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NECESSARY AND SUFFICIENT CONDITIONS FOR STABILITY OF VOLTERRA INTEGRO-DYNAMIC EQUATION ON TIME SCALES

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Abstract. In this research we establish necessary and sufficient conditions for the stability of the zero solution of scalar Volterra integro-dynamic equation on general time scales. Our approach is based on the construction of suitable Lyapunov functionals. We will compare our findings with known results and provides application to quantum calculus.

1. Introduction

Our aim is to use suitable Lyapunov functionals and arrive at necessary and sufficient conditions for the stability of the zero solution of the scalar Volterra integro-dynamic system on time scales

$$x^{\Delta} = A(t)x + \int_0^t C(t,s)x(s)\Delta s\,, \tag{1}$$

where $A(t)$ is continuous on $t \in [0,\infty)_{\mathbb{T}}$ and $C(t,s)$ is continuous on $t \in [0,\infty)_{\mathbb{T}}$ and rd-continuous with respect the second variable on $s \in [0,\infty)_{\mathbb{T}}$. Our results unify and extend some continuous results and their corresponding discrete analogues.

For the sake of stating general definitions of stability, we consider the functional dynamical system

$$x^{\Delta}(t) = G(t, x(s); 0 \le s \le t) := G(t, x(\cdot)) \tag{2}$$

on a time scale $\mathbb{T}$ that is unbounded above with $0 \in \mathbb{T}$, where $x \in \mathbb{R}$ and $G\colon [0,\infty)_{\mathbb{T}} \times \mathbb{R} \mapsto \mathbb{R}$ is a is rd-continuous function in t and x with $G(t,0) = 0$. Throughout this paper, for each continuous function $\phi\colon [0,t_0]_{\mathbb{T}} \mapsto \mathbb{R}$ there exists at least one continuous function $x(t) = x(t,t_0,\phi)$ on an interval $[t_0,a]_{\mathbb{T}}$ such that it satisfies (2) for $t \in [t_0,a]_{\mathbb{T}}$ and $x(t,t_0,\phi) = \phi(t)$ for $t \in [0,t_0]_{\mathbb{T}}$. For the existence and extendibility of solutions of (2) we refer the reader to [6] and to [14] for Volterra integral equations on time scales. When $\mathbb{T} = \mathbb{R}$, we refer the reader to [19], and [18] for results concerning boundedness of solutions of functional differential equations.

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In the paper [15], the authors used the notion of the resolvent equation and obtained results concerning boundedness and the exponential decay of solutions of Volterra integro-dynamic system on time scales, with forcing term. At the end of the paper, we make an attempt to compare the results of this paper with those obtained in [15].

For each $t_0 \in \mathbb{T}$ and for a given rd-continuous initial function $\psi\colon [0,t_0]_{\mathbb{T}} \to \mathbb{R}$, we say that $x(t) := x(t;t_0,\psi)$ is the solution of (2) if $x(t) = \psi(t)$ on $[0,t_0]_{\mathbb{T}}$ and satisfies (2) for all $t \in [t_0,\infty)_{\mathbb{T}}$.

2. Set up of Lyapunov functionals

We say $V\colon [\delta(t_0),\infty)_{\mathbb{T}} \times \mathbb{R} \mapsto [0,\infty)$ is a *type I Lyapunov functional* on $[\delta(t_0),\infty)_{\mathbb{T}} \times \mathbb{R}$ when

$$V(t,x) = \sum_{i=1}^{n} \big(V_i(x_i) + U_i(t)\big)\,,$$

where each $V_i\colon \mathbb{R} \mapsto \mathbb{R}$ and $U_i\colon [0,\infty)_{\mathbb{T}} \mapsto \mathbb{R}$ are continuously and Δ-differentiable. Next, we extend the definition of the derivative of a type I Lyapunov function to type I Lyapunov functionals. If V is a type I Lyapunov functional and x is a solution of equation (2), then we have

$$\begin{aligned}
[V(t,x)]^{\Delta} &= \sum_{i=1}^{n} \big(V_i(x_i(t)) + U_i(t)\big)^{\Delta} \\
&= \int_0^1 \nabla V\big[x(t) + h\mu(t)G(t,x(\cdot))\big] \cdot G(t,x(\cdot))\,dh + \sum_{i=1}^{n} U_i^{\Delta}(t)
\end{aligned}$$

where $\nabla = (\partial/\partial x_1,\ldots,\partial/\partial x_n)$ is the gradient operator. This motivates us to define $\dot V\colon [\delta(t_0),\infty)_{\mathbb{T}} \times \mathbb{R} \mapsto \mathbb{R}$ by

$$\dot V(t,x) = \big[V(t,x)\big]^{\Delta}\,.$$

Continuing in the spirit of [17], we have

$$\dot V(t,x) = \begin{cases} \sum\limits_{i=1}^{n} \frac{V_i(x_i+\mu(t)G_i(t,x(\cdot)))-V_i(x_i)}{\mu(t)} + \sum\limits_{i=1}^{n} U_i^{\Delta}(t)\,, & \text{when} \quad \mu(t) \neq 0, \\ \nabla V(x)\cdot G(t,x(\cdot)) + \sum\limits_{i=1}^{n} U_i^{\Delta}(t)\,, & \text{when} \quad \mu(t) = 0. \end{cases}$$

We also use a continuous strictly increasing function $W_i\colon [0,\infty) \mapsto [0,\infty)$ with $W_i(0) = 0$, $W_i(s) > 0$ if $s > 0$ for each $i \in \mathbb{Z}^+$.

For more on Lyapunov functions/functionals on time scales we refer the reader to [6], [16] and [17].

When the time scale is the set of reals the theory of Lyapunov functionals have been fully developed. However, there are few papers that deal with the concept of Lyapunov functionals on general time scales. In this paper, we attempt to close the gap and we declare that all the results of this paper are new even for the discrete case. In the continuous case, Burton, devoted all his book [11] to the development and applications of fixed point theory to study existence of solutions, existence of periodic solutions and stability of various types of functional differential equations.

The reason behind his book was to alleviate some of the difficulties that usually arise from the use of Lyapunov functionals. As we shall see later, Lyapunov functions allow us to establish the stability or instability of the system. The advantage of this method is the fact that no information need to be known on the actual solution $x(t)$. On the other hand, one of the most disturbing disadvantage is the fact that there is no general method of constructing such Lyapunov functions. In the particular case of homogeneous autonomous systems with constant coefficients, the Lyapunov function can be found as a quadratic form.

In the paper of [5], Adivar and Raffoul, developed the resolvent equations for system (1) and modified them to serve as Lyapunov functionals and obtained necessary and sufficient conditions for the uniform stability of the zero solution of (1). However, this paper offers more relaxed conditions for the stability only. Also, in the paper of [1] the author used the notion of fixed point theory and obtain functional bounds on solutions of Volterra integro-dynamic equations of convolution type. In addition, we shall compare our results to both [5], [12] and [15]. With respect to oscillation, the authors of [13] initiated the study of oscillation theory for integro-dynamic equations on time-scales. They presented sufficient conditions guaranteeing that the oscillatory character of the forcing term is inherited by the solutions. We will end this paper by applying our results to Volterra q-difference equations, or quantum calculus.

3. Calculus on time scales with preliminary results

An introduction with applications and advances in dynamic equations are given in [7, 8]. In this section, we only mention necessary basic results on time scales. We have two jump operators, namely the *forward jump operator* and the *backward jump operator*

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\}\,, \quad \rho(t) := \sup\{s < t : s \in \mathbb{T}\}$$

for all $t \in \mathbb{T}$, respectively. Therefore, there might be four types of points in a time scale, i.e., $\sigma(t) > t$ (*right-scattered point* t), $\rho(t) < t$ (*left-scattered* point t), $\sigma(t) = t$ (*right-dense* point t), and $\rho(t) = t$ (*left-dense* point t). Also $\mu\colon \mathbb{T} \mapsto [0,\infty)$ defined by $\mu(t) := \sigma(t) - t$ gives the distance between two points in a time scale.

Assume $x\colon \mathbb{T} \mapsto \mathbb{R}^n$. Then we define $x^{\Delta}(t)$ to be the vector (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\Big|[x_i(\sigma(t)) - x_i(s)] - x_i^{\Delta}(t)[\sigma(t) - s]\Big| \le \epsilon\,|\sigma(t) - s|$$

for all $s \in U$ and for each $i = 1, 2, \dots n$. We call $x^{\Delta}(t)$ the *delta derivative* of $x(t)$ at t, and it turns out that $x^{\Delta}(t) = x'(t)$ if $\mathbb{T} = \mathbb{R}$ and $x^{\Delta}(t) = x(t+1) - x(t)$ if $\mathbb{T} = \mathbb{Z}$. If $G^{\Delta}(t) = g(t)$, then the Cauchy integral is defined by

$$\int_a^t g(s)\Delta s = G(t) - G(a)\,.$$

It can be shown that for each continuous function $f\colon \mathbb{T} \mapsto \mathbb{R}^n$ at $t \in \mathbb{T}$

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \quad \text{for right-scattered point } t$$

and if the limit exists

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \quad \text{for right-dense point } t\,.$$

The product and quotient rules are given by

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) \tag{3}$$

and

$$\Big(\frac{f}{g}\Big)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)} \quad \text{if} \quad g(t)g^{\sigma}(t) \neq 0\,.$$

For differentiable functions f, $g\colon \mathbb{T} \mapsto \mathbb{R}^n$ at $t \in \mathbb{T}$. We also have the following simple useful formula

$$f^{\sigma}(t) = f(t) + \mu(t)f^{\Delta}(t)\,, \quad \text{where} \quad f^{\sigma} = f \circ \sigma\,. \tag{4}$$

We say $f\colon \mathbb{T} \mapsto \mathbb{R}$ is *rd-continuous* provided f is continuous at each right-dense point $t \in \mathbb{T}$ and whenever $t \in \mathbb{T}$ is left-dense $\lim_{s \to t^-} f(s)$ exists as a finite number.

The following chain rule is due to Poetzsche.

Theorem 3.1. *Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be continuously differentiable and suppose $g\colon \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable. Then $f \circ g\colon \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \Big\{ \int_0^1 f'\big(g(t) + h\mu(t)g^{\Delta}(t)\big)\, dh \Big\} g^{\Delta}(t) \tag{5}$$

holds.

We use the following result [7, Theorem 1.117] to calculate the derivative of the Lyapunov function in further sections. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$.

Theorem 3.2. *Let $t_0 \in \mathbb{T}^{\kappa}$ and assume $k\colon \mathbb{T} \times \mathbb{T}^{\kappa} \mapsto \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^{\kappa}$ with $t > t_0$. Also assume that $k(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose for each $\epsilon > 0$ there exists a neighborhood of t, independent U of $\tau \in [t_0, \sigma(t)]$, such that*

$$|k(\sigma(t), \tau) - k(s, \tau) - k^{\Delta}(t, \tau)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \textit{for all} \quad s \in U\,,$$

where k^{Δ} denotes the derivative of k with respect to the first variable. Then

$$g(t) := \int_{t_0}^{t} k(t, \tau)\Delta\tau \quad \textit{implies} \quad g^{\Delta}(t) = \int_{t_0}^{t} k^{\Delta}(t, \tau)\Delta\tau + k(\sigma(t), t)\,;$$

and

$$h(t) := \int_{t}^{b} k(t, \tau)\Delta\tau \quad \textit{implies} \quad k^{\Delta}(t) = \int_{t}^{b} k^{\Delta}(t, \tau)\Delta\tau - k(\sigma(t), t)\,.$$

We apply the following Cauchy–Schwarz inequality in [7, Theorem 6.15] to prove Theorem 4.1.

Theorem 3.3. *Let a, $b \in \mathbb{T}$. For rd-continuous f, $g\colon [a,b] \mapsto \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \sqrt{\Big\{ \int_a^b |f(t)|^2 \Delta t \Big\} \Big\{ \int_a^b |g(t)|^2 \Delta t \Big\}}\,.$$

We make use of the above expression in our examples. Following lemma is mentioned in [6].

Lemma 3.1. *Assume $\phi(t,s)$ is right-dense continuous (rd-continuous) with respect to the second variable and is delta-differentiable with respect to the first variable, and let*

$$V(t,x) = x^2 + \int_0^t \phi(t,s)W\big(|x(s)|\big)\Delta s\,.$$

If x is a solution of (2), then we have by using (4) and Theorem 3.2 that

$$\begin{aligned}\dot{V}(t,x) &= 2x\cdot G\big(t,x(\cdot)\big) + \mu(t)G^2\big(t,x(\cdot)\big)\\ &\quad + \int_0^t \phi^{\Delta}(t,s)W\big(|x(s)|\big)\Delta s + \phi\big(\sigma(t),t\big)W\big(|x(t)|\big)\,,\end{aligned}$$

where $\phi^{\Delta}(t,s)$ denotes the derivative of ϕ with respect to the first variable.

Definition 3.1. We say that a Type I Lyapunov functional $V\colon [0,\infty)_{\mathbb{T}}\times\mathbb{R}^n \mapsto [0,\infty)$ is *negative definite* if $V(t,x)\neq 0$ for $x\neq 0$, $x\in\mathbb{R}^n$, $V(t,x)=0$ for $x=0$ and along the solutions of (2) we have $\dot{V}(t,x)\le 0$. If the condition $\dot{V}(t,x)\le 0$ does not hold for all $(t,x)\in\mathbb{T}\times\mathbb{R}^n$, then the Lyapunov functional is said to be *non-negative definite*.

4. Stability

We begin this section by stating general stability definitions and then, we use Lyapunov functionals to obtain results concerning the stability of the zero solution of (1).

We define

$$E_{t_0} = [0,t_0]_{\mathbb{T}}$$

which we call *the initial interval.* Let $C(t)$ denote the set of rd-continuous functions $\phi\colon [0,t]_{\mathbb{T}}\to R$ and $\|\phi\| = \sup\{|\phi(s)| : 0\le s\le t\}$.

Definition 4.1. The zero solution of (2) is stable if for each $\varepsilon>0$ and each $t_0\ge 0$, there exists a $\delta=\delta(\varepsilon,t_0)>0$ such that $[\phi\in E_{t_0}\to\mathbb{R},\ \phi\in C(t)\colon \|\phi\|<\delta]$ implies $|x(t,t_0,\phi)|<\varepsilon$ for all $t_0\ge 0$.

Definition 4.2. The zero solution of (2) is uniformly stable (US) if it is stable and δ is independent of t_0.

Definition 4.3. The zero solution of (2) is asymptotically stable (AS) if it is stable and if for each $t_0\ge 0$ there is an $\eta>0$ such that $[\phi\in E_{t_0}\to\mathbb{R},\ \phi\in C(t)\colon \|\phi\|<\eta]$ implies that any solution of (2) $|x(t,t_0,\phi)|\to 0$ as $t\to\infty$.

We begin our main results by stating and proving necessary and sufficient conditions for the stability of the zero solution of (2).

Theorem 4.1. *Suppose $C(t,s)$ is rd-continuous with respect to the second variable. Let $\int_{\sigma(t)}^{\infty}|C(u,t)|\Delta u$ be continuous for $t\in[0,\infty)_{\mathbb{T}}$. Suppose A is a continuous* *function on* $[0,\infty)_{\mathbb{T}}$ *and that there are constants* $\nu > 1$ *and* $\alpha, \beta > 0$ *such that*

$$\begin{aligned}&\big(1+\mu(t)|A(t)|\big)\int_0^t |C(t,s)|\Delta s + \mu(t)A^2(t)\\ &\quad + \nu\int_{\sigma(t)}^{\infty}|C(u,t)|\Delta u - 2|A(t)| \le -\alpha \end{aligned} \tag{6}$$

and

$$\mu(t)\big(|A(t)| + \int_0^t |C(t,s)|\Delta s\big) - (\nu - 1) \le -\beta\,. \tag{7}$$

Then the zero solution of (1) *is stable if and only if* $A(t) < 0$ *for all* $t \in [0,\infty)_{\mathbb{T}}$*.*

Proof. Let

$$V(t,x) = x^2(t) + \nu\int_0^t\int_t^{\infty}|C(u,s)|\Delta u x^2(s)\Delta s\,. \tag{8}$$

Assume $A(t) < 0$ for all $t \in [0,\infty)_{\mathbb{T}}$. Using Theorem 3.2 and Lemma 3.1, we have along the solutions of (1) that

$$\begin{aligned}\dot V(t,x) &= 2x(t)\Big(A(t)x(t) + \int_0^t C(t,s)x(s)\Delta s\Big)\\ &\quad + \mu(t)\Big(A(t)x(t) + \int_0^t C(t,s)x(s)\Delta s\Big)^2\\ &\quad - \nu\int_0^t |C(t,s)|x^2(s)\Delta s + \nu\int_{\sigma(t)}^{\infty}|C(u,t)|x^2(t)\Delta u\\ &= -2|A(t)|x^2(t) + 2x(t)\int_0^t C(t,s)x(s)\Delta\\ &\quad + \mu(t)\Big(A^2(t)x^2(t) + 2A(t)x(t)\int_0^t C(t,s)x(s)\Delta s + \Big(\int_0^t C(t,s)x(s)\Delta s\Big)^2\Big)\\ &\quad - \nu\int_0^t |C(t,s)|x^2(s)\Delta s + \nu\int_{\sigma(t)}^{\infty}|C(u,t)|x^2(t)\Delta u\,. \end{aligned} \tag{9}$$

Using the fact that $ab \le a^2/2 + b^2/2$ for any real numbers a and b, we have

$$2\int_0^t |C(t,s)|\,|x(t)||x(s)|\Delta s \le \int_0^t |C(t,s)|\big(x^2(t) + x^2(s)\big)\Delta s\,.$$

Also, using Theorem 3.3 one obtains

$$\begin{aligned}\Big(\int_0^t |C(t,s)|x(s)\Delta s\Big)^2 &= \Big(\int_0^t |C(t,s)|^{1/2}|C(t,s)|^{1/2}x(s)\Delta s\Big)^2\\ &\le \int_0^t |C(t,s)|\Delta s\int_0^t |C(t,s)|x^2(s)\Delta s.\end{aligned}$$

A substitution of the above two inequalities into (9) yields

$$\begin{aligned}
\dot V(t,x) &\le \Big[\big(1+\mu(t)|A(t)|\big)\int_0^t |C(t,s)|\Delta s + \mu(t)A^2(t) \\
&\quad + \nu\int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u - 2|A(t)|\Big]x^2(t) \\
&\quad + \Big[\mu(t)\big(|A(t)| + \int_0^t |C(t,s)|\Delta s\big) - (\nu-1)\Big]\int_0^t |C(t,s)|x^2(s)\Delta s \\
&\le -\alpha x^2(t)\,. \qquad (10)
\end{aligned}$$

Let $\varepsilon > 0$ be given. We will find a $\delta > 0$ so that for any bounded initial function $\phi\colon E_{t_0} \to R$ with $\|\phi\| < \delta$, we have $|x(t,t_0,\phi)| < \varepsilon$. Due to (10), V is decreasing and hence for $t \in [0,\infty)_{\mathbb{T}}$. We have that

$$\begin{aligned}
x^2 \le V(t,x) &\le V(t_0,\phi) \\
&\le \|\phi\|^2 + \nu\int_0^{t_0}\int_{t_0}^{\infty} |C(u,s)|\Delta u\Delta s\|\phi|^2 \\
&= \Big(1+\nu\int_0^{t_0}\int_{t_0}^{\infty} |C(u,s)|\Delta u\Delta s\Big)\|\phi\|^2\,. \qquad (11)
\end{aligned}$$

Or,

$$|x(t,t_0,\phi)| \le \varepsilon \quad \text{for} \quad \delta = \Big\{\frac{\varepsilon}{1+\nu\int_0^{t_0}\int_{t_0}^{\infty}|C(u,s)|\Delta u\Delta s}\Big\}^{1/2}.$$

To prove the other part, we assume $A(t) > 0$ for some $t \in [0,\infty)_{\mathbb{T}}$. Define the functional

$$W(t,x) = x^2(t) - \nu\int_0^t\int_t^{\infty} |C(u,s)|\Delta u x^2(s)\Delta s\,.$$

Then along the solutions of (1), we have by a similar argument as for V that

$$\begin{aligned}
\dot W(t,x) &\ge \Big[2A(t) - \big(1+\mu(t)|A(t)|\big)\int_0^t |C(t,s)|\Delta s - \mu(t)A^2(t) \\
&\quad - \nu\int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u\Big]x^2(t) \\
&\quad + \Big[(\nu-1) - \mu(t)\big(|A(t)| + \int_0^t |C(t,s)|\Delta s\big)\Big]\int_0^t |C(t,s)|x^2(s)\Delta s \\
&\ge \alpha x^2(t)\,.
\end{aligned}$$

Given any $t_0 \ge 0$ and any $\delta > 0$, we can find a continuous function $\phi\colon E_{t_0} \to R$ with $\|\phi\| < \delta$ with $\|\phi\| < \delta$, and $W(t_0,\phi) > 0$ so that if we have $x(t) = x(t,t_0,\phi)$ is a solution of (1), then we have

$$\begin{aligned}
x^2(t) \ge W(t,x) &\ge W(t_0,\phi) + \alpha\int_{t_0}^t W(t_0,\phi)\Delta s \\
&= W(t_0,\phi) + \alpha W(t_0,\phi)(t-t_0)\,.
\end{aligned}$$

As $t \to \infty$, $|x(t)| \to \infty$, which is a contradiction. $\square$

Remark 4.1. When $\mathbb{T} = \mathbb{R}$, condition (7) becomes unnecessary if we take $\nu = 1$ and condition (6) reduces to

$$\int_0^t |C(t,s)|ds + \int_t^\infty |C(u,t)|du - 2|A(t)| \le -\alpha\,.$$

For more on this we refer the reader to [10].
We have the following corollary.

Corollary 4.1. *Assume the conditions of Theorem 4.1. In addition, if we ask that there exist a positive constant M such that*

$$\int_0^t \int_t^\infty |C(u,s)|\Delta u \Delta s \le M \quad \textit{for all} \quad t \in [0,\infty)_{\mathbb{T}}, \tag{12}$$

then the zero solution of (1) *is* (US).

Proof. The proof is an immediate consequence of (11) by taking

$$\delta = \left\{\frac{\varepsilon}{1+\nu M}\right\}^{1/2}.$$

□

As a consequence of Theorem 4.1 we are able to state and easily prove the next corollary.

Corollary 4.2. *Assume* (6) *and* (7) *hold with $A(t) < 0$ and $A(t)$ is bounded for all $t \in [0,\infty)_{\mathbb{T}}$. Then,*

a) *$x^2(t)$ is bounded,*
b) *$x^2(t) \in L^2\big([0,\infty)_{\mathbb{T}}\big)$,*
c) *$x^\Delta(t)$ is bounded,*

and

d) *the zero solution of* (1) *is* (AS).

Proof. The proof of a) is an immediate consequence of (11). To prove b) we make use of (10). Since V is positive and decreasing we have that

$$0 \le V(t,x) \le V(t_0,\phi) - \alpha \int_0^t x^2(s)\Delta s\,, \tag{13}$$

from which we obtain that

$$\int_0^t x^2(s)\Delta s \le \frac{1}{\alpha} V(t_0,\phi)\,.$$

For the proof of c) use (1) and the fact that $A(t)$ is bounded, condition (6) and the fact $x(t)$ is bounded. For part d) we have stability by Theorem 4.1. Left to show $|x(t,t_0,\phi)| \to 0$, as $t \to \infty$. We will prove this by contradiction. Suppose $|x(t,t_0,\phi)| \nrightarrow 0$, then there exists a large $T \in [0,\infty)_{\mathbb{T}}$ and small but positive ρ such that $|x(t,t_0,\phi)| > \rho$ for all $t \ge T$. Next we integrate (10) from T to t and get

$$\begin{aligned} 0 \le V(t,x) &\le V(T,x) - \alpha \int_T^t |x(s)|^2 \Delta s \\ &\le V(T,x) - \alpha(t-T)\rho \to -\infty\,, \quad \text{as} \quad t \to \infty, \end{aligned}$$

which is a contradiction. This completes the proves of (AS). □

In the next theorem, we provide a kind of an algorithm for constructing a suitable Lyapunov functional that decreases along the solutions. Let us begin by integrating (1) from 0 to t.

$$x(t) = x(0) + \int_0^t A(s)x(s)\Delta s + \int_0^t \int_0^u C(u,s)\Delta s x(s)\Delta u\,.$$

Interchanging the order of integrations yields to

$$\int_0^t \int_0^u C(u,s)\Delta u x(s)\Delta s = \int_0^t \int_{\sigma(s)}^t C(u,s)\Delta u x(s)\Delta s\,.$$

Define the functional

$$h(t,x) = x(t) + \int_0^t \Big[-A(s) - \int_{\sigma(s)}^t C(u,s)\Delta u\Big] x(s)\Delta s\,.$$

Finally, take the norms in the above expression and define the functional

$$V(t,x) = |x(t)| + \int_0^t \Big[|A(s)| - \int_{\sigma(s)}^t |C(u,s)|\Delta u\Big] |x(s)|\Delta s\,. \tag{14}$$

Note that the functional V defined by (14) may serve, under appropriate conditions, as a Lyapunov functional.
In order to Δ-differentiate V we need to use the identity

$$|x(t)|^\Delta = \frac{x(t) + x^\sigma(t)}{|x(t)| + |x^\sigma(t)|} x^\Delta(t)\,.$$

Its proof can be found in [9].

Theorem 4.2. *Consider* (1) *with* $A(t) \le 0$ *for all* $t \in [0,\infty)_{\mathbb{T}}$. *Assume*

$$|A(s)| - \int_{\sigma(s)}^t |C(u,s)|\Delta u \ge 0\,, \quad \text{for} \quad s \in [0,\infty)_{\mathbb{T}}\,. \tag{15}$$

Then the zero solution of (1) *is stable. Moreover, if there exists a* $t_2 \ge 0$ *and an* $\alpha > 0$ *with*

$$|A(s)| - \int_{\sigma(s)}^t |C(u,s)|\Delta u \ge \alpha\,, \quad \text{for} \quad s \in [t_2,\infty)_{\mathbb{T}}\,,$$

and if both $\int_0^t C(t,s)\Delta s$ *and* $A(t)$ *are bounded, then* $x = 0$ *is* (AS).

Proof. Let

$$f(t,s) = |A(s)|\,|x(s)| - \int_{\sigma(s)}^t |C(u,s)|x(s)\Delta u\,.$$

Then an application of Theorem 3.2 gives

$$\begin{aligned}\Big(\int_0^t f(t,s)\Delta s\Big)^{\Delta_t} &= \int_0^t f^{\Delta_t}(t,s)\Delta s + f(\sigma(t),t)\\ &= \int_0^t (-|C(t,s)|\,|x(s)|)\Delta s + |A(t)|\,|x(t)|\,.\end{aligned}$$

Hence,

$$\begin{aligned}\dot V(t,x) &= \frac{x(t)+x^{\sigma}(t)}{|x(t)|+|x^{\sigma}(t)|}x^{\Delta}(t)+\int_0^t |C(t,s)||x(s)|\Delta s\\ &\quad + |A(t)|\,|x(t)| - \int_0^t |C(t,s)||x(s)|\Delta s\\ &\le A(t)|x(t)| + \int_0^t |C(t,s)|\,|x(s)|\Delta s\\ &\quad - \int_0^t |C(t,s)|\,|x(s)|\Delta s + |A(t)|\,|x(t)| = 0\,,\end{aligned} \tag{16}$$

as expected. Now that we have showed V is decreasing, we let $\varepsilon > 0$ be given. We will find a $\delta > 0$ so that for any bounded initial function $\phi\colon E_{t_0} \to R$ with $\|\phi\| < \delta$, we have $|x(t, t_0, \phi)| < \varepsilon$. Due to (16), V is decreasing and hence for $t \in [0,\infty)_{\mathbb{T}}$, we have that

$$\begin{aligned}|x(t)| \le V(t,x) &\le V(t_0,\phi)\\ &\le |\phi(t_0)| + \int_0^{t_0}\Big[|A(s)| - \int_{\sigma(s)}^{t_0} |C(u,s)|\Delta u\Big]|\phi(s)|\Delta s\\ &\le \delta\Big\{1+\int_0^{t_0}\Big[|A(s)| - \int_{\sigma(s)}^{t_0} |C(u,s)|\Delta u\Big]\Delta s\Big\} < \varepsilon\,,\end{aligned} \tag{17}$$

for $\delta = \dfrac{\varepsilon}{1+\int_0^{t_0}\Big[|A(s)| - \int_{\sigma(s)}^{t_0}|C(u,s)|\Delta u\Big]\Delta s}$.

If t_2 and α exist, then $\dot V \le 0$, which implies that

$$\begin{aligned}|x(t)| + \int_{t_2}^t \alpha|x(s)|\Delta s &\le |x(t)| + \int_0^t \Big[|A(s)| - \int_{\sigma(s)}^t |C(u,s)|\Delta u\Big]|x(s)|\Delta s\\ &= V(t,x) \le V(t_0,\phi)\,.\end{aligned}$$

This implies that $x(t) \in L^1\big([0,\infty)_{\mathbb{T}}\big)$. Using the fact that $\displaystyle\int_0^t C(t,s)\Delta s$ and $A(t)$ are bounded, equation (1) gives that $x^{\Delta}(t)$ is bounded and hence $x(t) \to 0$, as $t \to \infty$. This completes the proof. □

We end this paper by applying our results to Volterra integro-dynamic q-difference equations.

5. Applications

Let $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \mathbb{N} \text{ and } q > 1\}$. On this time scale (1) takes the form

$$D_q x(t) = A(t)x(t) + \sum_{s\in[1,t)_{q^{\mathbb{N}}}} \mu_q(s)C(t,s)x(s))\,, \quad t \ge 1 \tag{18}$$

where $[1,t)_{q^{\mathbb{N}}} = [1,t) \cap q^{\mathbb{N}}$;

$$\mu_q(s) := (q-1)s\,;$$
$$D_q\varphi(t) = \frac{\varphi(qt)-\varphi(t)}{\mu_q(t)}\,, \quad t \in q^{\mathbb{N}}\,;$$

$C\colon [1,\infty)_{q^{\mathbb{N}}} \times [1,\infty)_{q^{\mathbb{N}}} \to \mathbb{R}$ is continuous function for $1 \le s \le t < \infty$, and $A\colon [1,\infty)_{q^{\mathbb{N}}} \to \mathbb{R}$ is also continuous.

Let $A(t) = -\frac{q}{t^3} - \frac{t+1}{t}$, $C(t,s) = \frac{1}{t^2s^2}$, for $(t,s) \in [1,\infty)_{q^{\mathbb{N}}} \times [1,\infty)_{q^{\mathbb{N}}}$. Let $t = q^n$. Since,

$$-\int_{\sigma(s)}^t |C(u,s)|\Delta u \ge -\int_1^t |C(u,s)|\Delta u$$

we have that

$$\begin{aligned}\int_1^t |C(u,s)|\Delta u &= \frac{1}{s^2}\Big\{\int_1^{q^n} \frac{1}{u^2} d_q u\Big\}\\
&= \Big\{\int_{q^0}^{q} \frac{1}{u^2} d_q u + \int_{q}^{q2} \frac{1}{u^2} d_q u + \int_{q^2}^{q^3} \frac{1}{u^2} d_q u + \cdots + \int_{q^{n-1}}^{q^n} \frac{1}{u^2} d_q u\Big\}\\
&= \frac{1}{s^2}\sum_{k=0}^{n-1} \int_{q^k}^{\sigma(q^k)} \frac{1}{u^2} d_q u = \frac{1}{s^2}\sum_{k=0}^{n-1} \mu(q^k)\frac{1}{q^{2k}} = \frac{1}{s^2}\sum_{k=0}^{n-1} (q-1)\,q^k\frac{1}{q^{2k}}\\
&\le \frac{1}{s^2}(q-1)\sum_{k=0}^{\infty}\frac{1}{q^k} = \frac{1}{s^2}q\,.\end{aligned}$$

Thus,

$$|A(s)| - \int_{\sigma(s)}^t |C(u,s)|\Delta u \ge \frac{1}{s^2}q + \frac{s+1}{s} - \frac{1}{s^2}q \ge 1\,.$$

All the conditions of Theorem 4.2 are satisfied with $t_2 = 1$, which implies that the zero solution of (18) is (AS).

After many failed attempts to construct an example as an application for Theorem 4.1, this author concluded that it might be very difficult, or if not impossible to apply the theorem to q-difference equations, due to condition (7) since $\mu_q(t) := (q-1)t$.

6. Comparison

In this section we compare our results to those obtained in [5] and [12]. Unlike the obtained results of [5] and [12], the results of this paper only imply stability and not uniform stability.

Theorem 6.1 ([5, Theorem 5.2])**.** *Suppose that $A(t)$ does not change sign. Then the zero solution of* (1) *is uniformly stable if and only if there exist a constant K such that*

$$A(t) + K \int_{t_0}^{\sigma(t)} |C(t,s)|\Delta s \leq 0\,, \tag{19}$$

and

$$\min\Big\{A(s)\Big(-1+\frac{1}{K}\Big), A(s)\Big(1+\frac{1}{K}\Big)\Big\} > 0\,. \tag{20}$$

Suppose $A(t) < 0$ for all $t \in [0,\infty)_{\mathbb{T}}$. Then from our condition (15) we have that $K = 1$. As a consequence, condition (20) can not hold since

$$\min\Big\{A(s)\Big(-1+\frac{1}{K}\Big), A(s)\Big(1+\frac{1}{K}\Big)\Big\} = 0\,.$$

Remark 6.1. In [12, Example 2.3], when $\mathbb{T} = \mathbb{R}$, the authors considered (1) and assumed the existence of a constant $K > 1$ such that for $t \geq 0$

$$A(t) + K \int_0^t |C(t,s)|\, ds \leq 0\,. \tag{21}$$

The above condition is needed to show uniform stability. Condition (21) imposes size limitation on $|C(t,s)|$. Precisely, we must have

$$\int_0^t |C(t,s)|\, ds \leq -\frac{1}{K}A(t) = \frac{1}{K}|A(t)|\,, \quad \text{for} \quad K > 1\,.$$

On the other hand, our condition (15) is less restrictive since it asks for

$$\int_0^t |C(t,s)|\, ds \leq |A(t)|\,.$$

Next we observe that in [15] the authors obtained exponential decay and boundedness of the solutions. However, in our work we provide necessary and sufficient conditions for the boundedness of solutions and stability of the zero solution. This forced us to require that $A(t) < 0$ and $A(t)$ is bounded for all $t \in [0,\infty)_{\mathbb{T}}$. On the other hand, the results of [15] are remarkable since they yield exponential stability, in the scalar case, with the coefficient $A(t)$ could be changing signs. It is easy, to find examples that will work for this paper but not for [15] and vice versa. The system given by (18) would serve as such an example.

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