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PSEUDOSYMMETRIC AND WEYL-PSEUDOSYMMETRIC (κ,μ)-CONTACT METRIC MANIFOLDS

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Abstract. In this paper we classify pseudosymmetric and Ricci-pseudosymmetric (κ,μ)-contact metric manifolds in the sense of Deszcz. Next we characterize Weyl-pseudosymmetric (κ,μ)-contact metric manifolds.

1. Introduction

Chaki [5] and Deszcz [11] introduced two different concept of a pseudosymmetric manifold. In both senses various properties of pseudosymmetric manifolds have been studied ([5]–[10]). We shall study properties of pseudosymmetric, Ricci-pseudosymmetric and Weyl-pseudostymmetric manifolds in the sense of Deszcz.

A Riemannian manifold is called semisymmetric if $R(X,Y)\cdot R=0$ where $X,\ Y\in\chi(M)$, [24]. Deszcz [11] generalized the concept of semisymmetry and introduced pseudosymmetric manifolds. Let (M^n,g), $n\geq 3$ be a Riemannian manifold. We denote by ∇, R and τ the Levi–Civita connection, the curvature tensor and the scalar curvature of (M,g), respectively. We define endomorphism $X\wedge Y$ for arbitrary vector field Z, $(0,k)$-tensor T and $(1,k)$-tensor T_1, $k\geq 1$, by

$$(X\wedge Y)Z=g(Y,Z)X-g(X,Z)Y\,, \tag{1}$$

$$\begin{aligned}\big((X\wedge Y)\cdot T\big)(X_1,X_2,\dots,X_k)&=-T\big((X\wedge Y)X_1,X_2,\dots,X_k\big)\\&\quad-\dots-T\big(X_1,\dots,X_{k-1},(X\wedge Y)X_k\big)\,,\end{aligned}\tag{2}$$

and

$$\begin{aligned}\big((X\wedge Y)\cdot T_1\big)(X_1,X_2,\dots,X_k)&=(X\wedge Y)T_1(X_1,X_2,\dots,X_k)\\&\quad-T_1\big((X\wedge Y)X_1,X_2,\dots,X_k\big)\\&\quad-\dots-T_1(X_1,\dots,X_{k-1},(X\wedge Y)X_k)\,,\end{aligned}\tag{3}$$

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respectively. For a $(0,k)$-tensor field T, the $(0,k+2)$ tensor fields $R\cdot T$ and $Q(g,T)$ are defined by ([1], [11])

$$\begin{aligned}(R\cdot T)(X_1,\dots,X_k;X,Y) &= \big(R(X,Y)\cdot T\big)(X_1,\dots,X_k)\\ &= -T\big(R(X,Y)X_1,X_2,\dots,X_k\big)\\ &\quad -\dots - T\big(X_1,\dots,X_{k-1},R(X,Y)X_k\big)\,,\end{aligned}\tag{4}$$

and

$$\begin{aligned}Q(g,T)(X_1,\dots,X_k;X,Y) &= -T\big((X\wedge Y)X_1,X_2,\dots,X_k\big)\\ &\quad -\dots - T\big(X_1,\dots,X_{k-1},(X\wedge Y)X_k\big)\,.\end{aligned}\tag{5}$$

A Riemannian manifold M is said to be pseudosymmetric if the tensors $R\cdot R$ and $Q(g,R)$ are linearly dependent at every point of M, i.e.

$$R\cdot R = L_R Q(g,R)\,.\tag{6}$$

This is equivalent to

$$\big(R(X,Y)\cdot R\big)(U,V,W) = L_R\big[\big((X\wedge Y)\cdot R\big)(U,V,W)\big]\tag{7}$$

holding on the set $U_R=\{x\in M : Q(g,R)\neq 0 \text{ at } x\}$, where L_R is some function on U_R, [11]. The manifold M is called pseudosymmetric of constant type if L is constant. Particularly if $L_R=0$ then M is a semisymmetric manifold. The manifold M is said to be locally symmetric if $\nabla R=0$. Obviously locally symmetric spaces are semisymmetric, [25].

Let S denote the Ricci tensor of M^{2n+1}. The Ricci operator Q is the symmetric endomorphism on the tangent space given by

$$S(X,Y) = g(QX,Y)\,.\tag{8}$$

If the tensors $R\cdot S$ and $Q(g,S)$ are linearly dependent at every point of M, i.e.

$$R\cdot S = L_S Q(g,S)\,,\tag{9}$$

then M is called Ricci-pseudosymmetric. This is equivalent to

$$\big(R(X,Y)\cdot S\big)(Z,W) = L_S\big[\big((X\wedge Y)\cdot S\big)(Z,W)\big]\tag{10}$$

holds on the set $U_S=\{x\in M : S-\frac{\tau}{n}g\neq 0 \text{ at } x\}$, for some function L_S on U_S ([7], [19]). We note that $U_S\subset U_R$ and on 3-dimensional Riemannian manifolds we have $U_S=U_R$. Every pseudosymmetric manifold is Ricci-pseudosymmetric but the converse statement is not true.

The Weyl conformal curvature operator C is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\Big\{(X\wedge QY)Z + (QX\wedge Y)Z - \frac{\tau}{2n}(X\wedge Y)Z\Big\}\,.\tag{11}$$

If $C=0$, $n\geq 3$, then M is called conformally flat. If the tensors $R\cdot C$ and $Q(g,C)$ are linearly dependent, then M is called Weyl-pseudosymmetric. This is equivalent to the statement that

$$(R\cdot C)(U,V,W,X,Y) = L_C\big[\big((X\wedge Y)\cdot C\big)(U,V)W\big]$$

holds on the set $U_C = \{x \in M : C \neq 0 \text{ at } x\}$, where L_C is defined on U_C. If $R \cdot C = 0$, then M is called Weyl-semisymmetric. If $\nabla C = 0$, then M is called conformally symmetric ([21], [23]).

3-dimensional pseudosymmetric spaces of constant type have been studied by Kowalski and Sekizawa ([16]–[17]). Conformally flat pseudosymmetric spaces of constant type were classified by Hashimoto and Sekizawa for dimension three, [14] and by Calvaruso for dimensions > 2, [4]. In dimension three, Cho and Inoguchi studied pseudosymmetric contact homogeneous manifolds, [6]. Cho et al. treated the conditions that 3-dimensional trans-Sasakians, non-Sasakian generalized (κ, μ)-spaces and quasi-Sasakians manifolds be pseudosymmetric, [1]. Belkhelfa et al. obtained some results on pseudosymmetric Sasakian space forms, [1]. Finally some classes of pseudosymmetric contact metric 3-manifolds have been studied by Gouli-Andreou and Moutafi ([12], [13]).

Papantoniou classified semisymmetric (κ, μ)-contact metric manifolds ([22, Theorem 3.4]). As a generalization, in this paper, we study pseudosymmetric (κ, μ)-contact metric manifolds.

This paper is organized as follows. After some preliminaries on (κ, μ)-contact metric manifolds, in Section 3 we study pseudosymmetric and Ricci-pseudosymmetric (κ, μ)-contact metric manifolds. Next in Section 4, we characterize Weyl-pseudosymmetric (κ, μ)-contact metric manifolds.

2. Preliminaries

A contact manifold is an odd-dimensional C^∞ manifold M^{2n+1} equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Since $d\eta$ is of rank $2n$, there exists a unique vector field ξ on M^{2n+1} satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any $X \in \chi(M)$ is called the Reeb vector field or characteristic vector field of η. A Riemannian metric g is said to be an associated metric if there exists a (1,1) tensor field φ such that

$$d\eta(X, Y) = g(X, \varphi Y), \qquad \eta(X) = g(X, \xi), \qquad \varphi^2 = -I + \eta \otimes \xi\,.$$

The structure (φ, ξ, η, g) is called a contact metric structure and a manifold M^{2n+1} with a contact metric structure is said to be a contact metric manifold. Given a contact metric structure (φ, ξ, η, g), we define a $(1,1)$ tensor field h by $h = (1/2)\mathcal{L}_\xi \varphi$ where $\mathcal{L}$ denotes the operator of Lie differentiation. A contact metric manifold for which ξ is a Killing vector field is called a K-contact manifold. It is well known that a contact manifold is K-contact if and only if $h = 0$. A contact metric manifold is said to be a Sasakian manifold if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$$

in which case

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y\,. \tag{12}$$

Note that a Sasakian manifold is K-contact, but the converse holds only if $\dim M = 3$.

A contact manifold is said to be η-Einstein if the Ricci operator Q satisfies the condition

$$Q = a\,\mathrm{Id} + b\eta \otimes \xi\,, \tag{13}$$

where a and b are smooth functions on M^{2n+1}.

The sectional curvature $K(\xi, X)$ of a plane section spanned by ξ and a vector X orthogonal to ξ is called a ξ-sectional curvature, while the sectional curvature $K(X, \varphi X)$ is called a φ-sectional curvature.

The (κ, μ)-nullity distribution of a contact metric manifold $M(\varphi, \xi, \eta, g)$ is a distribution, [3]

$$\begin{aligned} N(\kappa,\mu)\colon p \to N_p(\kappa,\mu) = \big\{ &W \in T_pM \mid R(X,Y)W \\ &= \kappa[g(Y,W)X - g(X,W)Y] + \mu[g(Y,W)hX - g(X,W)hY]\big\}\,, \end{aligned}$$

where κ, μ are real constants. Hence if the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution, then we have

$$R(X,Y)\xi = \kappa\big\{\eta(Y)X - \eta(X)Y\big\} + \mu\big\{\eta(Y)hX - \eta(X)hY\big\}\,. \tag{14}$$

A contact metric manifold satisfying (14) is called a (κ, μ)-contact metric manifold. If M be a (κ, μ)-contact metric manifold, then the following relations hold, [3]:

$$S(X,\xi) = 2nk\eta(X)\,, \tag{15}$$

$$Q\xi = 2nk\xi\,, \tag{16}$$

$$h^2 = (k-1)\varphi^2\,, \tag{17}$$

$$R(\xi,X)Y = \kappa\{g(X,Y)\xi - \eta(Y)X\} + \mu\{g(hX,Y)\xi - \eta(Y)hX\}\,, \tag{18}$$

$$\begin{aligned} S(X,Y) &= [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ &\quad + [2(1-n) + n(2\kappa+\mu)]\eta(X)\eta(Y)\,, \end{aligned} \tag{19}$$

$$\tau = 2n(2(n-1) + \kappa - n\mu)\,, \tag{20}$$

$$Q\varphi - \varphi Q = 2[2(n-1) + \mu]h\varphi\,. \tag{21}$$

We note that if M^{2n+1} be a (κ, μ)-contact metric manifold, then $\kappa \leq 1$, [3]. When $\kappa < 1$, the nonzero eigenvalues of h are $\pm\sqrt{1-\kappa}$ each with multiplicity n. Let λ and D denote the positive eigenvalue of h and the distribution $\operatorname{Ker}\eta$ respectively. Then M^{2n+1} admits three mutually orthogonal and integrable distributions $D(0), D(\lambda)$ and $D(-\lambda)$ defined by the eigenspaces of h, [26]. We easily check that Sasakian manifolds are contact (κ, μ)-manifolds with $\kappa = 1$ and $h = 0$, [3]. In particular, if $\mu = 0$, then we obtain the condition of k-nullity distribution introduced by Tanno, [26].

3. Pseudosymmetric and Ricci-pseudosymmetric (κ,μ)-manifolds

We know that [2] if M^{2n+1} be a contact metric manifold and $R_{XY}\xi = 0$ for all vector fields X and Y, then M^{2n+1} is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant curvature 4.

In [3] Blair et al. studied the condition of (κ,μ)-nullity distribution on a contact manifold and obtained the following theorem.

Theorem 1. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a contact manifold with ξ belonging to the (κ,μ)-nullity distribution. If $\kappa<1$, then for any X orthogonal to ξ the following formulas hold:*

1. *The ξ-sectional curvature $K(X,\xi)$ is given by*

$$K(X,\xi)=\kappa+\mu g(hX,X)=\begin{cases}\kappa+\lambda\mu & \text{if}\quad X\in D(\lambda)\\ \kappa+\lambda\mu & \text{if}\quad X\in D(-\lambda)\end{cases}$$

2. *The sectional curvature of a plan section $\{X,Y\}$ normal to ξ is given by*

$$K(X,Y)=\begin{cases}\text{i)}\quad 2(1+\lambda)-\mu & \text{if}\quad X,Y\in D(\lambda)\\ \text{ii)}\quad -(\kappa+\mu)[g(X,\varphi Y)]^2 & \text{for any unit vectors}\\ & X\in D(\lambda), Y\in D(-\lambda)\\ \text{iii)}\quad 2(1-\lambda)-\mu & \text{if}\quad X,Y\in D(-\lambda),\ n>1\,.\end{cases}\tag{22}$$

Pseudosymmetric contact 3-manifold were studied in [6] and following result obtained.

Theorem 2. *Contact Riemannian 3-manifolds such that $Q\varphi=\varphi Q$ are pseudo-symmetric. In particular, every Sasakian 3-manifold is a pseudosymmetric space of constant type.*

Firstly we give the following propositions.

Proposition 1. *Let M^{2n+1} be a (κ,μ)-contact metric pseudosymmetric manifold. Then for any unit vector fields X, $Y\in\chi(M)$ orthogonal to ξ and such that $g(X,Y)=0$ we have:*

$$\begin{aligned}&\big\{(\kappa-L_R)g\big(X,R(X,Y)Y\big)+\mu g\big(hX,R(X,Y)Y\big)-\kappa(\kappa-L_R)\\&\quad-\mu(\kappa-L_R)g(hY,Y)-\kappa\mu g(hX,X)-\mu^2g(hX,X)g(hY,Y)\\&\quad+\mu^2g^2(hX,Y)\big\}\xi\\&\quad-(\kappa-L_R)g(R(X,Y)Y,\xi)X-\mu g(R(X,Y)Y,\xi)hX=0\,.\end{aligned}\tag{23}$$

Proof. Since M is pseudosymmetric then

$$\big(R(\xi,X)\cdot R\big)(U,V)W=L_R\big[\big((\xi\wedge X)\cdot R\big)(U,V)W\big]\,.\tag{24}$$

Putting $U = X$ and $V = W = Y$ in (24) and using (3) and (4), we get

$$\begin{aligned} R(\xi, X) \cdot R(X,Y)Y - R(R_{\xi X}X,Y)Y - R(X, R_{\xi X}Y)Y - R(X,Y)R_{\xi X}Y \\ = L_R\{(\xi \wedge X) \cdot R(X,Y)Y - R\big((\xi \wedge X)X,Y\big)Y \\ - R\big(X,(\xi \wedge X)Y\big)Y - R(X,Y)\big((\xi \wedge X)Y\big)\}e\,. \end{aligned} \tag{25}$$

From (1) and (18) one can easily get the result. □

Proposition 2. *Every pseudosymmetric Sasakian manifold with $L_R \neq 1$ is of constant curvature* 1.

Proof. Let X and Y be tangent vectors such that $\eta(X) = \eta(Y) = 0$ and $g(X,Y) = 0$. Since M is Sasakian then $\kappa = 1$ and $h = 0$. Using (12) and (18) in equation (25) and direct computations we get

$$(1 - L_R)\{\eta\big(R(X,Y)Y\big)X - g\big(X, R(X,Y)Y\big)\xi + g(X,X)g(Y,Y)\xi\} = 0\,.$$

Since $L_R \neq 1$ then

$$\eta\big(R(X,Y)Y\big)X - g\big(X, R(X,Y)Y\big)\xi + g(X,X)g(Y,Y)\xi = 0\,. \tag{26}$$

Taking the inner product with ξ gives

$$g\big(X, R(X,Y)Y\big) = g(X,X)g(Y,Y)\,. \tag{27}$$

Then (M^{2n+1}, g) is of constant φ-sectional curvature 1 and hence it is of constant curvature 1, [19]. □

Theorem 3. *Let M^{2n+1}, $n > 1$ be a (κ,μ)-contact metric pseudosymmetric manifold. Then M^{2n+1} is either*

1) *A Sasakian manifold of constant sectional curvature* 1 *if* $L_R \neq 1$ *or*
2) *Locally isometric to the product of a flat $(n+1)$-dimensional Euclidean manifold and an n-dimensional manifold of constant curvature* 4.

Proof. If $\kappa = 1$ then M is a Sasakian manifold and result get from Proposition 2. Let $\kappa < 1$ and X, Y are orthonormal vectors of the distribution $D(\lambda)$. Applying the relation (23) for $hX = \lambda X$, $hY = \lambda Y$ we get

$$\begin{aligned} \{(\kappa - L_R + \mu\lambda)g\big(X, R(X,Y)Y\big) - \kappa(\kappa - L_R) - \mu\lambda(\kappa - L_R) - \kappa\mu\lambda - \mu^2\lambda^2\}\xi \\ - (\kappa - L_R + \mu\lambda)g\big(R(X,Y)Y, \xi\big)X = 0\,. \end{aligned} \tag{28}$$

Considering ξ-component of (28) gives

$$\text{i) } K(X,Y) = \kappa + \lambda\mu \quad \text{or} \quad \text{ii) } \kappa = -\lambda\mu + L_R\,. \tag{29}$$

Comparing part (i) of equations (22) and (29) gives

$$\mu = 1 + \lambda\,. \tag{30}$$

Let X, $Y \in D(-\lambda)$ and $g(X,Y) = 0$. Putting $hX = -\lambda X$ and $hY = -\lambda Y$ in (23) and taking the inner product with ξ we get

$$\text{i) } K(X,Y) = \kappa - \lambda\mu \quad \text{or} \quad \text{ii) } \kappa = \lambda\mu + L_R\,. \tag{31}$$

Comparing the equations (22)(iii) and (31)(i) we have

$$\text{i) } \mu = 1 - \lambda \quad \text{or} \quad \text{ii) } \lambda = 1\,. \tag{32}$$

In the case $X \in D(\lambda)$ and $Y \in D(-\lambda)$ equation (23) is reduced to

$$\begin{aligned}&\{(\kappa - L_R + \mu\lambda)g(X, R(X,Y)Y) - \kappa(\kappa - L_R) + \mu\lambda(\kappa - L_R) - \kappa\mu\lambda + \mu^2\lambda^2\}\xi \\ &- (\kappa - L_R + \mu\lambda)g(R(X,Y)Y,\xi)X = 0\,,\end{aligned} \tag{33}$$

from which taking the inner products with ξ we have

$$\text{i) } K(X,Y) = \kappa - \lambda\mu \quad \text{or} \quad \kappa = -\lambda\mu + L_R\,, \tag{34}$$

while if $X \in D(-\lambda)$ and $Y \in D(\lambda)$ we similarly prove that

$$\text{i) } K(X,Y) = \kappa + \lambda\mu \quad \text{or} \quad \kappa = \lambda\mu + L_R\,. \tag{35}$$

By the combination now of the equation (29)(ii), (30), (31)(ii), (32), (34) and (35) we establish the following nine systems among the unknowns κ, λ, μ and L_R.

1) $\{\mu = 1-\lambda,\ \mu = 1+\lambda,\ \lambda = 0\}$
2) $\{\kappa = -\lambda\mu + L_R,\ \kappa = \lambda\mu + L_R,\ \mu = 0,\ \lambda > 0\}$
3) $\{\kappa = -\lambda\mu + L_R,\ \lambda = 1,\ \mu = 0\}$
4) $\{\kappa = -\lambda\mu + L_R,\ \lambda = 1,\ \mu = L_R\}$
5) $\{K(X,Y) = \kappa + \lambda\mu,\ K(X,Y) = \kappa - \lambda\mu,\ \mu = 1-\lambda,\ \kappa = -\lambda\mu + L_R\}$
6) $\{\mu = 1+\lambda,\ \lambda = 1,\ L_R = \pm 2\}$
7) $\{\mu = 1+\lambda,\ K(X,Y) = \kappa - \lambda\mu,\ K(X,Y) = \kappa + \lambda\mu\}$
8) $\{\kappa = -\lambda\mu + L_R,\ \mu = 1-\lambda,\ K(X,Y) = \kappa + \lambda\mu\}$
9) $\{\mu = 1+\lambda,\ \kappa = \lambda\mu + L_R,\ K(X,Y) = \kappa - \lambda\mu\}$

From the first system we get easily $\mu = 1$ and since $\lambda^2 = 1-\kappa$ we have $\kappa = 1$, which is a contradiction, since we required that $\kappa < 1$.

The systems 2, 3, 4 and 5 have as the only solution $\kappa = 0,\ \mu = 0, \lambda = 1,\ L_R = 0$. Then $R_{XY}\xi = 0$ for any $X,\, Y \in \chi(M)$ and M is locally isometric to the product $E^{n+1}(0) \times S^n(4)$, [2]. We show that remainder systems can not occur.

In system 6, from $\lambda = 1$ we have $\mu = 0$ and $\kappa = 0$. Using equation (34) (or (35)) and (22)(ii) we have $[g(X,\varphi Y)]^2 = -1$ and this is a contradiction.

From system 7, one can get easily $\lambda\mu = 0$. But $\lambda \neq 0$ (since $\kappa < 1$) and then $\mu = 0$. Therefore $\lambda = \mu - 1 = -1$ and this is a contradiction with $\lambda > 0$.

In two last systems for all $X,\, Y \in \chi(M)$ we have

$$K(X,Y) = L_R\,. \tag{36}$$

Let $Y = \varphi X$ in (36) and comparing it with equation (22)(ii) we get

$$L_R = -(\kappa + \mu)\,, \tag{37}$$

Replacing κ and μ of two last systems in (37) we get two equation

$$(1-\lambda)^2 = -2L_R\,, \tag{38}$$

and

$$(1+\lambda)^2 = -2L_R\,, \tag{39}$$

respectively. Then in systems 8 and 9 $L_R \leq 0$.

In system 8, by virtue of $\kappa = -\lambda\mu + L_R$ and $\kappa = 1-\lambda^2$, we have

$$2\lambda^2 - \lambda + (L_R - 1) = 0\,.$$

This quadratic equation has two roots $\lambda = 1 \pm \sqrt{9-8L_R}$. If $\lambda = 1+\sqrt{9-8L_R}$ and replacing it in (38) we get $L_R = 1.5$ and if $\lambda = 1-\sqrt{9-8L_R}$, since λ is positive, we get $L_R > 1$. Then in the both case we get contradiction whit $L_R \leq 0$. The roots of equation (39) in last system are $\lambda = -1 \pm \sqrt{-2L_R}$ and since $\lambda > 0$ then $\lambda = -1+\sqrt{-2L_R}$ and hence $\mu = \sqrt{-2L_R}$. Substituting λ and μ in $\kappa = \lambda\mu + L_R$ and $\kappa = 1-\lambda^2$ we get $L_R = -2$ and then $\lambda = 1, \mu = 2$ and $\kappa = 0$ which are not acceptable since from (34) (or (35)) we get a contradiction from (22)(ii) and this complete the proof. $\square$

Theorem 4. *Every 3-dimensional (κ,μ)-contact metric manifold is pseudosymmetric manifold.*

Proof. From the combination of the equations (34) and (35) we get four systems with respect to the κ, λ, μ , L_R and the sectional curvature $K(X,Y)$, from which we have the following possibilities:

1) $K(X,Y) = \kappa$, $\lambda\mu = 0$,
2) $\kappa = L_R$, $\lambda\mu = 0$,
3) $\kappa = \lambda\mu + L_R$ or $\kappa = \lambda\mu - L_R$ and $K(X,Y) = L_R$.

In two first cases we have $\lambda\mu = 0$. If $\mu = 0$ then equation (21) leads to $Q\varphi = \varphi Q$ and result get from Theorem 2. If $\lambda = 0$ then M^3 being a Sasakian manifold and from Theorem 2 every Sasakian 3-manifold is a pseudosymmetric space of constant type.

In the last case, let $Y = \varphi X$ then $K(X,\varphi X) = L_R$. On the other hand, from (22)(ii) $K(X,\varphi X) = -(\kappa+\mu)$. Then $L_R = -(\kappa+\mu)$ and manifold is of constant sectional curvature. Every Riemannian manifold of constant sectional curvature is locally symmetric ([20] page 221) and then pseudosymmetric. Thus M^3 is pseudosymmetric manifold of constant type. $\square$

Theorem 5. *Let M^{2n+1} be a Ricci-pseudosymmetric (κ,μ)-contact metric manifold. Then M^{2n+1} is either*

(i) *locally isometric to $E^{n+1} \times S^n(4)$, or*
(ii) *an Einstein-Sasakian manifold if $\kappa \neq L_S$, or*
(iii) *an η-Einstein manifold provided*
$2n\kappa\mu - (\kappa - L_S)[2(n-1)+\mu] - \mu[2(n-1)-n\mu] \neq 0.$

Proof. (i) If $\kappa = 0, \mu = 0$ then we have $R_{XY}\xi = 0$ for any tangent vector fields X, Y and hence M is locally isometric to $E^{n+1} \times S^n(4)$, [2].

(ii) Let $\kappa \neq 0$.

Since M is a Ricci-pseudosymmetric (κ,μ)-contact metric manifold for any $X, Y, U, V \in \chi(M)$ we have

$$(R(X,Y)\cdot S)(U,V) = L_S Q(g,S)(U,V;X,Y)\,. \tag{40}$$

Then from (4) and (5) we can write

$$-S\big(R(\xi,X)Y,Z\big) - S\big(Y,R(\xi,X)Z\big) = L_S\big[-S\big((\xi\wedge X)Y,Z\big) - S\big(Y,(\xi\wedge X)Z\big)\,. \tag{41}$$

Replacing Z with ξ and using (1), (15) and (14) one can get

$$-2n\kappa(\kappa-L_S)g(X,Y) - 2n\kappa\mu g(hX,Y) + (\kappa-L_S)S(X,Y) + \mu S(hX,Y) = 0. \tag{42}$$

If $\mu=0$ then since $\kappa\neq 0, L_S$, we get that the manifold is Einstein and then M is a Sasakian manifold ([26] Theorem 5.2).

(iii) Suppose now that $\kappa\neq 0, \mu\neq 0$. Then, using the equation (19) and (17), $\kappa\leq 1$, we have

$$\begin{aligned}S(hX,Y)&=[2(n-1)-n\mu]g(hX,Y)-(\kappa-1)[2(n-1)+\mu]g(X,Y)\\ &\quad+(\kappa-1)[2(n-1)+\mu]\eta(X)\eta(Y)\,.\end{aligned}\tag{43}$$

Replacing equation (43) in equation (42) gives

$$\begin{aligned}&\{2n\kappa\mu-(\kappa-L_S)[2(n-1)+\mu]-\mu[2(n-1)-n\mu]\}\,g(hX,Y)\\ &=\{-2n\kappa(\kappa-L_S)+(\kappa-L_S)[2(n-1)-n\mu]-\mu(\kappa-1)[2(n-1)+\mu]\}\,g(X,Y)\\ &\quad+\{(\kappa-L_S)[2(1-n)+n(2\kappa+\mu)]+\mu(\kappa-1)[2(n-1)+\mu]\}\,\eta(X)\eta(Y)\,.\end{aligned}\tag{44}$$

From (19) and (44), we get

$$S(X,Y)=\alpha g(X,Y)+\beta\eta(X)\eta(Y)$$

where

$$\alpha=\frac{[2(n-1)+\mu][-2n\kappa(\kappa-L_S)+(\kappa-L_S)[2(n-1)-n\mu]-\mu(\kappa-1)(2(n-1)+\mu)]}{2n\kappa\mu-(\kappa-L_S)[2(n-1)+\mu]-\mu[2(n-1)-n\mu]}+[2(n-1)-\mu n]\,.$$

$$\beta=\frac{[2(n-1)+\mu][(\kappa-L_S)[2(1-n)+n(2\kappa+\mu)+\mu(\kappa-1)(2(n-1)+\mu)]}{2n\kappa\mu-(\kappa-L_S)[2(n-1)+\mu]-\mu[2(n-1)-n\mu]}+[2(1-n)+n(2\kappa+\mu)]\,.$$

So, the manifold is an η-Einstein manifold with constant coefficients and the proof is complete. ☐

4. Weyl-pseudosymmetric (κ,μ)-contact metric manifolds

In the present section our aim is to find the characterization of (κ,μ)-contact metric manifolds satisfying the condition $R\cdot C=L_CQ(g,C)$.

Theorem 6. *Let M^{2n+1}, $n>1$ be a non-Sasakian (κ,μ)-contact metric manifold. If M is Weyl-pseudosymmetric manifold then either $\mu=0$ and then $L_C=\kappa$ or $\mu=\frac{2n-1}{2n-2}$ holds on M.*

Proof. Since M is a Weyl-pseudosymmetric then

$$\big(R(X,Y)\cdot C\big)(U,V,W)=L_CQ(g,C)(U,V,W;X,Y)\,.\tag{45}$$

Using (4) and (5) in (45) we can write

$$\begin{aligned} R(X,Y)C(U,V)W - C\big(R(X,Y)U,V\big)W - C\big(U,R(X,Y)V\big)W \\ - C(U,V)R(X,Y)W \\ = L_C\big[(X\wedge Y)C(U,V)W - C\big((X\wedge Y)U,V\big)W \\ - C\big(U,(X\wedge Y)V\big)W - C(U,V)(X\wedge Y)W\big]\,. \end{aligned} \tag{46}$$

Replacing X with ξ and Y with U in (46) we have

$$\begin{aligned} R(\xi,U)C(U,V)W - C\big(R(\xi,U)U,V\big)W - C\big(U,R(\xi,U)V\big)W \\ - C(U,V)R(\xi,U)W \\ = L_C\big[(\xi\wedge U)C(U,V)W - C\big((\xi\wedge U)U,V\big)W \\ - C\big(U,(\xi\wedge U)V\big)W - C(U,V)(\xi\wedge U)W\big]\,. \end{aligned} \tag{47}$$

Substituting (1) and (18) in (47) and taking the inner product with ξ, we get

$$\begin{aligned} (\kappa - L_C)g\big(U,C(U,V)\,W\big) + \mu g\big(hU,C(U,V)W\big) - (\kappa - L_C)g(U,U)g\big(C(\xi,V)W,\xi\big) \\ - \mu g(hU,U)g\big(C(\xi,V)W,\xi\big) + \mu\eta(U)g\big(C(hU,V)W,\xi\big) \\ - (\kappa - L_C)g(U,V)g\big(C(U,\xi)W,\xi\big) - \mu g(hU,V)g\big(C(U,\xi)W,\xi\big) \\ + \mu\eta(V)g\big(C(U,hU)W,\xi\big) + (\kappa - L_C)\eta(W)g\big(C(U,V)U,\xi\big) \\ + \mu\eta(W)g\big(C(U,V)hU,\xi\big) = 0\,. \end{aligned} \tag{48}$$

Let $U \in D(\lambda)$ and contraction of (48) with respect to U we have

$$\big(-2n\kappa + (1-2n)\lambda\mu + 2nL_C\big)g\big(C(\xi,V)W,\xi\big) = 0\,. \tag{49}$$

Similarity for $U \in D(-\lambda)$ and contraction of (48) with respect to U we get

$$\big(-2n\kappa - (1-2n)\lambda\mu + 2nL_C\big)g\big(C(\xi,V)W,\xi\big) = 0\,. \tag{50}$$

Suppose $\mu = 0$. Then from the equation (49) we obtain

$$(L_C - \kappa)g\big(C(\xi,V)W,\xi\big) = 0\,. \tag{51}$$

If $g(C(\xi,V)W,\xi) = 0$. Using (20), (11) and straightforward computation, we have

$$\begin{aligned} S(X,Y) = \big[2(n-1) - n\mu\big]g(X,Y) + \big[2(n-1)\mu\big]g(hX,Y) \\ + \big[2(1-n) + n(2\kappa+\mu)\big]\eta(X)\eta(Y)\,. \end{aligned} \tag{52}$$

Comparing equation (52) with (19) one can get

$$\mu = \frac{2n-1}{2n-2} \tag{53}$$

and this is a contradiction. Then $\kappa = L_C$.

Suppose now that $\mu \neq 0$ and substracting equations (49) and (50), we get

$$\lambda\mu g\big(C(\xi,V)W,\xi\big) = 0\,. \tag{54}$$

But $\lambda\mu \neq 0$ since $\kappa < 1$ and $\mu \neq 0$. Hence $g(C(\xi,V)W,\xi) = 0$ and then $\mu = \dfrac{2n-1}{2n-2}$. □

Therefore we have the following corollary.

Corollary 1. *If M be a Weyl-pseudosymmetric Sasakian manifold then either $L_C = 1$ or $\mu = \dfrac{2n-1}{2n-2}$ holds on M.*

Proof. Since M is Sasakian then $\kappa = 1$ and $\lambda = 0$. From equation (49) one can easily get the results. □

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