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ASYMPTOTIC INTEGRATION OF DIFFERENTIAL EQUATIONS WITH SINGULAR p-LAPLACIAN

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Dedicated to professor Miroslav Bartušek on the occasion of his 70th birthday

Abstract. In this paper we deal with the problem of asymptotic integration of nonlinear differential equations with $p-$Laplacian, where $1 < p < 2$. We prove sufficient conditions under which all solutions of an equation from this class are converging to a linear function as $t \to \infty$.

1. Introduction

In the asymptotic theory of n-th order nonlinear ordinary differential equations

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}) \tag{1}$$

the classical problem is to establish conditions for the existence of a solution which asymptotically behaves as a polynomial of degree $1 \le m \le n-1$ as $t \to \infty$. The first paper concerning this problem was published by D. Caligo [5] in 1941 (see also [1]). He proved a result for that type of a linear second order differential equation. Since then many results concerning this problem for nonlinear differential equations have been proved, e.g. in the papers by D.S. Cohen [6], A. Constantin [7], [9] and [8], F.M. Dannan [10], T. Kusano and W.F. Trench [11] and [12], O. Lipovan [13], O.G. Mustafa, Y.V. Rogovchenko [17], Ch.G. Philos, I.K. Purnaras and P.Ch. Tsamatos [20], Y.V. Rogovchenko [22], S.P. Rogovchenko [21], J. Tong [23], F. Trench [24]. The paper by R.P. Agarwal, S.D. Djebali, T. Moussaoui and O.G. Mustafa [1] surveys the literature concerning the topic in asymptotic integration theory of ordinary differential equations. Several conditions under which all solutions of the one dimensional p-Laplacian equation

$$(|y'|^{p-1}y')' = f(t, y, y')\,, \quad p > 1 \tag{2}$$

behave asymptotically as $a + bt$ as $t \to \infty$ for some real numbers a, b are proved in [16] and some sufficient conditions for the existence of such solutions of the

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equation

$$\big(\Phi(y^{(n)})\big)' = f(t,y)\,, \quad n \ge 1\,, \tag{3}$$

where $\Phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with a locally Lipschitz inverse satisfying $\Phi(0) = 0$ are given in the paper [14]. We remark that in the papers [2], [3], [15] and [19] problems of the global existence, extendability and non-extendability of solutions of systems of equations with p-Laplacian are studied.

In this paper we prove sufficient conditions under which all solutions of a p-Laplace equation behave asymptotically as a linear function for $t \to \infty$. In its proof we apply the Bihari inequality. This technique was applied also in the paper [16] concerning a p-Laplace equation. In some of the above mentioned papers, also in the paper [14] concerning a p-Laplace equation, some results on the existence of solutions behaving like linear functions near the infinity are proved by using the Schauder fixed point theorem.

2. Asymptotic properties of one-dimensional singular p-Laplace equations

Consider the initial problem

$$(Q(t)\Phi_p(u'))' + f(t,u,u') = 0\,, \tag{4}$$

$$u(t_0) = u_0, u'(t_0) = u_1\,, \quad t_0 \ge 1\,, \tag{5}$$

where $\Phi_p(v) = |v|^{p-2}v$, $Q(t)$ is a continuous positive function. If $p > 1$ and $q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\Phi_q(v) = \Phi_p^{-1}(v)$. We need to assume $q > 2$. However in this case $1 < p < 2$ and this means that the p-Laplacian $\Phi_p(v)$ is singular.

Theorem 1. *Let the following conditions be satisfied:*

(C1) $1 < p < 2$*;*

(C2) *There exists a continuous nonnegative function* $h\colon \mathbb{R}_+ = [0,\infty) \to \mathbb{R}$*, continuous positive nondecreasing functions* $g_i\colon \mathbb{R}_+ \to \mathbb{R}$*,* $i = 1,2$ *and a positive number* k *such that*

$$|f(t,u,v)| \le H(t)\Big[g_1\Big(\Big[\frac{|u|}{t}\Big]^k\Big) + g_2(|v|^k)\Big]$$

for all $(t,u,v) \in (0,\infty)\times\mathbb{R}\times\mathbb{R}$*;*

(C3)

$$\int_0^\infty H(s)^{\frac{1}{p-1}}\,ds < \infty\,;$$

(C4)

$$\int_{v_0}^\infty \frac{d\sigma}{g_1(\sigma^k)^{\frac{1}{p-1}} + g_2(\sigma^k)^{\frac{1}{p-1}}} = \frac{1}{k}\int_{v_0^k}^\infty \frac{\tau^{\frac{1}{k}-1}d\tau}{g_1(\tau)^{\frac{1}{p-1}} + g_2(\tau)^{\frac{1}{p-1}}} = \infty\,, \quad v_0 \ge 0\,;$$

(C5) *There exists a constant* $K > 0$ *such that*

$$Q(t) \ge Kt\,, \quad t \ge t_0 \ge 1\,.$$

Then for any solution $u(t)$ of the initial value problem (4), (5) *there exist $a, b \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} |u(t) - (a + bt)| = 0 .$$

Proof. First let us write the equation (4) in the form

$$\big(\Phi_p(h(t)u')\big)' + f(t, u, u') = 0 , \tag{6}$$

where $h(t) = Q(t)^r = Q(t)^{q-1} = Q(t)^{\frac{1}{p-1}}$ $(r = q - 1 = \frac{1}{p-1})$. From condition (C5) it follows that

$$h(t) \ge K^r t^r , \quad t \ge t_0 \ge 1 . \tag{7}$$

If $u(t)$ is a solution of equation (4) satisfying the initial value condition (5), then

$$u'(t) = \frac{1}{h(t)}\Big\{\Phi_q\Big(\Phi_p(h(t_0)u_1) - \int_{t_0}^{t} f(s, u(s), u'(s))ds\Big)\Big\} , \tag{8}$$

$$u(t) = u_0 + \int_{t_0}^{t} \frac{1}{h(\tau)}\Big\{\Phi_q\Big(\Phi_p(h(t_0)u_1) - \int_{t_0}^{\tau} f(s, u(s), u'(s))ds\Big)\Big\}d\tau . \tag{9}$$

Using condition (C5) we obtain

$$\frac{1}{h(t)} = \frac{1}{Q(t)^r} \le L\frac{1}{t^r} , \quad L = \frac{1}{K^r}$$

and

$$|u(t)| \le |u_0|t + L\int_{t_0}^{t} \frac{1}{\tau^r}\Big(|\Phi_p(h(t_0)u_1)| + \int_{t_0}^{\tau} |f(s, u(s), u'(s))|\, ds\Big)^r d\tau .$$

Using the Hölder inequality (with r and $\frac{r}{r-1}$) and the inequality $(a_1 + a_2 + \dots + a_m)^n \le m^{n-1}(a_1^n + a_2^n + \dots + a_m^n)$, $a_1, a_2, \dots, a_m \ge 0$, $n \in \mathbb{N}$, and condition (C2) we obtain for $t \ge t_0 \ge 1$:

$$\begin{aligned}
|u(t)| &\le |u_0|t + L\int_{t_0}^{t} \frac{1}{\tau^r}\Big(2^{r-1}|\Phi_p(h(t_0)u_1)|^r + 2^{r-1}\tau^{r-1}\int_0^{\tau} |f(s, u(s), u'(s))|^r\, ds\Big)\, d\tau \\
&\le |u(t_0)|t + Lt2^{r-1}|\Phi_p(h(t_0)u_1)|^r + L2^{r-1}\int_0^{t}\int_{t_0}^{s} |f(\tau, u(\tau), u'(\tau))|^r d\tau\, ds \\
&\le |u(t_0)|t + Lt2^{r-1}|\Phi_p(h(t_0)u_1)|^r \\
&\quad + L2^{r-1}t\int_{t_0}^{t} H(s)^r\Big(g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^k\Big) + g_2(|u'(s)|^k)\Big)^r ds \\
&\le |u(t_0)|t + Lt2^{r-1}|\Phi_p(h(t_0)u_1)|^r \\
&\quad + L4^{r-1}t\int_{t_0}^{t} H(s)^r\Big(g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^k\Big)^r + g_2(|u'(s)|^k)^r\Big)\, ds .
\end{aligned}$$

This yields

$$\frac{|u(t)|}{t} \le A_1 + B\int_{t_0}^{t} H(s)^r\Big(g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^k\Big)^r + g_2(|u'(s)|^k)^r\Big)\, ds ,$$

where $A_1 = |u(t_0)| + L2^{r-1}|\Phi_p(h(t_0)u_1)|^r$, $B = 4^{r-1}L$. One can show that

$$\frac{|u(t)|}{t} \le z(t)\,, \quad |u'(t)| \le z(t)\,, \tag{10}$$

where

$$z(t) = A + B\int_{t_0}^{t} H(s)^r \Big(g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^k\Big)^r + g_2(|u'(s)|^k)^r\Big)\,ds\,,$$

$A = A_1 + |u_1|$. Since the functions g_1, g_2 are nondecreasing, the inequalities (10) yield

$$z(t) \le A + B\int_{t_0}^{t} H(s)^r \big(g_1(z(s)^k)^r + g_2(z(s)^k)^r\big)\,ds$$

and from the Bihari inequality it follows

$$\Omega(z(t)) \le K_1 := \Omega(A) + B\int_{t_0}^{\infty} H(s)^r\,ds < \infty\,,$$

where

$$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{g_1(\sigma^k)^r + g_2(\sigma^k)^r}\,, \quad r = q-1\,.$$

From inequalities (10) we have

$$\frac{|u(t)|}{t} \le K := \Omega^{-1}(K_1) < \infty\,, \quad |u'(t)| \le K, \quad t \ge t_0\,. \tag{11}$$

Since

$$\int_{t_0}^{t} |f(s,u(s),u'(s))|\,ds \le \int_{t_0}^{t} H(s)\Big(g_1\Big(\Big[\frac{|u(s)|}{s}\Big]^k\Big) + g_2(|u'(s)|^k)\Big)\,ds$$
$$\le z(t) \le K\,, \quad t \ge t_0\,,$$

the integral $\int_{t_0}^{\infty} |f(s,u(s),u'(s))|\,ds$ exists.

From (11) it follows that there exists $a \in \mathbb{R}$ such that

$$\lim_{t\to\infty} u'(t) = a$$

and by using the L'Hospital rule we obtain

$$\lim_{t\to\infty} \frac{|u(t)|}{t} = \lim_{t\to\infty} u'(t) = a\,.$$

Therefore there exist a, $b \in \mathbb{R}$ such that $u(t) = at + b + o(t)$ as $t \to \infty$. □

Example. Let $t_0 = 1$, $1 < p < 2$, $0 < k \le 1$, $H(t)$ be a nonnegative, continuous function on $[0,\infty)$ with $\int_1^{\infty} H(s)^{\frac{1}{p-1}}\,ds < \infty$ and

$$f(t,u,v) = H(t)\Big(u^{\frac{(p-1)(1-k)}{k}} \ln^{p-1} u + v^{\frac{(p-1)(1-k)}{k}}\Big)\,, \quad u, v > 0,\ t \in [0,\infty)\,.$$

If $g_1(u) := u^{\frac{(p-1)(1-k)}{k}} \ln^{p-1} u$, $g_2(v) := v^{\frac{(p-1)(1-k)}{k}}$, $Q(t) := t$, $t \ge 1$, then

$$\int_{v_0^k}^{\infty} \frac{\tau^{\frac{1}{k}-1}\,d\tau}{g_1(\tau)^{p-1} + g_2(\tau)^{p-1}} = \int_{v_0^k}^{\infty} \frac{d\tau}{\ln\tau + \tau} = \infty$$

(see [7]) and thus all conditions of Theorem 1 are satisfied.

Remark 1. Let us define the following classes of functions defined on the region $D \subset (0,\infty)\times\mathbb{R}\times\mathbb{R}$:

$$\mathcal{C}_i = \{f(t,u,v) : f \in C(D) \text{ and satisfies the condition } (Ki)\}\,,\quad i=0,1,2\,,$$

where (K0) is given by the conditions (C2), (C3), (C4) from Theorem 1,

(K1)

$$|f(t,u,v)| \le h_1(t)\Big[g_1\Big(\Big[\frac{|u|}{t}\Big]^k\Big) + h_2(t)g_2(|v|^k) + h_3(t)\Big]$$

for all $(t,u,v)\in(0,\infty)\times\mathbb{R}\times\mathbb{R}$ with

$$\int_0^\infty h_j(s)^{\frac{1}{p-1}}ds < \infty\,,\quad j=1,2,3$$

and

$$\int_{v_0}^\infty \frac{d\sigma}{g_1(\sigma^k)^{\frac{1}{p-1}} + g_2(\sigma^k)^{\frac{1}{p-1}}} = \frac{1}{k}\int_{v_0^k}^\infty \frac{\tau^{\frac{1}{k}-1}d\tau}{g_1(\tau)^{\frac{1}{p-1}} + g_2(\tau)^{\frac{1}{p-1}}} = \infty\,,\quad v_0\ge 0\,;$$

(K2)

$$|f(t,u,v)| \le h_4(t)\Big[g_1\Big(\Big[\frac{|u|}{t}\Big]^k\Big)g_2(|v|^k) + h_5(t)\Big]$$

for all $(t,u,v)\in(0,\infty)\times\mathbb{R}\times\mathbb{R}$ with

$$\int_0^\infty h_j(s)^{\frac{1}{p-1}}ds < \infty\,,\quad j=4,5$$

and

$$\int_{v_0}^\infty \frac{d\sigma}{g_1(\sigma^k)^{\frac{1}{p-1}} g_2(\sigma^k)^{\frac{1}{p-1}}} = \frac{1}{k}\int_{v_0^k}^\infty \frac{\tau^{\frac{1}{k}-1}d\tau}{g_1(\tau)^{\frac{1}{p-1}} g_2(\tau)^{\frac{1}{p-1}}} = \infty\,,\quad v_0\ge 0\,.$$

Proposition 2. *It holds*

$$\mathcal{C}_1\subset\mathcal{C}_0\,,\quad \mathcal{C}_2\subset\mathcal{C}_0\,.$$

This proposition is a corollary of Proposition 2 from [18]. If we substitute conditions (K1) or (K2) instead of conditions (C1), (C2), (C3) in Theorem 1 we obtain results which are corollaries of Theorem 1. This type of results with these classes of nonlinearities are proved in [22], [21] and also in [16], separately.

Remark 2. Since we study equation (6) with $1<p<2$ we need condition (C5). This condition is not necessary in the case studied in [16].

Theorem 3. *Let conditions* (C1)–(C5) *of Theorem 1 be satisfied. Then any solution* $u\colon [0,T)\to\mathbb{R}$ *with* $0<T<\infty$ *can be extended to the right beyond* T.

Proof. Let $u\colon[0,T)\to\mathbb{R}$ be a solution of equation (4) with $0<T<\infty$ satisfying the initial value condition (5), which cannot be extended to the right beyond T. Then $\lim_{t\to T^-}|u(t)|=\infty$. However from inequality (10) we have

$$|u(t)| \le t|z(t)|\,,\quad t\ge 1\,, \tag{12}$$

where

$$z(t) \le A + B\int_{t_0}^{t} H(s)^r \left(g_1(z(s)^k)^r + g_2(z(s)^k)^r\right)\, ds\,, \tag{13}$$

and by applying the Bihari inequality we obtain that $|z(t)| \le K$ for all $t \in [1,\infty)$, where $K > 0$ is a constant. However from the inequality (12) we have $|u(t)| \le TK$ for all $t \in [1,\infty)$ and it is a contradiction. □

Theorem 4. *Let conditions* (C1)–(C4) *of Theorem 1 be satisfied and suppose that there exists a solution* $u\colon [1,T) \to \mathbb{R}$ *of equation* (4) *with* $0 < T < \infty$ *which cannot be extended to the right of* T*. Then* $G(+\infty) < \infty$*, where*

$$G(v) = \int_{v_0}^{v} \frac{d\sigma}{g_1(\sigma^k)^{\frac{1}{p-1}} + g_2(\sigma^k)^{\frac{1}{p-1}}}\,, \quad v \ge v_0 \ge 0\,.$$

This theorem can be proved by a modification of the procedure used in the proof of Lemma 3.6 from [18].

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