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WARD IDENTITIES FROM RECURSION FORMULAS FOR CORRELATION FUNCTIONS IN CONFORMAL FIELD THEORY

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Abstract. A conformal block formulation for the Zhu recursion procedure in conformal field theory which allows to find n-point functions in terms of the lower correlations functions is introduced. Then the Zhu reduction operators acting on a tensor product of VOA modules are defined. By means of these operators we show that the Zhu reduction procedure generates explicit forms of Ward identities for conformal blocks of vertex operator algebras. Explicit examples of Ward identities for the Heisenberg and free fermionic vertex operator algebras are supplied.

1. Introduction

Algebraic methods in computation of the partition and n-point functions in Conformal Field Theory/Vertex Operator Super Algebras proved their effectiveness. The Zhu reduction formula allowing to express $n+1$-point correlation functions as finite sums of n-point functions constitute the main algebraic tool for calculations. In this paper we give a the Zhu recursion procedure formulation for chiral Ward identities for corresponding conformal blocks for vertex operator algebras considered on Riemann surfaces of genus $g \geq 0$. First we recall the notion of a vertex operator algebra, introduce correlation functions on Riemann surfaces, and review the Zhu recursion formulas for correlation functions. Then we provide examples of explicit Ward identities for the Heisenberg and free fermionic vertex operator algebras on Riemann surfaces of genuses one and two. Finally an appropriate definition for conformal blocks related to chiral vertex operator algebra correlation functions is given. We then define the Zhu reduction operators generating Ward explicit identities for corresponding conformal blocks.

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2. Vertex operator algebras

A vertex operator algebra (VOA) [3, 4, 5, 6, 11, 18] is determined by a quadruple $(V, Y, \mathbf{1}, \omega)$, where V is a linear space endowed with a $\mathbb{Z}$-grading

$$V = \bigoplus_{r\in\mathbb{Z}} V_r\,,$$

with $\dim V_r < \infty$. The state $\mathbf{1} \in V_0$, $\mathbf{1} \neq 0$, is the vacuum vector and $\omega \in V_2$ is the conformal vector with properties described below. The vertex operator Y is a linear map

$$Y\colon V \to \mathrm{End}(V)[[z, z^{-1}]]\,,$$

for formal variable z so that for any vector $u \in V$ we have a vertex operator

$$Y(u, z) = \sum_{n\in\mathbb{Z}} u(n) z^{-n-1}\,.$$

The linear operators (modes) $u(n)\colon V \to V$ satisfy creativity

$$Y(u, z)\mathbf{1} = u + O(z)\,,$$

and lower truncation

$$u(n)v = 0\,,$$

conditions for each u, $v \in V$ and $n \gg 0$. For the conformal vector ω one has

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}\,,$$

where $L(n)$ satisfies the Virasoro algebra for some central charge C

$$[\, L(m), L(n)\,] = (m-n)L(m+n) + \frac{C}{12}(m^3 - m)\delta_{m,-n}\mathrm{Id}_V\,,$$

where Id_V is identity operator on V. Each vertex operator satisfies the translation property

$$Y(L(-1)u, z) = \partial_z Y(u, z)\,.$$

The Virasoro operator $L(0)$ provides the $\mathbb{Z}$-grading with

$$L(0)u = ru\,,$$

for $u \in V_r$, $r \in \mathbb{Z}$. Finally, the vertex operators satisfy the Jacobi identity

$$\begin{aligned} z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big) Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big) Y(v, z_2)Y(u, z_1) \\ = z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big) Y\Big(Y(u, z_0)v, z_2\Big)\,, \end{aligned}$$

where $\delta\big(\frac{x}{y}\big) = \sum\limits_{r\in\mathbb{Z}} x^r y^{-r}$. We use the formal expansion:

$$(z_1 + z_2)^m = \sum_{n\geq 0} \binom{m}{n} z_1^{m-n} z_2^n\,,$$

i.e., for $m < 0$ we formally expand in the second parameter z_2. These axioms imply locality, skew-symmetry, associativity and commutativity conditions:

$$(z_1 - z_2)^N Y(u, z_1)Y(v, z_2) = (z_1 - z_2)^N Y(v, z_2)Y(u, z_1)\,, \tag{2.1}$$

$$Y(u, z)v = e^{zL(-1)}Y(v, -z)u\,,$$

$$(z_0 + z_2)^N Y(u, z_0 + z_2)Y(v, z_2)w = (z_0 + z_2)^N Y(Y(u, z_0)v, z_2)w\,,$$

$$u(k)Y(v, z) - Y(v, z)u(k) = \sum_{j\geq 0} \binom{k}{j} Y(u(j)v, z)z^{k-j}\,, \tag{2.2}$$

for u, v, $w \in V$ and integers $N \gg 0$ [5,11]. For $v = \mathbf{1}$ one has

$$Y(\mathbf{1}, z) = \mathrm{Id}_V\,.$$

Note also that modes of homogeneous states are graded operators on V, i.e., for $v \in V_k$,

$$v(n)\colon V_m \to V_{m+k-n-1}\,.$$

In particular, let us define the zero mode $o(v)$ of a state of weight $wt(v) = k$, i.e., $v \in V_k$, as

$$o(v) = v(wt(v) - 1)\,,$$

extending to V additively. There is a number of equivalent sets of axioms for vertex operator algebra theory. In [5,6] Frenkel-Huang-Lepowsky have proven that one can describe a vertex operator algebra by the set of all its correlation functions.

3. Correlation functions for vertex operator algebras

3.1. Matrix elements on the sphere.

Let us first define matrix elements on the sphere. Assume that our VOA is of CFT-type, i.e.,

$$V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus \dots\,.$$

We define the restricted dual space of V by [5]

$$V' = \bigoplus_{n\geq 0} V_n^*\,,$$

where V_n^* is the dual space of linear functionals on the finite dimensional space V_n with respect to the canonical pairing $\langle \cdot, \cdot \rangle$ between V' and V. Define matrix elements for $v' \in V'$, $v \in V$ and n vertex operators $Y(v_1, z_1), \dots, Y(v_n, z_n)$ by

$$\langle v', Y(v_1, z_1) \dots Y(v_n, z_n)v \rangle\,. \tag{3.1}$$

In particular, choosing $v = \mathbf{1}$ and $v' = \mathbf{1}'$ we obtain the n-point correlation function on the sphere:

$$F_V^{(0)}(v_1, z_1; \dots; v_n, z_n) = \langle \mathbf{1}', Y(v_1, z_1) \dots Y(v_n, z_n)\mathbf{1} \rangle\,.$$

Here the upper index of $F_V^{(0)}$ stands for the genus.

One can show in general that every matrix element is a homogeneous rational function of $z_1, \dots, z_n$, [5]. Thus the formal parameters of VOA theory can be replaced by complex parameters on (appropriate subdomains of) the genus zero Riemann sphere $\mathbb{CP}^1$.

We illustrate [15] this by considering matrix elements containing one or two vertex operators. Recall that for $u \in V_n$,

$$u(k) \colon V_m \to V_{m+n-k-1} \, .$$

Hence it follows that for $v' \in V'_{m'}$, $v \in V_m$ and $u \in V_n$ we obtain a monomial

$$\langle v', Y(u,z) v \rangle = C^u_{v'v} z^{m'-m-n} \, ,$$

where

$$C^u_{v'v} = \langle v', u(m+n-m'-1)v \rangle \, .$$

Next consider the matrix element of two vertex operators. We then find in [5]

Theorem 1. *Let $v' \in V'_{m'}$, $v \in V_m$, $u_1 \in V_{n_1}$ and $u_2 \in V_{n_2}$. Then*

$$\left\langle v', Y(u_1,z_1) Y(u_2,z_2) v \right\rangle = \frac{f(z_1,z_2)}{z_1^{m+n_1} z_2^{m+n_2} (z_1-z_2)^{n_1+n_2}} \, ,$$

$$\left\langle v', Y(u_2,z_2) Y(u_1,z_1) v \right\rangle = \frac{f(z_1,z_2)}{z_1^{m+n_1} z_2^{m+n_2} (-z_2+z_1)^{n_1+n_2}} \, ,$$

where $f(z_1,z_2)$ is a homogeneous polynomial of degree $m+m'+n_1+n_2$.

In general, let us recall the Proposition 3.5.1 of [5]:

Proposition 1. *For $v_1, \ldots, v_n$, $v \in V$, and $v' \in V'$, with any permutation of $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$, the matrix element*

$$\left\langle v', Y(u_{i_1}, z_{i_1}) \ldots Y(u_{i_n}, z_{i_n}) v \right\rangle$$

is independent of the permutation can be expressed via a (uniquely determined) function $f(z_1, \ldots, z_n)$ of the form

$$f(z_1, \ldots, z_n) = \frac{g(z_1, \ldots, z_n)}{\prod\limits_{i=1}^{n} z_i^{r_i} \prod\limits_{j<k} (z_j - z_k)^{s_{jk}}} \, ,$$

for some rational function $g(z_1, \ldots, z_n)$ in $\{z_1, \ldots, z_n\}$, and r_i, $s_{jk} \in \mathbb{Z}$.

3.2. **Genus zero Zhu reduction.** Using the vertex commutator property (2.2), i.e.,

$$[u(m), Y(v,z)] = \sum_{i \geq 0} \binom{m}{i} Y\left(u(i)v, z\right) z^{m-i} \, ,$$

one derives [26] a recursive relationship in terms of rational functions for $n+1$ vertex operators and a finite sum of matrix elements for n vertex operators.

One has [26] the following

Lemma 1. *For $v_1, \ldots, v_n \in V$, and a homogeneous $v \in V$, we find*

$$\begin{aligned} (3.2) \quad & \langle v', Y(v_1,z_1) \ldots Y(v_n,z_n) v \rangle \\ &= \sum_{r=2}^{n} \sum_{m \geq 0} f_{wt(v_1),m}(z_1,z_r) \, \langle v', Y(v_2,z_2) \ldots Y(v_1(m)\, v_r, z_r) \ldots Y(v_n,z_n) v \rangle \\ &\qquad + \langle v', o(v_1)\, Y(v_2,z_2) \ldots Y(v_n,z_n) v \rangle, \end{aligned}$$

where $f_{wt(v_1),m}(z_1, z_r)$ are some rational functions.

3.3. **Correlation functions on the torus.** One would like to ask now for generalizations at higher genus. In order to consider modular-invariance of n-point functions at genus one, Zhu introduced a second "square-bracket" VOA $(V, Y[\ ,\ ], \mathbf{1}, \tilde{\omega})$ associated to a given VOA $(V, Y(\ ,\ ), \mathbf{1}, \omega)$. The new square bracket vertex operators are defined by a change of coordinates, namely

$$Y[v, z] = \sum_{n\in\mathbb{Z}} v[n]z^{-n-1} = Y\left(q_z^{L(0)}v, q_z - 1\right),$$

with $q_z = e^z$, while the new conformal vector is $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$. For v of $L(0)$ weight $wt(v) \in \mathbb{R}$ and $m \geq 0$,

$$v[m] = m! \sum_{i\geq m} c(wt(v), i, m)v(i),$$

$$\sum_{m=0}^{i} c(wt(v), i, m)x^m = \binom{wt(v) - 1 + x}{i}.$$

In particular we note that

$$v[0] = \sum_{i\geq 0} \binom{wt(v) - 1}{i} v(i).$$

At genus one, instead of matrix elements of the form (3.1), one considers [26] (see also [13, 15, 17]) traces over corresponding vertex operator algebra. Vertex operator algebra formal parameters are associated now with local coordinates around insertion points on the torus. For $v_1, \ldots, v_n \in V$ the genus one n-point function has the form:

$$\begin{aligned} F_V^{(1)}&(v_1, z_1; \ldots; v_n, z_n; \tau) \\ &= \mathrm{Tr}_V \left(Y(q_1^{L(0)}v_1, q_1) \ldots Y(q_n^{L(0)}v_n, q_n)\, q^{L(0)-C/24}\right), \end{aligned} \tag{3.3}$$

for $q = e^{2\pi i\tau}$ and $q_i = e^{z_i}$, where τ is the torus modular parameter. Let us also introduce an extra notation for the insertion of a state $v \in V$ inside the trace in (3.3)

$$\begin{aligned} F_V^{(1)}&(v; v_1, z_1; \ldots; v_n, z_n; \tau) \\ &= \mathrm{Tr}_V \left(v\, Y(q_1^{L(0)}v_1, q_1) \ldots Y(q_n^{L(0)}v_n, q_n)\, q^{L(0)-C/24}\right). \end{aligned}$$

Then the genus one Zhu recursion formula is given by [26]

Theorem 2. *For any v, $v_1, \ldots, v_n \in V$ we find for an $n+1$-point function*

$$\begin{aligned} F_V^{(1)}&(v, z; v_1, z_1; \ldots; v_n, z_n; \tau) \\ &= \sum_{r=1}^{n} \sum_{m\geq 0} P_{m+1}(z - z_r, \tau)\, F_V^{(1)}(v_1, z_1; \ldots; v[m]v_r, z_r; \ldots; v_n, z_n; \tau) \\ &\quad + F_V^{(1)}\left(o(v); v_1, z_1; \ldots; v_n, z_n; \tau\right). \end{aligned} \tag{3.4}$$

In this theorem $P_m(z,\tau)$ denote higher Weierstrass functions [19] defined by

$$P_m(z,\tau) = \frac{(-1)^m}{(m-1)!} \sum_{n\in\mathbb{Z}_{\neq 0}} \frac{n^{m-1} q_z^n}{1-q^n} \,.$$

The genus two version of the genus zero and one Zhu reduction procedure and the formula (3.4) described in Lemma 1 and Theorem 2 are also available [8].

Let X be a smooth projective curve X over $\mathbb{C}$ and $p_1, \ldots, p_n$ an n-tuple of points of X with local coordinates $z_1, \ldots, z_n$. We consider the case when we have a finite number n of irreducible modules for a vertex operator algebra V. We attach to this points modules M_{λ_i} parameterized by complex parameters λ_i, $i=1,\ldots,n$ (see for instance [24] for the description of such a parameterization in case of the generalized vertex operator algebra constructed as an extension of the Heisenberg subalgebra by its irreducible modules). For $v_i \in V$ and a point p_i we insert the vertex operator $Y_{\lambda_i}(v_i, z_i)$ which belong to V-module M_{λ_i} with the formal parameter z_i near p_i.

4. Ward identities for vertex operator algebras: examples

4.1. **Heisenberg vertex operator algebra on the torus.** For the Heisenberg vertex operator algebra M, the Zhu reduction gives all n-pt functions [15], and in particular,

$$F_M^{(1)}(a, z_1; a; z_2; q) = P_2(z_1 - z_2, \tau)\; F_M^{(1)}(q) \,,$$

where $P_2(z,\tau) = -\frac{d}{dz} P_1(z,\tau)$. Then,

$$\begin{aligned} F_M^{(1)}(\widetilde{\omega}, q) &= \frac{1}{2} \lim_{x\to y} \left(F_M^{(1)}(a, z_1; a, z_2; q) - (z_1 - z_2)^{-2} F_M^{(1)}(q) \right) \\ &= \frac{1}{2} E_2(q) F_M^{(1)}(q) \,. \end{aligned}$$

Thus one has an ordinary differential equation [8]:

$$\Big(q\partial_q - \frac{1}{2} E_2(q) \Big) F_M^{(1)}(q) = 0 \,.$$

Then the genus one partition function for the Heisenberg vertex operator algebra is given by $F_M^{(1)}(q) = \frac{1}{\eta(q)}$.

For the Virasoro vector $\widetilde{\omega} = \omega - \frac{C}{24}$ one finds [13, 15]

$$F_V^{(1)}(\widetilde{\omega}, q) = \mathrm{Tr}_V \left((L(0) - C/24)\; q^{L(0)-C/24} \right) = q\partial_q F_V^{(1)}(q) \,.$$

For n primary vectors $u_1, \ldots, u_n$ (i.e., satisfying $L(n)u = 0$, for all $n \geq 1$), the Zhu reduction gives the Ward identity [8] for the $n+1$-point function:

$$\begin{aligned} &F_V^{(1)}(\widetilde{\omega}, x; u_1, z_1; \ldots; u_n, z_n; q) \\ &= \Big(q\partial_q + \sum_{1\leq i\leq n} \big(P_1(x - z_i)\partial_{z_i} + \mathrm{wt}[u_i]\; \partial_{z_i} P_1(x - z_i) \big) \Big) F_V^{(1)}(u_1; z_1; \ldots; u_n; z_n; q) \,. \end{aligned}$$

4.2. **Ward identity for Virasoro one point function of the Heisenberg VOA.** One can derive [8] the genus two Ward identities for the Heisenberg vertex operator algebra. The genus two counterpart of the relation for the genus one Virasoro one-point function

$$F_V^{(1)}(\omega, q) = q \frac{\partial}{\partial_q} F_V^{(1)}(q)\,,$$

is given for $\widetilde{\omega} \in V_{[2]}$ by the following expression [8]

$$F_V^{(2)}(\widetilde{\omega}, z)\; dz^2 = D_z F_V^{(2)}(q)\,,$$

where

$$D_z = \sum_{i=1}^{3} D_i(z)\; q_i \partial_{q_i}\,,$$

for specific local two-forms $D_i(z)$, and $q_3 \equiv \epsilon \in \mathbb{C}$ is a parameter describing sewing of two tori to form a genus two Riemann surface [14, 15].

4.3. **Genus two Ward identities for the Heisenberg VOA.** Recall the notion of the genus g projective connection $s^{(g)}$ is defined by [9]

$$s^{(g)}(x) = 6 \lim_{x \to y} \left(\omega^{(g)}(x, y) - \frac{dxdy}{(x-y)^2} \right),$$

where $\omega^{(g)}$ is the meromorphic differential of the second kind on a Riemann surface. The projective connection $s^{(g)}(x)$ transforms under a general conformal transformation $x \to \phi(x)$ as follows $s^{(g)}(\phi(x)) = s^{(g)}(x) - \{\phi; x\} dx^2$, where $\{\phi; x\} = \frac{\phi'''}{\phi'} - \frac{3}{2} \left(\frac{\phi''}{\phi'} \right)^2$ is the Schwarzian derivative.

If $v \in V$ is a primary vector, one obtains the Ward identity [8]

$$\begin{aligned} &Z_V^{(2)}(\widetilde{\omega}, z_1, v, z_2, \tau_1, \tau_2, \epsilon) \\ &\qquad = \big(D_{z_1} + P_1(z_1, z_2) \partial_{z_2} + wt[v]\; \partial_{z_2} P_1(z_1, z_2) \big) Z_V^{(2)}(v, z_2, \tau_1, \tau_2, \epsilon)\,, \end{aligned}$$

for a specific $P_1(z_1, z_2)$ which depends on $2wt[v] - 2$. Thus for the Heisenberg VOA M we find that $Z_M^{(2)}$ satisfies the PDE:

$$\Big(D_z - \frac{1}{12} s^{(2)}(z) \Big) Z_M^{(2)}(q) = 0\,,$$

which is the genus two counterpart of the relation

$$\Big(q\partial_q - \frac{1}{2} E_2(q) \Big) Z_M^{(1)}(q) = 0\,.$$

4.4. **Ward identities for fermionic genus two correlation functions.** Introduce the differential operator [21, 22, 23, 25]

$$\mathcal{D}^{(g)} = \frac{1}{2\pi i} \sum_{1\leq i\leq j\leq g} \nu_i^{(g)}(x)\, \nu_j^{(g)}(x) \frac{\partial}{\partial \Omega_{ij}^{(g)}} .$$

It includes holomorphic 1-forms $\nu_i^{(g)}$ as well as derivative with respect to the genus g Riemann surface period matrix $\Omega^{(g)}$.

Let $Z_M^{(2)}$ be the genus two partition function for the rank one Heisenberg VOA M [14, 16]. We proved in [23] that the Virasoro one-point normalized differential form for the rank two fermion VOSA satisfies the genus two Ward identity

$$\frac{\mathcal{F}_{\mathcal{V},1}^{(2)} \begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix} (\tilde{\omega}, z)}{Z_M^{(2)}} = e^{-2\pi i \alpha^{(2)}\cdot\beta^{(2)}} \left[\mathcal{D}^{(2)} + \frac{1}{12} s^{(2)}(z)\right] \cdot \vartheta^{(2)} \begin{bmatrix} \alpha^{(2)} \\ \beta^{(2)} \end{bmatrix} \left(\Omega^{(2)}\right) . \tag{4.1}$$

Here the expression for the normalized genus two one-point differential form is represented as the action of a differential operator on an automorphic function. One can generalize (4.1) to get a genus g formula. In contrast to the pure algebraic–geometry [12, 20, 25] and physics approaches [1, 7], the Ward identity shows up from algebraic properties of corresponding vertex algebra.

5. Ward identities from Zhu reduction

5.1. **Conformal blocks for vertex operator algebras.** Let us introduce the space of *chiral conformal blocks* as the space $C_V(M_{\lambda_1}, \ldots, M_{\lambda_n})$ of linear functionals

$$\varphi\colon \bigotimes_{i=1}^{n} M_{\lambda_i} \to \mathbb{C} .$$

A conformal block is called *invariant* when

$$\varphi\left(\eta \cdot \mathbf{v}\right) = 0 ,$$

for all $\mathbf{v} \in \bigotimes_{i=1}^n M_{\lambda_i}$, and $\eta \in V \otimes \mathbb{C}[X \setminus \{x_1, \ldots, x_n\}]$.

The set of genus g correlation functions for a vertex operator algebra forms a conformal block, i.e., we consider as φ

$$\varphi = \varphi_n(\mathbf{v}) = F^{(g)}_{M_{\lambda_1},\ldots,M_{\lambda_n}}(v_1, z_1; \ldots; v_n, x_n) , \tag{5.1}$$

a genus g n-point correlation function.

5.2. **Generation of Ward identities.** The general structure of the Zhu reduction formulas have the form of the insertion of an extra state v on the right hand side of the formula into lower n-point functions. This plays the role of the action of corresponding vertex operator algebra in the general definition of the conformal block for vertex operator algebras.

Taking into account properties of genus g for $g = 0, 1$ correlation functions we introduce the operators $J^{(g)}_{n+1}$, $n \geq 1$, $g = 0, 1$ corresponding to the simplest version

Zhu reduction operators given by

$$J_{n+1}^{(g)} = \Big(I + o(v_1). - \sum_{r=1}^{n} \sum_{m\geq 0} f_r^{(g)} \cdot \widehat{_1} \otimes \bigotimes_{j=1}^{n} \left(v(m)\right)^{\delta_{j,r}} . \Big) \,,$$

which act on $n+1$-correlation functions, and where $f_r^{(g)}$ are specific autmorphic functions corresponding to the genus g, and $v \in V$ acts on the tensor product, and $\widehat{_1}$ denotes skipping the first multiplier in the tensor product. Notice that in the case of $g = 1$, modes $v(m)$ have to be replaced with $v[m]$ as we see from (3.2) and (3.4).

Proposition 2. *For $\eta = J_{n+1}^{(g)}$, $n \geq 1$, $g = 0, 1$ a conformal block φ (5.1) for vertex is invariant,*

$$\varphi_{n+1}\big(J_{n+1}^{(0,1)} \cdot v\big) = 0 \,,$$

i.e., it satisfies the left chiral Ward identity.

For $g = 2$ an appropriate expression for the operator $J_{n+1}^{(2)}$ can be derived taking into account a version of the Zhu reduction procedure for $g = 2$ presented in [8]. The Zhu reduction procedure can be generalized to the genus g case various examples of vertex operator algebras. For higher genus cases we formulate the following

Conjecture 1. *The genus g Zhu reduction procedure generates chiral Ward identities of the form $\varphi_{n+1}\left(J_{n+1}^{(g)} \cdot v\right) = 0$, for $v \in V$ with an appropriate operator $J_{n+1}^{(g)}$.*

The Zhu reduction formulas results in corresponding relations for conformal blocks.

Corollary 1. *The $n+1$-point conformal blocks expand into (finite) sum of the n-point conformal blocks:*

$$C_V(M_{\lambda_1}, \ldots, M_{\lambda_{n+1}}) \subset \mathcal{M} \times C_V(M_{\lambda_1}, \ldots, M_{\lambda_n}),$$

where $\mathcal{M}$ is the space of genus g automorphic forms.

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