ARCHIVUM MATHEMATICUM (BRNO)
Tomus 51 (2015), 331–346

BRANCHING PROBLEMS AND $\mathfrak{sl}(2,\mathbb{C})$-ACTIONS

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Abstract. We study certain $\mathfrak{sl}(2,\mathbb{C})$-actions associated to specific examples of branching of scalar generalized Verma modules for compatible pairs $(\mathfrak{g},\mathfrak{p})$, $(\mathfrak{g}',\mathfrak{p}')$ of Lie algebras and their parabolic subalgebras.

1. Introduction

The notion of composition series or branching rules for geometrically realized algebraic objects of representation theoretical origin, lies at the heart of many problems on the intersection of representation theory, algebraic analysis and differential geometry.

The present letter attempts to address several concrete questions belonging to this line of research, originating in the series of articles [6], [5], [7]. Namely, motivated by questions in differential geometry and harmonic analysis on differential invariants associated to pairs of generalized flag manifolds, a constructive method (the F-method) was developed there. It is based on algebraic analysis applied to generalized Verma modules, realized as $\mathcal{D}$-modules on the point orbit of the nilpotent group on the generalized flag manifold. The methods of algebraic analysis, which replace the standard combinatorial approach, allow to find the singular vectors responsible for the composition structure of generalized Verma modules in a striking way. In many cases, the factorization identities yield the answer not only in the Grothendieck group of the Bernstein-Gelfand-Gelfand category $\mathcal{O}^{\mathfrak{p}}(\mathfrak{g})$, but also allow to recognize the composition structure and the extension classes.

In the present note we discuss several questions left untouched in [6], [5]. To describe these questions, we first introduce some notation. Let $\mathfrak{g}$ be a simple Lie algebra and let $\mathfrak{p}\subset\mathfrak{g}$ be a parabolic subalgebra. Let $(\mathfrak{g}',\mathfrak{p}'),(\mathfrak{g},\mathfrak{p})$ be a compatible pair of Lie algebras and their parabolic subalgebras with $\mathfrak{g}'\subset\mathfrak{g},\mathfrak{p}'\subset\mathfrak{p}$. Let $\mathfrak{n}$, respectively $\mathfrak{n}'$, be the nilradical of $\mathfrak{p}$ and $\mathfrak{p}'$, respectively.

We focus on the role of the generators of the complement of $\mathfrak{n}'$ in $\mathfrak{n}$. As we shall see in our two examples, each of them is characterized by the one dimensional quotient $\mathfrak{n}/\mathfrak{n}'$, the root vector generating this complement acts on the space of $\mathfrak{g}'$-singular vectors. We shall observe in one of these examples that there is moreover a $\mathfrak{sl}(2,\mathbb{C})$-module structure on the space of $\mathfrak{g}'$-singular vectors, but the action of

2010 *Mathematics Subject Classification*: primary 22E47.
Key words and phrases: representation theory of simple Lie algebra, generalized Verma modules, singular vectors and composition series, relative Lie algebra and Dirac cohomology.
DOI: 10.5817/AM2015-5-331

the opposite root space does not directly follow from any representation theoretical construction.

The article is organized as follows. In the beginning of Section 2 we review some basic notation, and we pass to the two examples – one is given by the Hermitean symmetric space associated to the first node of the Dynkin diagram of an orthogonal Lie algebra, and the second example is the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ diagonally embedded in $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$. As we already mentioned, in the first example we find an $\mathfrak{sl}(2,\mathbb{C})$-module structure on the set of $\mathfrak{g}'$-singular vectors which allows detailed analysis related to the structure of composition series. In Section 3, we pass to a natural question on the relative Lie algebra or Dirac cohomology associated to our branching problem, which does not seem to be discussed in the literature. We finish the letter by several useful conventions and formulas related to Gegenbauer and Jacobi polynomials, which realize the $\mathfrak{g}'$-singular vectors in the examples of Section 2.

2. Main examples

We shall start with a brief review of several basic notions, relying on the conventions in [6], [5].

We denote by $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{l}$, $\mathfrak{n}$, $\mathfrak{n}_-$ a real simple Lie algebra, its parabolic subalgebra, the Levi factor and the nilradical of the parabolic subalgebra, and the opposite (negative) nilradical. There is an isomorphism of vector spaces $\mathfrak{g}\simeq\mathfrak{n}_-\oplus\mathfrak{p}\simeq\mathfrak{n}_-\oplus\mathfrak{l}\oplus\mathfrak{n}$. The connected and simply connected groups corresponding to these Lie algebras are denoted by G, P, L, N, N_-.

The $\mathfrak{g}$-modules we consider are the scalar generalized Verma modules, geometrically realized by $\mathcal{D}$-modules supported at the closed orbit eP of the nilpotent group N on the generalized flag manifold G/P. These modules are then identified by distribution Fourier transform with the underlying vector space of the polynomial algebras in Fourier dual variables ξ_i, $i=1,\cdots,\dim_{\mathbb{R}}(\mathfrak{n}_-)$. We can consider another collection of Lie algebras $\mathfrak{g}'$, $\mathfrak{p}'$, $\mathfrak{l}'$, $\mathfrak{n}'$, $\mathfrak{n}'_-$ with the same properties as in the un-primed case, such that $\mathfrak{g}'\subset\mathfrak{g}$ induces the primed to un-primed inclusions for all the other Lie subalgebras. The compatibility of the two collections through the compatible grading of $\mathfrak{p}'$ and $\mathfrak{p}$ realized by the adjoint action of the grading element in $\mathfrak{l}'$ is also required.

The $\mathfrak{g}'$-singular vectors in $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)$ describe the generators of $\mathfrak{g}'$-submodules in the Grothendieck group $K(\mathcal{O}^{\mathfrak{p}'})$ of the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}'}$, and consequently determine its $\mathfrak{g}'$-composition structure. They are the solution spaces of the system of partial differential equations corresponding to the action of $\mathfrak{n}'$ by differential operators on $\mathbb{C}[\xi_1,\dots,\xi_{\dim_{\mathbb{R}}(\mathfrak{n}_-)}]$.

In the two examples of our interest, the $\mathfrak{g}'$-singular vectors are the Gegenbauer and Jacobi polynomials, respectively. Our main concern in these simple (but representative) examples is the action of the generator of $\mathfrak{n}/\mathfrak{n}'$ on the span of $\mathfrak{g}'$-singular vectors and its consequences, e.g. a lift of its structure to a $\mathfrak{sl}(2,\mathbb{C})$-structure. We construct such a lift in one of our examples, but at the cost of having to leave the setting of the universal enveloping algebra $U(\mathfrak{g})$.

This raises a representation theoretical question whether, say even for the Hermitean symmetric spaces $(\mathfrak{g},\mathfrak{p})$ characterized by parabolic subalgebras $\mathfrak{p}$ with commutative nilradical, the action of elements in $\mathfrak{n}/\mathfrak{n}'$ on the $U(\mathfrak{l}')$-submodule $\mathrm{Ker}(\mathfrak{n}') \subset M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)$ of a generalized Verma module can be lifted to a non-trivial action of a bigger subalgebra in the Weyl algebra on the opposite nilradical $\mathfrak{n}$. We do not know the answer to this question, and the present article is a modest attempt to get an elementary insight into some of its aspects.

Throughout the article we use the notation $\langle\,,\rangle$ for the linear span of a subset of a vector space, U applied to an algebra denotes its universal enveloping algebra, and $\mathbb{N}_0$ the set of natural integers including 0.

2.1. **The pair of Lie algebras $(\mathfrak{so}(n{+}1,1,\mathbb{R}),\mathfrak{so}(n,1,\mathbb{R}))$ and the Gegenbauer polynomials.** Let us consider the case of compatible pair of Lie algebras

$$\begin{aligned}\mathfrak{g} &= \mathfrak{so}(n+1,1,\mathbb{R}),\ \mathfrak{p} = (\mathfrak{so}(n,\mathbb{R})\times\mathbb{R})\ltimes\mathbb{R}^n\,,\\ \mathfrak{g}' &= \mathfrak{so}(n,1,\mathbb{R}),\ \mathfrak{p}' = (\mathfrak{so}(n-1,\mathbb{R})\times\mathbb{R})\ltimes\mathbb{R}^{n-1}\,,\end{aligned}\tag{1}$$

such that the opposite nilradicals are given by $\mathfrak{n}'_-\simeq\mathbb{R}^{n-1}\subset\mathfrak{n}_-\simeq\mathbb{R}^n$, and the one-dimensional complement of $\mathfrak{n}'_-$ in $\mathfrak{n}_-$ is generated by the lowest root space of $\mathfrak{n}_-$ (i.e., the lowest root space of $\mathfrak{g}$.) In particular, the nilradicals $\mathfrak{n}_-,\mathfrak{n}'_-$ are commutative. The Iwasawa-Langlands decomposition of $\mathfrak{so}(n+1,1,\mathbb{R})$ is realized by the block decomposition

$$\begin{pmatrix} a & Y & 0\\ X & A & -Y\\ 0 & -X & -a\end{pmatrix}\,,\quad a\in\mathbb{R},\, A\in\mathfrak{so}(n,\mathbb{R}),\, Y\in\mathfrak{n}, X\in\mathfrak{n}_-.\tag{2}$$

Let us consider the family of scalar generalized Verma $\mathfrak{g}$-modules $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)$ induced from complex characters $\xi_\lambda:\mathfrak{p}\to\mathbb{C}$, $\lambda\in\mathbb{C}$. The branching problem for the pair $\mathfrak{g},\mathfrak{g}'$ applied to this class of modules was solved in [6], and we have

$$M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)|_{(\mathfrak{g}',\mathfrak{p}')}\simeq\bigoplus_{j=0}^{\infty}M^{\mathfrak{g}'}_{\mathfrak{p}'}(\lambda-j)\tag{3}$$

in the Grothendieck group $K(\mathcal{O}^{\mathfrak{p}'})$ of the BGG parabolic category $\mathcal{O}^{\mathfrak{p}'}$. In what follows we construct an $\mathfrak{sl}(2,\mathbb{C})$-module structure on $\mathfrak{g}'$-singular vectors (the generators of the $\mathfrak{g}'$-modules on the right hand side of (3)) in the family of $\mathfrak{g}$-modules $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)$.

Let $C^\alpha_l(x)$ be the l-th Gegenbauer polynomial in the variable x with spectral parameter $\alpha\in\mathbb{C}$, $l\in\mathbb{N}_0$; we also set $C^\alpha_{-1}(x)=0$. See the Appendix for basic properties of Gegenbauer polynomials. The recurrence relations for Gegenbauer polynomials imply

$$\begin{aligned}&\big((1-x^2)\partial_x+lx\big)C^\alpha_l(x)=(l+2\alpha-1)C^\alpha_{l-1}(x)\,,\\ &\big((1-x^2)\partial_x-(l+2\alpha)x\big)C^\alpha_l(x)=-(l+1)C^\alpha_{l+1}(x)\,.\end{aligned}\tag{4}$$

Lemma 2.1. *Let $\alpha \in \mathbb{C}$, $l \in \mathbb{N}_0$, and let* Deg *be the degree operator acting on a polynomial of degree l with the eigenvalue l. The linear operators $\{e(l), f(l), h(l)\}_{l\in\mathbb{N}_0}$,*

$$\begin{aligned} e(l) &= (1-x^2)\partial_x - (l+2\alpha)x : \langle C_l^\alpha(x)\rangle \to \langle C_{l+1}^\alpha(x)\rangle, \\ f(l) &= (1-x^2)\partial_x + lx : \langle C_l^\alpha(x)\rangle \to \langle C_{l-1}^\alpha(x)\rangle, \\ h(l) &= 2(\mathrm{Deg}+\alpha) : \langle C_l^\alpha(x)\rangle \to \langle C_l^\alpha(x)\rangle \end{aligned} \tag{5}$$

act on the vector space spanned by Gegenbauer polynomials $\{C_l^\alpha(x)\}_{l\in\mathbb{N}_0}$ and furnish it with the structure of a lowest weight $\mathfrak{sl}(2,\mathbb{C})$-module.

Proof. By direct computation, we have for $l \in \mathbb{N}_0$:

$$\begin{aligned} C_l^\alpha(x) &= (e(l-1)f(l) - f(l+1)e(l))C_l^\alpha(x) \\ &= 2(l+\alpha)C_l^\alpha(x) = h(l)C_l^\alpha(x), \\ \text{and} \qquad [h,e]C_l^\alpha(x) &= (h(l+1)e(l) - e(l)h(l))C_l^\alpha(x) \\ &= 2e(l)C_l^\alpha(x)\,. \end{aligned} \tag{6}$$

The same computation applies to the commutator $[h,f]$. Notice that the collection of all $e(l)$ defines an operator e, and the same for h and f. $\square$

Let us now briefly review the relation of the set of singular vectors generating the $\mathfrak{g}'$-submodules on the right hand side of (3) to Gegenbauer polynomials, [6]. Denoting by $\xi_1,\dots,\xi_{n-1},\xi_n$ the Fourier transforms of the root spaces in $\mathfrak{n}_-$ such that $\xi_1,\dots,\xi_{n-1}$ correspond to $\mathfrak{n}'_-$, $M_\mathfrak{p}^\mathfrak{g}(\lambda) \simeq \mathbb{C}[\xi_1,\dots,\xi_{n-1},\xi_n]$ as a vector space and the $\mathfrak{g}'$-singular vectors are given by homogeneous polynomials

$$\tilde{F}_l(\xi_1,\dots,\xi_{n-1},\xi_n) = \xi_n^l \tilde{C}_l^\alpha(-t^{-1})\,,$$

where $l \in \mathbb{N}_0$ denotes the homogeneity of the polynomial, $\alpha = -\lambda - \frac{n-1}{2}$, $t = \frac{1}{\xi_n^2}\sum_{j=1}^{n-1}\xi_j^2$ and $\tilde{C}_l^\alpha(-t^{-1})$ is defined as follows: due to the fact that $x^{-l}C_l^\alpha(x)$ is an even rational function, we define $x^{-l}C_l^\alpha(x) = \tilde{C}_l^\alpha(x^2) = \tilde{C}_l^\alpha(-t^{-1})$ with $x^2 = -t^{-1}$. The space of all singular vectors is exhausted by $l \in \mathbb{N}_0$.

Example 2.2. *In what follows we use the normalized singular vectors $F_l(\xi_1,\dots,\xi_{n-1},\xi_n)$, whose coefficient by the highest power of the quadratic invariant $\sum_{i=1}^{n-1}\xi_i^2$ is λ-independent:*

$$\begin{aligned} F_0(\xi_1,\dots,\xi_{n-1},\xi_n) &= 1\,, \\ F_1(\xi_1,\dots,\xi_{n-1},\xi_n) &= \xi_n\,, \\ F_2(\xi_1,\dots,\xi_{n-1},\xi_n) &= -(2\lambda+n-3)\xi_n^2 + \sum_{i=1}^{n-1}\xi_i^2\,, \\ F_3(\xi_1,\dots,\xi_{n-1},\xi_n) &= -(2\lambda+n-5)\xi_n^3 + 3\xi_n\sum_{i=1}^{n-1}\xi_i^2\,, \end{aligned} \tag{7}$$

and so

$$\begin{aligned}
&\tilde{C}_0^\alpha(-t^{-1}) = 1\,,\\
&\tilde{C}_1^\alpha(-t^{-1}) = 2\alpha\,,\\
&\tilde{C}_2^\alpha(-t^{-1}) = \alpha\big(t+2(1+\alpha)\big)\,,\\
&\tilde{C}_3^\alpha(-t^{-1}) = \frac{2}{3}\alpha(\alpha+1)\big(3t+2(2+\alpha)\big)\,,
\end{aligned}\tag{8}$$

where the Gegenbauer polynomials are

$$\begin{aligned}
&C_0^\alpha(x) = 1\,,\\
&C_1^\alpha(x) = 2\alpha x\,,\\
&C_2^\alpha(x) = -\alpha + 2\alpha(1+\alpha)x^2\,,\\
&C_3^\alpha(x) = -2\alpha(1+\alpha)x + \frac{4}{3}\alpha(1+\alpha)(2+\alpha)x^3\,.
\end{aligned}\tag{9}$$

The relation between $C_l^\alpha(x)$ and $\tilde{C}_l^\alpha(-t^{-1})$ implies that the operator identities in (4) transform in the variable t into

$$\begin{aligned}
&\big(-2(t+1)\partial_t + l\big)\tilde{C}_l^\alpha(-t^{-1}) = (l+2\alpha-1)\tilde{C}_{l-1}^\alpha(-t^{-1})\,,\\
&\big(2t(t+1)\partial_t - lt - 2(l+\alpha)\big)\tilde{C}_l^\alpha(-t^{-1}) = -(l+1)\tilde{C}_{l+1}^\alpha(-t^{-1})\,.
\end{aligned}\tag{10}$$

The proof of the following claim is an elementary consequence of the commutativity of the nilradical $\mathfrak{n}$.

Lemma 2.3. *Let* $\Box^\xi = \sum_{i=1}^n \partial_{\xi_i}^2$, $\mathbb{E}_\xi = \sum_{i=1}^n \xi_i\partial_{\xi_i}$, $\alpha = -\lambda - \frac{n-1}{2}$, $l\in\mathbb{N}_0$. *Then the root space in* $\mathfrak{n}/\mathfrak{n}'$ *acts on the generalized Verma module* $M_\mathfrak{p}^\mathfrak{g}(\lambda)$ *by the operator* $P(\lambda)\equiv P_n(\lambda) = i(\frac{1}{2}\xi_n\Box^\xi + (\lambda-\mathbb{E}_\xi)\partial_{\xi_n})$, *and descends to the map*

$$P(\lambda)\colon \langle F_l(\xi_1,\dots,\xi_{n-1},\xi_n)\rangle \to \langle F_{l-1}(\xi_1,\dots,\xi_{n-1},\xi_n)\rangle.$$

In the variable t, $P(\lambda)$ *acts by the operator*

$$\big(-2(t+1)\partial_t + l\big) : \langle\tilde{C}_l^\alpha(-t^{-1})\rangle \to \langle\tilde{C}_{l-1}^\alpha(-t^{-1})\rangle.\tag{11}$$

Let us notice that due to the fact that the annihilator ideal of $\mathfrak{g}'$-singular vectors in the algebraic Weyl algebra on $\mathbb{C}^n$ is rather large, there is a plenty of algebraic differential operators inducing linear action on the vector space of $\mathfrak{g}'$-singular vectors (e.g., the same as the operator $P(\lambda)$.) In general, we can not expect to get $\mathfrak{sl}(2,\mathbb{C})$-actions staying entirely inside $U(\mathfrak{g})$. We show an example demonstrating this phenomenon that comes as close to this as possible but still fails by considering the operator

$$\begin{aligned}
Q(\lambda) : = &\Big(\sum_{i=1}^{n-1}\xi_i^2\Big)\Big(\frac{1}{2}\xi_n\Box^\xi + (\lambda-\mathbb{E}_\xi)\partial_{\xi_n}\Big)\\
&-(\lambda-\mathbb{E}_\xi+2)(n+2\lambda-2\mathbb{E}_\xi+1)\xi_n\,,
\end{aligned}\tag{12}$$

which induces an action on singular vectors $F_l(\xi_1,\dots,\xi_n)$, $l\in\mathbb{N}_0$,

$$Q(\lambda)\colon \langle F_l(\xi_1,\dots,\xi_n)\rangle \to \langle F_{l+1}(\xi_1,\dots,\xi_n)\rangle\,.\tag{13}$$

In particular, we have

$$Q(\lambda)(1) = -(\lambda+1)(n+2\lambda-1)\xi_n\,,$$

$$Q(\lambda)(\xi_n) = \lambda\Big(\Big(\sum_{i=1}^{n-1}\xi_i^2\Big) - (n+2\lambda-3)\xi_n^2\Big)\,,$$

$$Q(\lambda)\Big(\Big(\sum_{i=1}^{n-1}\xi_i^2\Big) - (n+2\lambda-3)\xi_n^2\Big) = -(\lambda-1)(n+2\lambda-3) \times \Big(3\xi_n\Big(\sum_{i=1}^{n-1}\xi_i^2\Big) - (n+2\lambda-5)\xi_n^3\Big)\,, \tag{14}$$

$$Q(\lambda)\Big(3\xi_n\Big(\sum_{i=1}^{n-1}\xi_i^2\Big) - (n+2\lambda-5)\xi_n^3\Big) = (\lambda-2)\Big(3\Big(\sum_{i=1}^{n-1}\xi_i^2\Big)^2 - 6(n+2\lambda-5)\xi_n^2\Big(\sum_{i=1}^{n-1}\xi_i^2\Big) + (n+2\lambda-7)(n+2\lambda-5)\xi_n^4\Big)\,.$$

Notice that the operator $Q(\lambda)$ is an element of the universal enveloping algebra $U(\mathfrak{g})$. A disadvantage of $Q(\lambda)$ is that the pair of operators $P(\lambda), Q(\lambda)$ together with the homogeneity operator do not close in an $\mathfrak{sl}(2,\mathbb{C})$-algebra realized in $U(\mathfrak{g})$. A straightforward but tedious computation shows

$$\begin{aligned}\Big[\frac{1}{2}\xi_n\Box^\xi + (\lambda-\mathbb{E}_\xi)\partial_{\xi_n}, Q(\lambda)\Big] &= -\frac{1}{2}(4\lambda+10+2\mathbb{E}_\xi)\xi_n^2\Box^\xi \\ &+ [(2\mathbb{E}_\xi+n-3)(\lambda+1-\mathbb{E}_\xi) + (\lambda-\mathbb{E}_\xi+2)(n+2\lambda+1-2\mathbb{E}_\xi) \\ &- (\lambda-\mathbb{E}_\xi)(n+4\lambda+3-4\mathbb{E}_\xi) - (n+4\lambda+7+4\mathbb{E}_\xi)]\xi_n\partial_{\xi_n} \\ &- \Big(\Big(\sum_{i=1}^{n-1}\xi_i^2\Big)+\xi_n^2\Big)(\xi_n\partial_{\xi_n}+1)\Box^\xi - 2(\lambda+1-\mathbb{E}_\xi)\Big(\Big(\sum_{i=1}^{n-1}\xi_i^2\Big)+\xi_n^2\Big)\partial_{\xi_n}^2 \\ &- (\lambda-\mathbb{E}_\xi)\big[(\lambda-\mathbb{E}_\xi+2)(n+2\lambda+1-2\mathbb{E}_\xi) - (n+4\lambda+3-4\mathbb{E}_\xi)\big]\,,\end{aligned} \tag{15}$$

where the operator on the right hand side of the last equality does not act in the algebraic Weyl algebra on $\mathbb{C}^n$ as a multiple of identity on both $P(\lambda)$ and $Q(\lambda)$.

In what follows we construct two operators in the variables $\xi_1,\dots,\xi_n$, which are not the elements of $U(\mathfrak{g})$, but they fulfill $\mathfrak{sl}(2,\mathbb{C})$-commutation relations when their action is restricted to the set of $\mathfrak{g}'$-singular vectors in $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$.

Theorem 2.4. *Let $l \in \mathbb{N}_0$. Then the collection of operators $\{e_\xi(l), f_\xi(l), h_\xi(l)\}_{l\in\mathbb{N}_0}$ in the variables $\xi_1,\dots,\xi_n$,*

$$\begin{aligned} e_\xi(l) &:= -\Big(\sum_{i=1}^{n}\xi_i^2\Big)\partial_{\xi_n} - (l+2\alpha)\xi_n \colon \langle F_l(\xi_1,\dots,\xi_n)\rangle \to \langle F_{l+1}(\xi_1,\dots,\xi_n)\rangle\,, \\ f_\xi(l) &:= \frac{1}{\xi_n}\Big(\frac{(\sum_{i=1}^{n}\xi_i^2)}{(\sum_{i=1}^{n-1}\xi_i^2)}(\xi_n\partial_{\xi_n}-l)+l\Big) \colon \langle F_l(\xi_1,\dots,\xi_n)\rangle \\ &\qquad \to \langle F_{l-1}(\xi_1,\dots,\xi_n)\rangle\,, \\ h_\xi(l) &:= 2(l+\alpha)\,\mathrm{Id} \colon \langle F_l(\xi_1,\dots,\xi_n)\rangle \to \langle F_l(\xi_1,\dots,\xi_n)\rangle\,,\end{aligned} \tag{16}$$

fulfill the $\mathfrak{sl}(2,\mathbb{C})$*-commutation relations, and* $F_l(\xi_1,\dots,\xi_n), l\in\mathbb{N}_0$*, are the one-dimensional weight spaces of a highest weight* $\mathfrak{sl}(2,\mathbb{C})$*-Verma module.*

Proof. Let $f=f(\xi_1,\dots,\xi_n)$, $l\in\mathbb{N}_0$, and $t=\frac{1}{\xi_n^2}\sum\limits_{i=1}^{n-1}\xi_i^2$. We have

$$\partial_{\xi_n}(\xi_n^{-l}f)=-l\xi_n^{-l-1}f+\xi_n^{-l}\partial_{\xi_n}f\,,$$

and

$$\partial_t=-\frac{\xi_n^3}{2(\sum\limits_{i=1}^{n}\xi_i^2)}\partial_{\xi_n}=\frac{\xi_n^2}{2\xi_i}\partial_{\xi_i}$$

for all $i=1,\dots,n-1$. By direct substitution for t, the first operator equals to

$$\xi_n^{l+1}\left(2\frac{(\sum\limits_{i=1}^{n-1}\xi_i^2)}{\xi_n^2}\left(\frac{(\sum\limits_{i=1}^{n-1}\xi_i^2)}{\xi_n^2}+1\right)\left(\frac{-\xi_n^3}{2(\sum\limits_{i=1}^{n-1}\xi_i^2)}\partial_{\xi_n}\right)-l\frac{(\sum\limits_{i=1}^{n-1}\xi_i^2)}{\xi_n^2}-2(\alpha+l)\right)\xi_n^{-l}\,, \tag{17}$$

and standard manipulations give the required result $e_\xi(l)$. Analogously, the second operator is equal to

$$\xi_n^{l-1}\left(-2\left(\frac{(\sum\limits_{i=1}^{n-1}\xi_i^2)}{\xi_n^2}+1\right)\left(\frac{-\xi_n^3}{2(\sum\limits_{i=1}^{n-1}\xi_i^2)}\partial_{\xi_n}\right)+l\right)\xi_n^{-l}\,, \tag{18}$$

and this gives $f_\xi(l)$.

As for the $\mathfrak{sl}(2,\mathbb{C})$-commutation relations, it is again straightforward to check that

$$\begin{aligned}
&f_\xi(l+1)e_\xi(l)-e_\xi(l-1)f_\xi(l)=\\
&\left[\frac{1}{\xi_n}\left(\frac{(\sum\limits_{i=1}^{n}\xi_i^2)}{(\sum\limits_{i=1}^{n-1}\xi_i^2)}\big(\xi_n\partial_{\xi_n}-(l+1)\big)+(l+1)\right)\right]\\
&\quad\circ\Big[-\Big(\sum_{i=1}^{n}\xi_i^2\Big)\partial_{\xi_n}-(l+2\alpha)\xi_n\Big]\\
&-\Big[-\Big(\sum_{i=1}^{n}\xi_i^2\Big)\partial_{\xi_n}-(l-1+2\alpha)\xi_n\Big]\circ\left[\frac{1}{\xi_n}\left(\frac{(\sum\limits_{i=1}^{n}\xi_i^2)}{(\sum\limits_{i=1}^{n-1}\xi_i^2)}(\xi_n\partial_{\xi_n}-l)+l\right)\right]
\end{aligned} \tag{19}$$

is equal to $-h_\xi(l)$. The two remaining relations involving $h_\xi(l)$ are the consequence of the explicit formula for $h_\xi(l)$ and the homogeneity of $e_\xi(l)$, $f_\xi(l)$.

Finally, the structure of $\mathfrak{sl}(2,\mathbb{C})$-Verma module is a consequence of $\mathfrak{sl}(2,\mathbb{C})$-commutation relations and the action of $e_\xi(l), f_\xi(l)$ and $h_\xi(l)$ on the weight vectors $F_l(\xi_1,\dots,\xi_n)$. $\square$

Remark 2.5. *The normalization of singular vectors $F_l(\xi_1,\dots,\xi_n)$, $l\in\mathbb{N}_0$, is chosen in such a way that the action of particular basis elements $e_\xi(l), f_\xi(l)$ produces the constants:*

$$\begin{array}{c} F_{2l} \\ {\scriptstyle -(2\alpha+2l)}\big\downarrow\big\uparrow{\scriptstyle 2l+1} \\ F_{2l+1} \\ {\scriptstyle -1}\big\downarrow\big\uparrow{\scriptstyle (2l+2)(2\alpha+2l+1)} \\ F_{2l+2} \end{array}$$

and their commutator gives the weight $2\alpha+4l-2$.

The operator $\frac{1}{2}\xi_n\Box^\xi+(\lambda-\mathbb{E}_\xi)\partial_{\xi_n}$ does not have $\mathfrak{sl}(2,\mathbb{C})$-commutation relation with $e_\xi(l):=-(\sum_{i=1}^n\xi_i^2)\partial_{\xi_n}-(l+2\alpha)\xi_n$. In fact, in the algebraic Weyl algebra we have

$$\begin{aligned}\Big[-\Big(\sum_{i=1}^n\xi_i^2\Big)\partial_{\xi_n}-(\mathbb{E}_\xi-1+2\alpha)\xi_n,\frac{1}{2}\xi_n\Box^\xi+(\lambda-\mathbb{E}_\xi)\partial_{\xi_n}\Big]&\\ =-\frac{1}{2}\Big(\sum_{i=1}^{n-1}\xi_i^2\Big)\Box^\xi-\Big(\sum_{i=1}^{n-1}\xi_i^2\Big)\partial_{\xi_n}\partial_{\xi_n}+\frac{1}{2}\xi_n^2\Box^\xi&\\ +(n+\lambda+\mathbb{E}_\xi)\xi_n\partial_{\xi_n}+(\mathbb{E}_\xi+2\alpha)(\lambda-\mathbb{E}_\xi)\,.&\end{aligned}\tag{20}$$

Notice that the operator $f_\xi(l)$ introduced in (16) does not belong to the universal enveloping algebra $U(\mathfrak{g})$, but rather to its localization with respect to the subalgebra of invariants in $U(\mathfrak{n}_-)$ with respect to the simple part of the Levi factor $\mathfrak{l}'$.

Let us finally examine the action of the $\mathfrak{sl}(2,\mathbb{C})$-Casimir operator on $\mathfrak{g}'$-singular vectors. Recall that for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ generated by elements e,f,h with commutation relations $[e,f]=h$, $[h,e]=2e$ and $[h,f]=-2f$, the Casimir operator is $\mathrm{Cas}=ef+fe+\frac{1}{2}h^2$.

Theorem 2.6. (1) *The Casimir operator of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ realized by $e_\xi(l), f_\xi(l), h_\xi(l)$, $l\in\mathbb{N}_0$, is*

$$\begin{aligned}\mathrm{Cas}:&=f_\xi(l+1)e_\xi(l)+e_\xi(l-1)f_\xi(l)+\frac{1}{2}h_\xi^2(l)\\ &=-2\frac{(\sum\limits_{i=1}^n\xi_i^2)^2}{(\sum\limits_{i=1}^{n-1}\xi_i^2)}\partial_{\xi_n}^2-2(2\alpha+1)\frac{(\sum\limits_{i=1}^n\xi_i^2)}{(\sum\limits_{i=1}^{n-1}\xi_i^2)}\xi_n\partial_{\xi_n}\\ &\quad-2\frac{\alpha(\sum\limits_{i=1}^{n-1}\xi_i^2)-l(l+2\alpha)\xi_n^2}{(\sum\limits_{i=1}^{n-1}\xi_i^2)}+2(\mathbb{E}_\xi+\alpha)^2\end{aligned}\tag{21}$$

when acting on $F_l(\xi_1,\dots,\xi_n)$.

(2) *The Casimir operator* Cas *acts by* $2\alpha(\alpha-1)$*-multiple of identity on the* $\mathfrak{sl}(2,\mathbb{C})$*-Verma module with weight spaces* $\{\langle F_l(\xi_1,\dots,\xi_n)\rangle\}_{l\in\mathbb{N}_0}$.

Proof. We have

$$\begin{aligned}
\mathrm{Cas} &= f_\xi(l+1)e_\xi(l)+e_\xi(l-1)f_\xi(l)+\frac{1}{2}h_\xi^2(l)\\
&= \Big[\frac{1}{\xi_n}\Big(\frac{(\sum_{i=1}^{n}\xi_i^2)}{(\sum_{i=1}^{n-1}\xi_i^2)}\big(\xi_n\partial_{\xi_n}-(l+1)\big)+(l+1)\Big)\Big]\circ\Big[-\Big(\sum_{i=1}^{n}\xi_i^2\Big)\partial_{\xi_n}-(l+2\alpha)\xi_n\Big]\\
&\quad+\Big[-\Big(\sum_{i=1}^{n}\xi_i^2\Big)\partial_{\xi_n}-(l-1+2\alpha)\xi_n\Big]\circ\Big[\frac{1}{\xi_n}\Big(\frac{(\sum_{i=1}^{n}\xi_i^2)}{(\sum_{i=1}^{n-1}\xi_i^2)}(\xi_n\partial_{\xi_n}-l)+l\Big)\Big]\\
&\quad+\frac{1}{2}\big(-2(\mathbb{E}_\xi+\alpha)\big)^2 .
\end{aligned}\tag{22}$$

Expanding the compositions and recollecting all terms according to the power of ∂_{ξ_n}, we arrive at (21). As for the proof of the second claim, it follows from

$$\mathrm{Cas}(1)=(-2\alpha+2\alpha^2)1=2\alpha(\alpha-1)1\,.$$

The proof is complete. □

2.2. **The pair of Lie algebras $(\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{sl}(2,\mathbb{R}),\mathrm{diag}(\mathfrak{sl}(2,\mathbb{R})))$ and the Jacobi polynomials.** Let us consider the pair of compatible Lie algebras and their Borel subalgebras,

$$\begin{aligned}
&\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{sl}(2,\mathbb{R}), &&\mathfrak{g}\supset\mathfrak{b}=(\mathbb{R}\times\mathbb{R})\ltimes(\mathbb{R}\times\mathbb{R}),\\
&\mathfrak{g}'=\mathrm{diag}\big(\mathfrak{sl}(2,\mathbb{R})\big), &&\mathfrak{g}'\supset\mathfrak{b}'=\mathrm{diag}\big((\mathbb{R}\times\mathbb{R})\ltimes(\mathbb{R}\times\mathbb{R})\big),
\end{aligned}$$

such that $\mathfrak{n}'_-\subset\mathfrak{n}_-$ ($\dim_{\mathbb{R}}(\mathfrak{n}_-)=2,\dim_{\mathbb{R}}(\mathfrak{n}'_-)=1$) and its one dimensional complement is denoted by F. In particular, the nilradicals $\mathfrak{n}_-,\mathfrak{n}'_-$ are commutative and X denotes the generator of $\mathfrak{n}'_-$.

Let us consider the family of scalar Verma $\mathfrak{g}$-modules $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda,\mu)$, induced from complex characters $\xi_{\lambda,\mu}\colon\mathfrak{b}\to\mathbb{C}$ with $\lambda,\mu\in\mathbb{C}$. The branching of scalar Verma modules for the pair $(\mathfrak{g},\mathfrak{b})$ and $(\mathfrak{g}',\mathfrak{b}')$ is given by

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda,\mu)|_{(\mathfrak{g}',\mathfrak{p}')}\simeq\bigoplus_{j=0}^{\infty}M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda+\mu-2j)\tag{23}$$

in the Grothendieck group $K(\mathcal{O}(\mathfrak{g}))$ of the BGG category $\mathcal{O}(\mathfrak{g})$.

In [5], the generators of $\mathfrak{g}'$-submodules generating the summands on the right hand side of (23) are determined and the non-trivial composition structure related to factorization properties of Jacobi polynomials is discussed in the non-generic case $\lambda+\mu\in\mathbb{N}_0$. Here we restrict to the case $\lambda,\mu\notin\mathbb{N}_0$. For $\nu\in\mathbb{N}_0$, we define (see [4, Chapter 3]) a $\mathfrak{g}'$-module $P_{\mathfrak{b}'}^{\mathfrak{g}'}(\nu)$ as the non-split extension

$$0\to M_{\mathfrak{b}'}^{\mathfrak{g}'}(\nu)\to P_{\mathfrak{b}'}^{\mathfrak{g}'}(\nu)\to M_{\mathfrak{b}'}^{\mathfrak{g}'}(-\nu-2)\to 0.\tag{24}$$

We introduce an involution

$$\iota\colon \mathbb{C} \to \mathbb{C}, \quad \nu \mapsto -\nu - 2\,.$$

and for $N \in \mathbb{N}_0$ set

$$\begin{aligned} \Lambda \ &\equiv \Lambda(N) \ := \{N - 2l : l \in \mathbb{N}_0\}\,, \\ \Lambda_s &\equiv \Lambda_s(N) := \{N - 2l : l \in \mathbb{N}_0,\ 2l \le \lambda + \mu\}, \\ \Lambda_r &\equiv \Lambda_r(N) := \Lambda \setminus (\Lambda_s \cup \iota(\Lambda_s)) \\ &\qquad\qquad = \begin{cases} \{-1\} \cup \{-N, -N-2, -N-4, \dots\} & (N\text{: even})\,, \\ \{-N, -N-2, -N-4, \dots\} & (N\text{: odd})\,. \end{cases} \end{aligned}$$

Theorem 2.7. [5] *Suppose that $\lambda + \mu \in \mathbb{N}_0$ with $\lambda, \mu \notin \mathbb{N}_0$, i.e., the scalar Verma modules $M^{\mathfrak{sl}(2,\mathbb{R})}_{\mathfrak{b}'}(\lambda)$ resp. $M^{\mathfrak{sl}(2,\mathbb{R})}_{\mathfrak{b}'}(\mu)$ are irreducible $\mathfrak{g}'$-modules. Then the tensor product of two scalar Verma modules $M^{\mathfrak{sl}(2)}_{\mathfrak{b}'}(\lambda) \otimes M^{\mathfrak{sl}(2)}_{\mathfrak{b}'}(\mu)$ decomposes as* $\mathrm{diag}(\mathfrak{sl}(2,\mathbb{R})) \simeq \mathfrak{sl}(2,\mathbb{R})$*-module*

$$\bigoplus_{\nu \in \Lambda_r(\lambda+\mu)} M^{\mathfrak{sl}(2)}_{\mathfrak{b}'}(\nu) \ \oplus \bigoplus_{\nu \in \Lambda_s(\lambda+\mu)} P^{\mathfrak{sl}(2)}_{\mathfrak{b}'}(\nu)\,. \tag{25}$$

Here $P^{\mathfrak{sl}(2)}_{\mathfrak{b}'}(\nu)$ are the projective objects defined in (24)*.*

It is straightforward to see that the generator X of $\mathfrak{n}'_+$ acts in the non-compact model of the representation $\mathcal{C}^\infty(\mathfrak{n}_-, \mathbb{C}_{\lambda,\mu})$, induced from the character (λ, μ) of $\mathfrak{b}$ on the 1-dimensional vector space $\mathbb{C}_{\lambda,\mu} \simeq \mathbb{C}$, by the first order differential operator

$$d\pi(X) = \lambda x + x^2\partial_x + \mu y + y^2\partial_y\,. \tag{26}$$

The action on the scalar Verma module, induced from the dual representation to $\mathbb{C}_{\lambda,\mu}$ and realized in the Fourier dual picture, results into the second order differential operator

$$d\tilde{\pi}(X) = i(-\lambda\partial_\xi + \xi\partial_\xi^2 - \mu\partial_\eta + \eta\partial_\eta^2) \tag{27}$$

acting on polynomial algebra $\mathbb{C}[\xi, \eta]$. As a $\mathfrak{g}$-module, $\mathbb{C}[\xi, \eta]$ can be identified with the Verma module induced from the dual representation to $\mathbb{C}_{\lambda,\mu}$. Analogously, the generator F of $\mathfrak{n}_+/\mathfrak{n}'_+$ acts on $\mathcal{C}^\infty(\mathfrak{n}_-, \mathbb{C}_{\lambda,\mu})$ and $\mathbb{C}[\xi, \eta]$ by differential operators

$$d\pi(F) = \lambda x + x^2\partial_x - \mu y - y^2\partial_y \tag{28}$$

and

$$d\tilde{\pi}(F) = i(-\lambda\partial_\xi + \xi\partial_\xi^2 + \mu\partial_\eta - \eta\partial_\eta^2)\,, \tag{29}$$

respectively. The Levi factor $\mathfrak{l} \subset \mathfrak{b}$ contains the Euler homogeneity operator and the operator $d\tilde{\pi}(X)$ preserves the space of homogeneous polynomials. We define $t = \frac{\xi}{\eta}$ and write a homogeneous polynomial as $\eta^l Q(t)$ for some polynomial $Q = Q(t)$ of degree l. We easily compute

$$
\begin{aligned}
&\partial_\xi\big(\eta^l Q(t)\big) = \eta^{l-1}Q'(t)\,,\\
&\partial_\eta\big(\eta^l Q(t)\big) = \eta^{l-1}lQ - \xi\eta^{l-2}Q'(t)\,,\\
&\partial_\xi^2\big(\eta^l Q(t)\big) = \eta^{l-2}Q''(t)\,,\\
&\partial_\eta^2\big(\eta^l Q(t)\big) = \eta^{l-2}l(l-1)Q(t) - 2(l-1)\xi\eta^{l-3}Q'(t) + \xi^2\eta^{l-4}Q''(t)\,.
\end{aligned}
\tag{30}
$$

The substitution into (27) resp. (29) yields the differential equation consisting of polynomials $\eta^l Q(\frac{\xi}{\eta})$, where $Q(t)$ is a polynomial solution to

$$
[t(t+1)\partial_t^2 + (t(\mu-2(l-1))-\lambda)\partial_t + l(l-1-\mu)]Q(t) = 0\,,
\tag{31}
$$

resp. the differential equation representing the action of $F\in\mathfrak{n}_+/\mathfrak{n}'_+$:

$$
-\,t(t-1)\partial_t^2 + (t(2l-\mu-2)-\lambda)\partial_t + l(\mu-l+1)\,.
\tag{32}
$$

The ordinary second order hypergeometric differential equation (31) is the Jacobi differential equation, and its polynomial solutions are the Jacobi polynomials:

Theorem 2.8 ([5])**.** *Let $P_l^{\alpha,\beta}(x)$ denote the degree l polynomial solution of the Jacobi hypergeometric equation* (44)*, see* (42)*. Let us define homogeneous polynomials $\tilde P_l^{-\lambda-1,\mu+\lambda-2l+1}(\xi,\eta)$ by*

$$
\tilde P_l^{-\lambda-1,\mu+\lambda-2l+1}(\xi,\eta) := \eta^l P_l^{-\lambda-1,\mu+\lambda-2l+1}\Big(\frac{2\xi}{\eta}+1\Big)
\tag{33}
$$

for all $l\in\mathbb{N}_0$. Then for any $\lambda,\mu\in\mathbb{C}$,

$$
\bigoplus_{l=0}^{\infty}\big\langle \tilde P_l^{-\lambda-1,\mu+\lambda-2l+1}\big\rangle
\tag{34}
$$

is the complete set of polynomial solutions of (27) *representing the singular vectors in the Fourier dual picture.*

By abuse of notation, we denote by $d\tilde\pi(X)$, $d\tilde\pi(F)$ the two mutually commuting operators acting on degree l-polynomials in the non-homogeneous variable t:

$$
\begin{aligned}
d\tilde\pi(X) &:= t(t+1)\partial_t^2 + (t(\mu-2(l-1))-\lambda)\partial_t + l(l-1-\mu)\,,\\
d\tilde\pi(F) &:= -t(t-1)\partial_t^2 + (t(2l-\mu-2)-\lambda)\partial_t + l(\mu-l+1)\,.
\end{aligned}
\tag{35}
$$

Theorem 2.9. *Let $P_l^{\alpha,\beta}(x)$ be the Jacobi polynomial of degree $l\in\mathbb{N}_0$ and*

$$
d\tilde\pi(X)(P_l^{-\lambda-1,\mu+\lambda-2l+1}(2t+1)) = 0
$$

for $x = 2t+1$. Then

$$
\begin{aligned}
d\tilde\pi(F)\big(P_l^{-\lambda-1,\mu+\lambda-2l+1}(2t+1)\big)&\\
&= 2(l-1-\lambda)(\mu-l+1)P_{l-1}^{-\lambda-1,\mu+\lambda-2l+3}(2t+1)\,,
\end{aligned}
\tag{36}
$$

i.e., $d\tilde\pi(F)$ maps the homogeneity l polynomial solution of the Jacobi differential equation $d\tilde\pi(X)$ to (*a multiple depending on α,β of*) *the homogeneity* $(l-1)$ *polynomial solution of the Jacobi differential equation.*

Proof. We first observe that $\sum_{i=0}^{l} a_i^l t^i$ is the degree l Jacobi polynomial in the variable t provided the recursion relations

$$\big[i(i-2l+\mu+1)+l(l-\mu-1)\big]a_i^l+(i-\lambda)(i+1)a_{i+1}^l=0 \tag{37}$$

are satisfied for all $i=0,\dots,l$. The operator $d\tilde{\pi}(F)$ maps the degree l Jacobi polynomial to the space of polynomials of degree $(l-1)$. The reason is that the coefficient of the monomial t^l in $d\tilde{\pi}(F)(P_l^{-\lambda-1,\mu+\lambda-2l+1}(2t+1))$ is equal to

$$-l(l-1)+l(2l-\mu-2)+l(\mu-l+1)=0\,.$$

In particular, we have

$$d\tilde{\pi}(F)\big(P_l^{-\lambda-1,\mu+\lambda-2l+1}(2t+1)\big)=\sum_{i=0}^{l}\big[2i(-i+2l-\mu-1)+2l(\mu-l+1)\big]a_i^l t^i$$

for $i=0,\dots,l-1$, and it remains to prove that this polynomial is, up to a multiple, the degree $(l-1)$ Jacobi polynomial. Assuming the recursion relation (37) holds, we prove that $[2i(-i+2l-\mu-1)+2l(\mu-l+1)]a_i^l$ are the coefficients of the degree $l-1$ Jacobi polynomial (see again (37)):

$$\begin{aligned}&\big[i(i-2l+\mu+3)+(l-1)(l-\mu-2)\big]\big[2i(-i+2l-\mu-1)+2l(\mu-l+1)\big]a_i^l\\ &\qquad=-(i-\lambda)(i+1)\big[2(i+1)(-i+2l-\mu-2)+2(\mu-l+1)\big]a_{i+1}^l\,.\end{aligned}\tag{38}$$

However, the last equality is equivalent to

$$\big[i(i-2l+\mu+3)+(l-1)(l-\mu-2)\big]=-\big[(i+1)(-i+2l-\mu-2)+l(\mu-l+1)\big]\,,$$

which is easy to verify and the claim follows.

It remains to compute the explicit polynomial in λ, μ as a coefficient of the proportionality. By definition,

$$\begin{aligned}a_l^l&=\frac{1}{l!}\mu(\mu-1)\dots(\mu-l+2)(\mu-l+1)\,,\\ a_{l-1}^{l-1}&=\frac{1}{(l-1)!}\mu(\mu-1)\dots(\mu-l+2)\,,\end{aligned}$$

and so we get for $i=l-1$

$$d\tilde{\pi}(F)\Big(\sum_{i=0}^{l}a_i^l t^i\Big)=2\mu a_l^{l-1}t^{l-1}+\cdots\,.$$

Because $-\mu a_l^{l-1}=-l(l-1-\lambda)a_l^l$, a direct comparison yields the required form $2(l-1-\lambda)(\mu-l+1)$ of the coefficient of proportionality. The proof is complete. $\square$

Example 2.10. *Let us present a first few low degree polynomials:*

$$P_0^{-\lambda-1,\mu+\lambda+1}(2t+1) = 1\,,$$
$$P_1^{-\lambda-1,\mu+\lambda-1}(2t+1) = \mu t - \lambda\,,$$
$$P_2^{-\lambda-1,\mu+\lambda-3}(2t+1) = -\frac{1}{2}\big[\mu(1-\mu)t^2 + 2(1-\mu)(1-\lambda)t + \lambda(1-\lambda)\big]\,,$$
$$P_3^{-\lambda-1,\mu+\lambda-5}(2t+1) = \frac{1}{6}\big[\mu(1-\mu)(2-\mu)t^3 + 3(2-\mu)(\mu-1)(2-\lambda)t^2 + 3(2-\mu)(1-\lambda)(2-\lambda)t + \lambda(2-\lambda)(1-\lambda)\big]\,.$$

It is straightforward to check

$$d\tilde{\pi}(F)\big(P_1^{-\lambda-1,\mu+\lambda-1}(2t+1)\big) = -2\lambda\mu P_0^{-\lambda-1,\mu+\lambda+1}(2t+1)\,,$$
$$d\tilde{\pi}(F)\big(P_2^{-\lambda-1,\mu+\lambda-3}(2t+1)\big) = -2(\lambda-1)(\mu-1)P_1^{-\lambda-1,\mu+\lambda-1}(2t+1)\,,$$
$$d\tilde{\pi}(F)\big(P_3^{-\lambda-1,\mu+\lambda-5}(2t+1)\big) = -2(\lambda-2)(\mu-2)P_2^{-\lambda-1,\mu+\lambda-3}(2t+1)\,.$$

We remark that because of the assumption of irreducibility, λ, $\mu \notin \mathbb{N}_0$, the coefficients of proportionality $(2(l-1-\lambda)(\mu-l+1))$ are non-zero. In this example we do not attempt to construct the complete $\mathfrak{s}l(2,\mathbb{C})$-structure on the space of $\mathfrak{g}'$-singular vectors.

3. Relative Lie and Dirac cohomology and $\mathfrak{g}'$-singular vectors

The Lie algebra (co)homology or the Dirac cohomology associated to a Lie algebra and its modules are among important algebraic invariants with applications in representation theory, see [8], [1], [2].

In fact, the two examples in Section 2, Section 3 are motivated by the following general problem. Let us consider the short exact sequence of pairs of Lie algebras and their parabolic subalgebras:

$$0 \to (\mathfrak{g}',\mathfrak{p}') \to (\mathfrak{g},\mathfrak{p}) \to (\mathfrak{g},\mathfrak{p})/(\mathfrak{g}',\mathfrak{p}') \to 0 \tag{39}$$

and a $\mathfrak{g}$-module V. In our applications, $\mathfrak{g}$ and $\mathfrak{g}'$ ($\mathfrak{g}' \subset \mathfrak{g}$) are simple Lie algebras, $\mathfrak{p} \subset \mathfrak{g}$ and $\mathfrak{p}' \subset \mathfrak{g}'$ their parabolic subalgebras ($\mathfrak{p}$ is $\mathfrak{g}'$-compatible), and the vector complements of $\mathfrak{p}$ and $\mathfrak{p}'$ in $\mathfrak{g}$ and $\mathfrak{g}'$ are the Lie algebras of the opposite nilradicals $\mathfrak{n}_-$ and $\mathfrak{n}'_-$.

Then the key question is an intrinsic definition of the relative Lie algebra (co)differential or relative Dirac operator associated to compatible couples of Lie algebras given by simple Lie algebra and its parabolic subalgebra, and their role in the compatibility of the branching problem applied to $\mathfrak{g}' \subset \mathfrak{g}$ and the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}(\mathfrak{g})$.

Namely, for $\mathfrak{n}_- = \mathfrak{n}'_- \oplus (\mathfrak{n}_-/\mathfrak{n}'_-)$ with $\mathfrak{n}'_-$ the ideal in $\mathfrak{n}$, we would like to define the relative Dirac operator such that the underlying relative Dirac cohomology functor $H_{D,\mathrm{rel}}(\mathfrak{n}_-,\mathfrak{n}'_-;-)$ abuts in a spectral sequence to the $(\mathfrak{g},\mathfrak{p})$-Dirac cohomology of $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$:

$$H_D\big(\mathfrak{g},\mathfrak{p};M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)\big) \Longrightarrow H_{D,\mathrm{rel}}\big(\mathfrak{n}_-,\mathfrak{n}'_-,H_D(\mathfrak{g}',\mathfrak{p}';M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}'})\big)\,. \tag{40}$$

Here $H_D(\mathfrak{g},\mathfrak{p};M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda))$ was determined in [3] for irreducible generalized Verma modules, while in the presence of non-trivial composition series the relevant higher Dirac cohomology was constructed in [10].

We do not have an answer to the previous question, and the examples in Section 2.1, Section 2.2 clearly demonstrate the difficulties. To be more explicit, we shall stick to the case discussed in Section 2.1 and assume that $\lambda\in\mathbb{C}$ is generic so that $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)$ as well as $M^{\mathfrak{g}'}_{\mathfrak{p}'}(\lambda-j)$ are irreducible highest weight modules for all $j\in\mathbb{N}_0$. Then the $(\mathfrak{g},\mathfrak{p})$-Dirac cohomology of the left hand side (3) equals to

$$H_D\big(\mathfrak{g},\mathfrak{p},M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)\big)\simeq\mathbb{C}_\lambda\otimes\mathbb{C}_{\rho(\mathfrak{n}_-)}\,,$$

while the $(\mathfrak{g}',\mathfrak{p}')$-Dirac cohomology of the right hand side of (3) is

$$H_D\Big(\mathfrak{g}',\mathfrak{p}',\bigoplus_{j=0}^{\infty}M^{\mathfrak{g}'}_{\mathfrak{p}'}(\lambda-j)\Big)\simeq\bigoplus_{j=0}^{\infty}\mathbb{C}_{\lambda-j}\otimes\mathbb{C}_{\rho(\mathfrak{n}'_-)}\,.$$

Here we used the notation $\rho(\mathfrak{n}_-)$ and $\rho(\mathfrak{n}'_-)$ for the half sum of roots of root spaces in $\mathfrak{n}_-$ and $\mathfrak{n}'_-$, respectively, and $\mathbb{C}_\lambda$ denotes the one dimensional inducing representation of ξ_λ. Then the relative Dirac operator D_{rel}

$$D_{\mathrm{rel}}(\mathfrak{n}_-,\mathfrak{n}'_-):=e_\xi\otimes f_\xi+f_\xi\otimes e_\xi\,, \tag{41}$$

based on Lemma 2.4 and the highest weight $\mathfrak{sl}(2,\mathbb{C})$-module with the action of $\{e_\xi(l),f_\xi(l),h_\xi(l)\}_{l\in\mathbb{N}_0}$, computes the expected result.

However, this approach clearly fails for non-generic values of λ because the $\mathfrak{s}l(2,\mathbb{C})$-module is no longer irreducible. Moreover, the construction (41) does not intrinsically proceed in $U(\mathfrak{g})$.

4. Appendix: Jacobi and Gegenbauer polynomials

In the present section we summarize for the reader's convenience a few basic conventions and properties related to the Jacobi and Gegenbauer polynomials.

We use the notation $\Gamma(z)$ for the Gamma function, $z\in\mathbb{C}$, and the analytical continuation of the binomial coefficient is given by

$$\binom{z}{l}:=\frac{\Gamma(z+1)}{\Gamma(l+1)\Gamma(z-l+1)}\,,\qquad\binom{z}{l}=0\quad\text{if}\quad l-z\in\mathbb{N}\quad\text{and}\quad z\notin-\mathbb{N}\,.$$

The Jacobi polynomials $P_l^{(\alpha,\beta)}(z)$ of degree $l\in\mathbb{N}_0$ with two spectral parameters $\alpha,\beta\in\mathbb{C}$ are defined as special values of the hypergeometric function

$$\begin{aligned}P_l^{(\alpha,\beta)}(z)&=\binom{l+\alpha}{l}\,{}_2F_1\left(-l,1+\alpha+\beta+l;\alpha+1;\frac{1-z}{2}\right)\\&=\frac{\Gamma(\alpha+l+1)}{l!\Gamma(\alpha+\beta+l+1)}\sum_{m=0}^{n}\binom{l}{m}\frac{\Gamma(\alpha+\beta+l+m+1)}{\Gamma(\alpha+m+1)}\left(\frac{z-1}{2}\right)^m\\&=\sum_{j=0}^{l}\binom{l+\alpha}{j}\binom{l+\beta}{l-j}\left(\frac{z-1}{2}\right)^{l-j}\left(\frac{z+1}{2}\right)^j,\end{aligned} \tag{42}$$

normalized by

$$P_l^{(\alpha,\beta)}(1) = \binom{l+\alpha}{l} = \frac{(\alpha+1)_l}{l!}\,.$$

Here $(\alpha+1)_l = (\alpha+1)(\alpha+2)\cdots(\alpha+l)$ denotes the Pochhammer symbol for the partially rising factorial. The Jacobi polynomials satisfy the orthogonality relations

$$\begin{aligned}&\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_k^{(\alpha,\beta)}(x)P_l^{(\alpha,\beta)}(x)\,dx\\ &\qquad = \frac{2^{\alpha+\beta+1}}{2l+\alpha+\beta+1}\frac{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{\Gamma(l+\alpha+\beta+1)l!}\delta_{kl}\end{aligned}$$

for $\mathrm{Re}(\alpha)>-1$ and $\mathrm{Re}(\beta)>-1$.

The k-th derivative of $P_l^{(\alpha,\beta)}(x)$, $k\in\mathbb{N}_0$, is

$$\frac{d^k}{dx^k}P_l^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\beta+l+1+k)}{2^k\Gamma(\alpha+\beta+l+1)}P_{l-k}^{(\alpha+k,\beta+k)}(x)\,. \tag{43}$$

The Jacobi polynomials $P_l^{(\alpha,\beta)}(x)$ are the polynomial solutions of the hypergeometric differential equation

$$\begin{aligned}&\Big((1-x^2)\frac{d^2}{dx^2}+(\beta-\alpha-(\alpha+\beta+2)x)\frac{d}{dx}+l(l+\alpha+\beta+1)\Big)\\ &\qquad\times P_l^{(\alpha,\beta)}(x)=0\,,\end{aligned} \tag{44}$$

and specialize for $\alpha=\beta$ to the Gegenbauer polynomials fulfilling recurrence relation

$$C_l^\alpha(x) = \frac{1}{l}\big(2x(l+\alpha-1)C_{l-1}^\alpha(x) - (l+2\alpha-2)C_{l-2}^\alpha(x)\big) \tag{45}$$

with $C_0^\alpha(x)=1, C_1^\alpha(x)=2\alpha x$. The Gegenbauer polynomials are solutions of the Gegenbauer differential equation

$$\Big((1-x^2)\frac{d^2}{dx^2}-(2\alpha+1)x\frac{d}{dx}+l(l+2\alpha)\Big)C_l^\alpha(x)=0\,, \tag{46}$$

and are represented by finite hypergeometric series

$$C_l^\alpha(z) = \frac{(2\alpha)_l}{l!}\,{}_2F_1\Big(-l,2\alpha+l;\alpha+\frac{1}{2};\frac{1-z}{2}\Big)\,.$$

More explicitly,

$$C_l^\alpha(z) = \sum_{k=0}^{[l/2]}(-1)^k\frac{\Gamma(l-k+\alpha)}{\Gamma(\alpha)k!(l-2k)!}(2z)^{l-2k}\,, \tag{47}$$

and their relation to the Jacobi polynomials is

$$C_l^\alpha(x) = \frac{(2\alpha)_l}{(\alpha+\frac{1}{2})_l}P_l^{(\alpha-1/2,\alpha-1/2)}(x)\,. \tag{48}$$

Acknowledgement. P. Pandžić acknowledges the financial support from the grant no. 4176 by the Croatian Science Foundation. P. Somberg acknowledges the financial support from the grant GA P201/12/G028.

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