end{aligned}$$

$$\begin{aligned}
\pi^w_\lambda(f_{12}) &= (xz + \tfrac{1}{2}x^2y)\partial_x + (yz - \tfrac{1}{2}xy^2)\partial_y + (z^2 + \tfrac{1}{4}x^2y^2)\partial_z - (\lambda_1+\lambda_2)z \\
&\quad + \tfrac{1}{2}(\lambda_1 - \lambda_2)xy\,, \\
\pi^w_\lambda(e_1) &= \partial_y + \tfrac{1}{2}x\partial_z\,, \\
\pi^w_\lambda(e_2) &= \partial_x - \tfrac{1}{2}y\partial_z\,, \\
\pi^w_\lambda(e_{12}) &= -\partial_z\,, \\
\pi^w_\lambda(h_1) &= x\partial_x - 2y\partial_y - z\partial_z + \lambda_1\,, \\
\pi^w_\lambda(h_2) &= -2x\partial_x + y\partial_y - z\partial_z + \lambda_2\,.
\end{aligned}$$

The vector $1 \in \mathbb{C}[x,y,z]$ has the weight $\lambda = (\lambda_1, \lambda_2)$.

3. Outlook and open questions

Let us finish by mentioning that many properties of (untwisted) Verma modules are not known for twisted Verma modules. For example, it is not clear which twisted Verma modules are over $U(\mathfrak{g})$ generated by one element. It is also desirable to understand when a given twisted Verma module $M^w_{\mathfrak{g}}(\lambda)$ belongs to a parabolic Bernstein-Gelfand-Gelfand category $\mathcal{O}^{\mathfrak{p}}$ associated to a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. This is a non-trivial task, because the choice of $\mathfrak{p}$ heavily depends on the twisting $w \in W$. Another question is related to the realization of twisted Verma modules on Schubert cells – as for $\mathfrak{sl}(3,\mathbb{C})$ there are six of them, but there exist altogether eight possibilities for an ideal defined by annihilating condition for three variables out of the collection x, ∂_x, y, ∂_y, z, ∂_z. What are the isomorphism classes of the remaining two $\mathfrak{sl}(3,\mathbb{C})$-modules?

Acknowledgement. L. Křižka is supported by PRVOUK p47, P. Somberg acknowledges the financial support from the grant GA P201/12/G028.

References

[1] Abe, N., *On the existence of homomorphisms between principal series representations of complex semisimple Lie groups*, J. Algebra **330** (1) (2011), 468–481.

[2] Andersen, H.H., Lauritzen, N., *Twisted Verma modules*, Studies in Memory of Issai Schur, Progress in Mathematics, vol. 210, Birkhäuser, Boston, 2003, pp. 1–26.

[3] Beilinson, A.A., Bernstein, J.N., *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 15–18.

[4] Feigin, B.L., Frenkel, E.V., *Affine Kac-Moody algebras and semi-infinite flag manifolds*, Comm. Math. Phys. **128** (1) (1990), 161–189.

[5] Fischer, E., *Über die Differentiationsprozesse der Algebra*, J. Reine Angew. Math. **148** (1918), 1–78.

[6] Frenkel, E.V., Ben-Zvi, D., *Vertex Algebras and Algebraic Curves*, Mathematical Surveys and Monographs, vol. 88, Amer. Math. Soc. Providence, 2004.

[7] Hotta, R., Takeuchi, K., Tanisaki, T., *$\mathcal{D}$-Modules, Perverse Sheaves, and Representation Theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston, 2008.

[8] Humphreys, J.E., *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*, Graduate Studies in Mathematics, Amer. Math. soc. Providence, 2008.

[9] Kashiwara, M., *Representaion theory and $\mathcal{D}$-modules on flag varieties*, Astérisque **173–174** (1989), 55–109.

[10] Kobayashi, T., *Restrictions of generalized Verma modules to symmetric pairs*, Transform. Groups **17** (2) (2012), 523–546.

[11] Kobayashi, T., Ørsted, B., Somberg, P., Souček, V., *Branching laws for Verma modules and applications in parabolic geometry. I*, Adv. Math. **285** (2015), 1–57.

[12] Křižka, L., Somberg, P., *Algebraic analysis on scalar generalized Verma modules of Heisenberg parabolic type I.: A_n-series*, (2015) `arXiv:1502.07095`.

[13] Mazorchuk, V., Stroppel, C., *Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module*, Trans. Amer. Math. Soc. **357** (2005), 2939–2973.

[14] Soergel, W., *Character formulas for tilting modules over Kac-Moody algebras*, Represent. Theory **2** (1998), 432–448.

Mathematical Institute of Charles University,
Sokolovská 83, 180 00 Praha 8,
Czech Republic
E-mail: `krizka@karlin.mff.cuni.cz` `somberg@karlin.mff.cuni.cz`