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BAUTIN BIFURCATION OF A MODIFIED GENERALIZED VAN DER POL-MATHIEU EQUATION

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Abstract. The modified generalized Van der Pol-Mathieu equation is generalization of the equation that is investigated by authors Momeni et al. (2007), Veerman and Verhulst (2009) and Kadeřábek (2012). In this article the Bautin bifurcation of the autonomous system associated with the modified generalized Van der Pol-Mathieu equation has been proved. The existence of limit cycles is studied and the Lyapunov quantities of the autonomous system associated with the modified Van der Pol-Mathieu equation are computed.

1. Introduction

The Van der Pol-Mathieu equation

$$\frac{d^2x}{dt^2} - \varepsilon(\alpha_0 - \beta_0 x^2)\frac{dx}{dt} + \omega_0^2(1 + \varepsilon h_0 \cos\gamma t)x = 0 \tag{1}$$

with a small detuning parameter ε describes the dynamics of dusty plasmas and it has recently been studied near 1: 2 resonance in [5]. In [7], using the averaging method, the existence of stable periodic and stable quasi-periodic solutions near the parametric frequency is proved. The autonomous system derived from Van der Pol-Mathieu equation is mathematically investigated also in [1], where the attracting sets of equilibrium points of this autonomous system are examined.

In [2] the generalized Van der Pol-Mathieu equation

$$\frac{d^2x}{dt^2} - \varepsilon(\alpha_0 - \beta_0 x^{2n})\frac{dx}{dt} + \omega_0^2(1 + \varepsilon h_0 \cos\gamma t)x = 0 \tag{2}$$

is investigated, where $n \in \mathbb{N}$, $\gamma = 2\omega_0 + 2d_0\varepsilon$, $\alpha_0 > 0$, $\beta_0 > 0$, $h_0 > 0$, $\omega_0 > 0$, $\varepsilon > 0$ and $d_0 \in \mathbb{R}$, under the effect of parametric resonance. Using the averaging method and the Bogoliubov theorem together with the method of complexification, the existence of periodic and quasi-periodic solutions of (2) has been proved.

In this work we consider the modified generalized Van der Pol-Mathieu equation

$$\frac{d^2x}{dt^2} - \varepsilon(\alpha_0 + \beta_{01}x^2 + \beta_{02}x^4 - \beta_{03}x^{2n})\frac{dx}{dt} + \omega_0^2(1 + \varepsilon h_0 \cos\gamma t)x = 0 \tag{3}$$

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where $n \in \mathbb{N}$, $n > 2$, $\gamma = 2\omega_0 + 2d_0\varepsilon$, $\alpha_0 \in \mathbb{R}$, $\beta_{01}, \beta_{02}, \beta_{03} \in \mathbb{R}$, $h_0 > 0$, $\omega_0 > 0$, $\varepsilon > 0$ and $d_0 \in \mathbb{R}$. We shall study the autonomous system associated with (3).

2. Preliminaries

First we shall state theorem that shows the connection between the solution of the averaged equation and the solution of the original equation. It can be found in [8, Theorem 11.1].

Theorem 1. *Consider the initial value problems*

$$x' = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon)\,, \quad x(0) = x_0\,, \tag{4}$$

$$y' = \varepsilon f^0(y)\,, \quad y(0) = x_0 \tag{5}$$

with $x,\ y,\ x_0 \in D \subseteq \mathbb{R}^n$, $t \geq 0$. *Suppose that*

(a) *the vector functions* $f,\ g,\ \frac{\partial f}{\partial x}$ *are defined, continuous and bounded by a constant* M *(independent of* ε*) in* $\langle 0,\infty) \times D$, $0 \leq \varepsilon \leq \varepsilon_0$*;*

(b) g *is Lipschitz-continuous in* x *for* $x \in D$*;*

(c) f *is* T*-periodic in* t *with average* $f^0(y) = 1/T \int_0^T f(t,y)\,dt$*;* T *is a constant independent of* ε*;*

(d) *the solution* $y(t)$ *of* (5) *is contained in an internal subset of* D.

Then the solution $x(t)$ *of* (4) *satisfies* $|x(t) - y(t)| \leq K\varepsilon$ *for* $t \in \langle 0, C/\varepsilon\rangle$, *where* K, C *are constants independent of* ε.

The following lemma will be useful for deriving the averaged equation. The proof of this lemma can be found in [2].

Lemma 1. *The following relations are true*

$$\begin{aligned}
&\int_0^{2\pi} (a\cos\tau + b\sin\tau)^{2n}(a\sin\tau - b\cos\tau)\sin\tau\,d\tau \\
&\qquad = a\,(a^2+b^2)^n \frac{2\pi(2n-1)!!}{2^{n+1}(n+1)!}\,, \\
&\int_0^{2\pi} (a\cos\tau + b\sin\tau)^{2n}(-a\sin\tau + b\cos\tau)\cos\tau\,d\tau \\
&\qquad = b\,(a^2+b^2)^n \frac{2\pi(2n-1)!!}{2^{n+1}(n+1)!}\,.
\end{aligned} \tag{6}$$

Now we shall state Theorems 2 and 3 which show the conditions for Generalized Andronov-Hopf (Bautin) bifurcation and the topological normal form for Bautin bifurcation. These theorems can be found in [4], page 311.

Theorem 2. *Suppose that a planar system* $\frac{dx}{dt} = f(x,\boldsymbol{\alpha})$, $x \in \mathbb{R}^2$, $\boldsymbol{\alpha} \in \mathbb{R}^2$ *with smooth* f, *has the equilibrium* $x = 0$ *with the eigenvalues*

$$\lambda_{1,2}(\boldsymbol{\alpha}) = \mu(\boldsymbol{\alpha}) \pm i\omega(\boldsymbol{\alpha})\,,$$

for all norms $\|\boldsymbol{\alpha}\|$ sufficiently small, where $\omega(0)=\omega_0>0$. For $\boldsymbol{\alpha}=0$, let the Bautin bifurcation conditions hold:

$$\mu(0)=0\,,\quad l_1(0)=0\,,$$

where $l_1(\boldsymbol{\alpha})$ is the first Lyapunov coefficient (*see* [4, page 309])*. Assume that the following genericity conditions are satisfied:*

(B.1) $l_2(0)\neq 0$*, where $l_2(\boldsymbol{\alpha})$ is the second Lyapunov coefficient given by* [4, pages 309-310]*;*

(B.2) *the map $\boldsymbol{\alpha}\mapsto(\mu(\boldsymbol{\alpha}),l_1(\boldsymbol{\alpha}))^T$ is regular at $\boldsymbol{\alpha}=0$.*

Then, by the introduction of a complex variable, applying smooth invertible coordinate transformations that depend smoothly on the parameters, and performing smooth parameter and time changes, the system can be reduced to the following complex form:

$$\frac{dz}{dt}=(\tilde{\beta}_1+i)z+\tilde{\beta}_2 z|z|^2+sz|z|^4+O(|z|^6)\,,\tag{7}$$

where $\tilde{\beta}_1,\tilde{\beta}_2\in\mathbb{R}$ and $s=\operatorname{sign} l_2(0)=\pm1$.

Remark 1. The condition (B.2) in Theorem 2 can be replaced with transversality condition

$$\begin{vmatrix}\frac{\partial\mu}{\partial\alpha_1} & \frac{\partial\mu}{\partial\alpha_2}\\ \frac{\partial l_1}{\partial\alpha_1} & \frac{\partial l_1}{\partial\alpha_2}\end{vmatrix}\neq 0 \quad\text{at}\quad \boldsymbol{\alpha}=0\,.\tag{8}$$

The following lemmas and theorem will be useful for the autonomous system associated with the differential equation (3) for $\beta_{03}\neq 0$. These lemmas and theorem are given in [4, pages 308-314].

Lemma 2 (Poincaré normal form for the Bautin bifurcation)**.** *The equation*

$$\frac{dz}{dt}=\lambda(\boldsymbol{\alpha})z+\sum_{2\le k+l\le 5}\frac{1}{k!l!}g_{kl}(\boldsymbol{\alpha})z^k\bar{z}^l+O(|z|^6)\,,\tag{9}$$

with smooth functions $g_{kl}(\boldsymbol{\alpha})$, where $\lambda(\boldsymbol{\alpha})=\mu(\boldsymbol{\alpha})+i\omega(\boldsymbol{\alpha})$, $\mu(0)=0$, $\omega(0)=\omega_0>0$ (μ and ω are smooth functions of their arguments), can be transformed by an invertible parameter-dependent change of the complex coordinate, smoothly depending on the parameters:

$$z=w+\sum_{2\le k+l\le 5}\frac{1}{k!l!}h_{kl}(\boldsymbol{\alpha})w^k\bar{w}^l\,,\quad h_{21}(\boldsymbol{\alpha})=h_{32}(\boldsymbol{\alpha})=0\,,\tag{10}$$

for all sufficiently small $\|\alpha\|$, into the equation

$$\frac{dw}{dt}=\lambda(\boldsymbol{\alpha})w+c_1(\boldsymbol{\alpha})w|w|^2+c_2(\boldsymbol{\alpha})w|w|^4+O(|w|^6)\,.\tag{11}$$

Lemma 3. *The system*

$$\frac{dz}{dt}=(\tilde{\beta}_1+i)z+\tilde{\beta}_2 z|z|^2\pm z|z|^4+O(|z|^6)\,,\tag{12}$$

is locally topologically equivalent near the origin to the system

$$\frac{dz}{dt} = (\tilde{\beta}_1 + i)z + \tilde{\beta}_2 z|z|^2 \pm z|z|^4 \,. \tag{13}$$

Theorem 3 (Topological normal form for Bautin bifurcation)**.** *Any generic planar two-parameter system* $\frac{dx}{dt} = f(x, \boldsymbol{\alpha})$*, having at* $\boldsymbol{\alpha} = 0$ *an equilibrium* $x = 0$ *that exhibits the Bautin bifurcation, is locally topologically equivalent near the origin to one of the following complex normal forms:*

$$\frac{dz}{dt} = (\tilde{\beta}_1 + i)z + \tilde{\beta}_2 z|z|^2 \pm z|z|^4 \,. \tag{14}$$

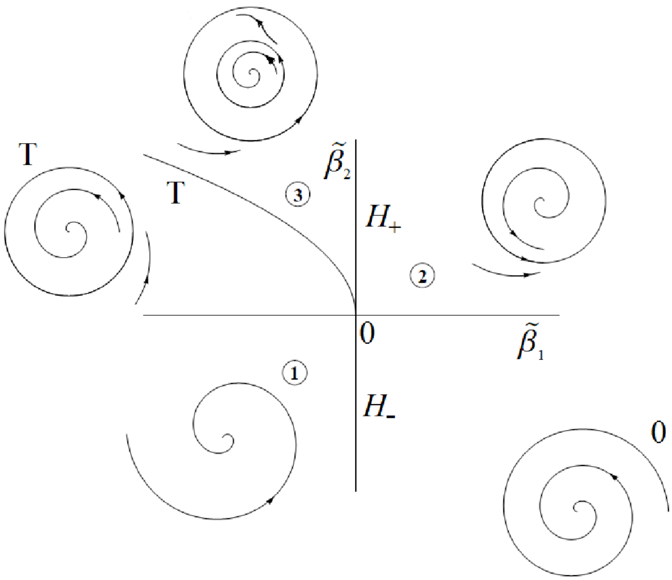


FIG. 1: The Bautin bifurcation.

Theorems 2 and 3 describe the *Bautin bifurcation* conditions. More precisely, the equilibrium $x = 0$ has purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, for $\boldsymbol{\alpha} = 0$ and the first Lyapunov coefficient vanishes: $l_1(0) = 0$.

For $s = -1$ the point $\tilde{\boldsymbol{\beta}} = (\tilde{\beta}_1, \tilde{\beta}_2) = (0, 0)$ separates two branches of the Andronov-Hopf bifurcation curve: the half-line $H_- = \{(\tilde{\beta}_1, \tilde{\beta}_2) \in \mathbb{R}^2 : \tilde{\beta}_1 = 0, \tilde{\beta}_2 < 0\}$ corresponds to the supercritical bifurcation that generates a stable limit cycle, while the half-line $H_+ = \{(\tilde{\beta}_1, \tilde{\beta}_2) \in \mathbb{R}^2 : \tilde{\beta}_1 = 0, \tilde{\beta}_2 > 0\}$ corresponds to the subcritical bifurcation that generates an unstable limit cycle. Two hyperbolic limit cycles (one stable and one unstable) exist in the region between H_+ and the curve $T = \{(\tilde{\beta}_1, \tilde{\beta}_2) \in \mathbb{R}^2 : \tilde{\beta}_1 = -\frac{1}{4}\tilde{\beta}_2^{\,2}, \tilde{\beta}_2 > 0\}$. The cycles collide and disappear on the curve T, corresponding to a nondegenerate fold bifurcation of the cycles.

The case $s = 1$ can be described similarly. It can be reduced to the first case by the transformation $(z, \tilde{\boldsymbol{\beta}}, t) \mapsto (\bar{z}, -\tilde{\boldsymbol{\beta}}, -t)$.

Remark 2. Figure 1 shows the Bautin bifurcation. The system has a single stable equilibrium and no cycles in region 1. Crossing the Hopf bifurcation boundary H_- from region 1 to region 2 implies the appearance of an unique and stable limit cycle. Crossing the Hopf bifurcation boundary H_+ creates an extra unstable cycle inside the first one, while the equilibrium regains its stability. Two cycles of opposite stability exist inside region 3 and disappear at the curve T.

The next theorem is very important for the proof of existence of limit cycles. The formulation of Dulac criteria can be found in [6].

Theorem 4 (Dulac criteria). *Suppose that $f(z, \bar{z})$ is a complex-valued function of the class C^1 in a region $\Omega \subseteq \mathbb{C}$ and that $q(z, \bar{z})$ is a real function of the class C^1 in Ω. Let $D \subseteq \Omega$ be a region such that the expression*

$$\operatorname{Re} \frac{\partial}{\partial z} \left[q(z, \bar{z}) f(z, \bar{z}) \right]$$

is non-negative or non-positive in D and is identically equal zero in no open subset of D. If D is a simply connected region, then the equation $z' = f(z, \bar{z})$ has no closed trajectory lying entirely in D. If D is a doubly connected annular region, then there exists at most one closed trajectory in D.

The computation of the first and second Lyapunov quantities is described in [3] where the authors consider a system in the case of expansion of the right-hand side up to the seventh order

$$\begin{aligned}
\frac{dx}{dt} &= -y + f_{20}x^2 + f_{11}xy + f_{02}y^2 + f_{30}x^3 + f_{21}x^2y + f_{12}xy^2 + f_{03}y^3 + f_{40}x^4 \\
&\quad + f_{31}x^3y + f_{22}x^2y^2 + f_{13}xy^3 + f_{04}y^4 + f_{50}x^5 + f_{41}x^4y + f_{32}x^3y^2 + f_{23}x^2y^3 \\
&\quad + f_{14}xy^4 + f_{05}y^5 + f_{60}x^6 + f_{51}x^5y + f_{42}x^4y^2 + f_{33}x^3y^3 + f_{24}x^2y^4 + f_{15}xy^5 \\
&\quad + f_{06}y^6 + f_{70}x^7 + f_{61}x^6y + f_{52}x^5y^2 + f_{43}x^4y^3 + f_{34}x^3y^4 + f_{25}x^2y^5 + f_{16}xy^6 \\
&\quad + f_{07}y^7\,,
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dt} &= x + g_{20}x^2 + g_{11}xy + g_{02}y^2 + g_{30}x^3 + g_{21}x^2y + g_{12}xy^2 + g_{03}y^3 + g_{40}x^4 \\
&\quad + g_{31}x^3y + g_{22}x^2y^2 + g_{13}xy^3 + g_{04}y^4 + g_{50}x^5 + g_{41}x^4y + g_{32}x^3y^2 + g_{23}x^2y^3 \\
&\quad + g_{14}xy^4 + g_{05}y^5 + g_{60}x^6 + g_{51}x^5y + g_{42}x^4y^2 + g_{33}x^3y^3 + g_{24}x^2y^4 + g_{15}xy^5 \\
&\quad + g_{06}y^6 + g_{70}x^7 + g_{61}x^6y + g_{52}x^5y^2 + g_{43}x^4y^3 + g_{34}x^3y^4 + g_{25}x^2y^5 + g_{16}xy^6 \\
&\quad + g_{07}y^7\,.
\end{aligned}$$

For the first Lyapunov quantity the following formula is stated:

$$\begin{aligned}
l_1(0) = \frac{\pi}{4}(&g_{21} + f_{12} + 3f_{30} + 3g_{03} + f_{20}f_{11} + f_{02}f_{11} \\
&- g_{11}g_{20} + 2g_{02}f_{02} - 2f_{20}g_{20} - g_{02}g_{11})\,.
\end{aligned} \tag{15}$$

The formula for the second Lyapunov quantity given in [3] is very complicated, we give here only its special case for $f_{20} = f_{02} = f_{11} = g_{20} = g_{02} = g_{11} = 0$:

$$\begin{aligned} l_2(0) = &-\frac{\pi}{72}(9g_{21}g_{30} + 9g_{21}f_{03} - 9f_{21}f_{30} + 27f_{30}g_{30} - 6f_{12}g_{12} \\ &+ 3f_{21}g_{21} - 9f_{30}g_{12} + 6f_{21}f_{12} + 27f_{30}f_{03} - 45g_{05} \\ &- 9g_{23} - 9g_{41} + 3g_{21}g_{12} - 45f_{50} - 9f_{14} - 9f_{32}) \,. \end{aligned} \tag{16}$$

Remark 3. The Lyapunov coefficient and Lyapunov quantity do not have the same value. The value of the Lyapunov quantity is 2π-multiple of the Lyapunov coefficient. The Lyapunov quantity can be found in [3] and the Lyapunov coefficient in [4].

3. The modified generalized Van der Pol-Mathieu equation and its averaging

Consider the modified generalized Van der Pol-Mathieu equation (3)

$$\frac{d^2x}{dt^2} - \varepsilon\big(\alpha_0 + \beta_{01}x^2 + \beta_{02}x^4 - \beta_{03}x^{2n}\big)\,\frac{dx}{dt} + \omega_0^2(1 + \varepsilon h_0 \cos\gamma t)x = 0\,,$$

where $n \in \mathbb{N}$, $n > 2$, $\gamma = 2\omega_0 + 2d_0\varepsilon$, $\alpha_0 \in \mathbb{R}$, $\beta_{01}, \beta_{02}, \beta_{03} \in \mathbb{R}$, $h_0 > 0$, $\omega_0 > 0$, $\varepsilon > 0$ and $d_0 \in \mathbb{R}$. In this section we derive the autonomous system from the differential equation (3) and we use the same method as in [2].

We carry out the substitution $\tau = (\omega_0 + d_o\varepsilon)t$ and get the equation

$$\begin{aligned} \frac{d^2x}{d\tau^2} + x = \frac{\varepsilon}{\omega_0}\Big[&(\alpha_0 + \beta_{01}x^2 + \beta_{02}x^4 - \beta_{03}x^{2n})\frac{dx}{d\tau} \\ &+ (2d_0 - h_0\omega_0\cos 2\tau)x\Big] + O(\varepsilon^2)\,. \end{aligned}$$

The averaging method supposes that the solution and derivative of solution of (3) are in the form

$$x(\tau) = a(\tau)\cos\tau + b(\tau)\sin\tau\,, \tag{17}$$

$$\frac{dx}{d\tau}(\tau) = -a(\tau)\sin\tau + b(\tau)\cos\tau\,, \tag{18}$$

where the functions $a(\tau)$, $b(\tau)$ are considered to be slowly varying. Using the equality

$$\frac{da}{d\tau}\cos\tau + \frac{db}{d\tau}\sin\tau = 0\,,$$

we obtain the system of two equations for $\frac{da}{d\tau}$ and $\frac{db}{d\tau}$. Using the averaging method, Theorem 1 and Lemma 1, we derive the autonomous system of two equations associated with the modified generalized Van der Pol-Mathieu equation:

$$
\begin{aligned}
\frac{da}{d\tau} &= \frac{\varepsilon}{\omega_0}\Big[\frac{\alpha_0}{2}a - \Big(d_0 + \frac{h_0\omega_0}{4}\Big)b + \frac{\beta_{01}}{8}a(a^2+b^2) + \frac{\beta_{02}}{16}a(a^2+b^2)^2 \\
&\quad - \beta_{03}a(a^2+b^2)^n \frac{(2n-1)!!}{2^{n+1}(n+1)!}\Big], \\
\frac{db}{d\tau} &= \frac{\varepsilon}{\omega_0}\Big[\Big(d_0 - \frac{h_0\omega_0}{4}\Big)a + \frac{\alpha_0}{2}b + \frac{\beta_{01}}{8}b(a^2+b^2) + \frac{\beta_{02}}{16}b(a^2+b^2)^2 \\
&\quad - \beta_{03}b(a^2+b^2)^n \frac{(2n-1)!!}{2^{n+1}(n+1)!}\Big].
\end{aligned} \tag{19}
$$

4. Autonomous system associated with the modified generalized Van der Pol-Mathieu equation

Using

$$
\alpha = \frac{\alpha_0\varepsilon}{2\omega_0},\quad d = \frac{d_0\varepsilon}{\omega_0},\quad \lambda = \frac{h_0\varepsilon}{4},\quad \beta_1 = \frac{\beta_{01}\varepsilon}{8\omega_0},\quad \beta_2 = \frac{\beta_{02}\varepsilon}{16\omega_0},\quad \beta_3 = \frac{\varepsilon\beta_{03}(2n-1)!!}{\omega_0 2^{n+1}(n+1)!},
$$

where $\alpha, \beta_1, \beta_2, \beta_3, d \in \mathbb{R}$, $\lambda > 0$, we can write the autonomous system (19) in the form

$$
\begin{aligned}
\frac{da}{d\tau} &= \alpha a - (d+\lambda)\, b + \beta_1 a(a^2+b^2) + \beta_2 a(a^2+b^2)^2 - \beta_3 a(a^2+b^2)^n, \\
\frac{db}{d\tau} &= (d-\lambda)\, a + \alpha b + \beta_1 b(a^2+b^2) + \beta_2 b(a^2+b^2)^2 - \beta_3 b(a^2+b^2)^n.
\end{aligned} \tag{20}
$$

This system has a focus in origin for case $|d| > \lambda$.

In some of our considerations it seems useful to use the polar coordinates ρ, φ that we receive by using the method of complexification and by rewriting the system (20) as one equation with complex-valued quantities by using $z = x + yi$:

$$
\frac{dz}{d\tau} = (\alpha + \mathrm{id})z - \lambda\bar{z}i + \beta_1 z|z|^2 + \beta_2 z|z|^4 - \beta_3 z|z|^{2n}. \tag{21}
$$

To introduce the polar coordinates ρ, φ we put $z = \rho e^{i\varphi}$ in (21) and by separating the real and imaginary parts, we get the system

$$
\begin{aligned}
\frac{d\rho}{d\tau} &= \rho\left(\alpha - \lambda\sin(2\varphi) + \beta_1\rho^2 + \beta_2\rho^4 - \beta_3\rho^{2n}\right) = h(\rho,\varphi), \\
\frac{d\varphi}{d\tau} &= d - \lambda\cos(2\varphi).
\end{aligned} \tag{22}
$$

The condition $|d| > \lambda$ implies that $d - \lambda\cos(2\varphi) \neq 0$ for all φ, therefore the system (20) has the unique stationary point at the origin. For computation of the Lyapunov quantities by [3] we change the autonomous system (20) by transformation $a = x$,

$b = \sqrt{\frac{d-\lambda}{d+\lambda}} \cdot y$, $t = \sqrt{d^2-\lambda^2} \cdot \tau$ to the form

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\alpha}{\sqrt{d^2-\lambda^2}} \cdot x - y + \frac{\beta_1}{\sqrt{d^2-\lambda^2}} \cdot x\Big(x^2 + \frac{d-\lambda}{d+\lambda} \cdot y^2\Big) \\
&\quad + \frac{\beta_2}{\sqrt{d^2-\lambda^2}} \cdot x\Big(x^2 + \frac{d-\lambda}{d+\lambda} \cdot y^2\Big)^2 - \frac{\beta_3}{\sqrt{d^2-\lambda^2}} \cdot x\Big(x^2 + \frac{d-\lambda}{d+\lambda} \cdot y^2\Big)^n, \\
\frac{dy}{dt} &= x + \frac{\alpha}{\sqrt{d^2-\lambda^2}} \cdot y + \frac{\beta_1}{\sqrt{d^2-\lambda^2}} \cdot y\Big(x^2 + \frac{d-\lambda}{d+\lambda} \cdot y^2\Big) \\
&\quad + \frac{\beta_2}{\sqrt{d^2-\lambda^2}} \cdot y\Big(x^2 + \frac{d-\lambda}{d+\lambda} \cdot y^2\Big)^2 - \frac{\beta_3}{\sqrt{d^2-\lambda^2}} \cdot y\Big(x^2 + \frac{d-\lambda}{d+\lambda} \cdot y^2\Big)^n.
\end{aligned} \tag{23}$$

The autonomous system (23) has Jacobi matrix at the origin

$$\mathbf{J}(0) = \begin{pmatrix} \frac{\alpha}{\sqrt{d^2-\lambda^2}} & -1 \\ 1 & \frac{\alpha}{\sqrt{d^2-\lambda^2}} \end{pmatrix} \tag{24}$$

with eigenvalues $\lambda_{1,2} = \frac{\alpha}{\sqrt{d^2-\lambda^2}} \pm i$. The modified generalized Van der Pol-Mathieu autonomous system has an unique equilibrium in $(0,0)$ which is the type of focus for $|d| > \lambda$.

5. Bautin bifurcation of the modified generalized Van der Pol-Mathieu system for $\beta_3 = 0$

In this section we investigate the system (23) for $\beta_3 = 0$. This system has a focus at the origin for $|d| > \lambda$. We consider the parameter $\boldsymbol{\alpha}$ in Theorem 2 in form $\boldsymbol{\alpha} = (\alpha, \beta_1)$.

Theorem 5. *Suppose $\beta_3 = 0$, $\beta_2 \neq 0$ and $|d| > \lambda$. The modified generalized Van der Pol-Mathieu autonomous system* (23) *exhibits Bautin bifurcation at the equilibrium* $(0,0)$ *for sufficiently small* $\|\boldsymbol{\alpha}\|$ *and for the second Lyapunov quantity it holds that*

$$l_2(0) = \frac{\pi\beta_2}{4\sqrt{d^2-\lambda^2}} \cdot \Big(3\frac{(d-\lambda)^2}{(d+\lambda)^2} + 2\frac{d-\lambda}{d+\lambda} + 3\Big). \tag{25}$$

Proof. We have to prove that the conditions of Theorem 2 are fulfilled with respect of parameter $\boldsymbol{\alpha} = (\alpha, \beta_1)$. It holds that $\mu(\boldsymbol{\alpha}) = \frac{\alpha}{\sqrt{d^2-\lambda^2}}$ equals zero for $\alpha = 0$ and $\omega(0) = 1 > 0$. We compute the first Lyapunov quantity of this system by (15) for the Bautin bifurcation:

$$\begin{aligned}
l_1(0) &= \frac{\pi}{4}\Big(\frac{\beta_1}{\sqrt{d^2-\lambda^2}} + \frac{\beta_1}{\sqrt{d^2-\lambda^2}} \cdot \frac{d-\lambda}{d+\lambda} + 3\frac{\beta_1}{\sqrt{d^2-\lambda^2}} \\
&\quad + 3\frac{\beta_1}{\sqrt{d^2-\lambda^2}} \cdot \frac{d-\lambda}{d+\lambda}\Big)\Big|_{\beta_1=0} = \frac{2\pi\beta_1 d}{\sqrt{d^2-\lambda^2}(d+\lambda)}\Big|_{\beta_1=0} = 0.
\end{aligned} \tag{26}$$

For the second Lyapunov quantity it is true by (16):

$$l_2(0) = \frac{\pi\beta_2}{4\sqrt{d^2-\lambda^2}} \cdot \Big(3\frac{(d-\lambda)^2}{(d+\lambda)^2} + 2\frac{d-\lambda}{d+\lambda} + 3\Big). \tag{27}$$

We get $l_2(0) \neq 0$ for $(\alpha, \beta_1) = (0,0)$ and $\beta_2 \neq 0$.
The transversality condition is satisfied for $d \neq 0$:

$$\begin{vmatrix} \frac{1}{\sqrt{d^2-\lambda^2}} & 0 \\ \frac{\partial l_1}{\partial \alpha} & \frac{2d\pi}{\sqrt{d^2-\lambda^2}(d+\lambda)} \end{vmatrix} \neq 0 \quad \text{at} \quad (\alpha, \beta_1) = (0,0)\,. \tag{28}$$

The expression $\frac{\partial l_1}{\partial \alpha}$ is not computed because it does not change the result. The modified generalized Van der Pol-Mathieu system (23) for $\beta_3 = 0$ satisfies the assumptions of Theorem 2, Lemma 2, Lemma 3, Theorem 3 and exhibits Bautin bifurcation at the equilibrium $(0,0)$ for sufficiently small $\|\boldsymbol{\alpha}\|$. □

In this section we shall apply Dulac criteria, Theorem 4, to the equation (21). Put

$$f(z,\bar{z}) = (\alpha + id)z - \lambda \bar{z} i + \beta_1 z|z|^2 + \beta_2 z|z|^4\,, \qquad q(z,\bar{z}) = \frac{1}{z\bar{z}}\,.$$

It holds that

$$\begin{aligned} \operatorname{Re}\frac{\partial}{\partial z}\left[q(z,\bar{z})f(z,\bar{z})\right] &= \operatorname{Re}\frac{\partial}{\partial z}\Big[\frac{1}{z\bar{z}}\left((\alpha+id)z - \lambda\bar{z}i + \beta_1 z|z|^2 + \beta_2 z|z|^4\right)\Big] \\ &= \operatorname{Re}\left(\frac{i\lambda}{z^2} + \beta_1 + 2\beta_2|z|^2\right) = \frac{\lambda \operatorname{Im} z^2}{|z|^4} + \beta_1 + 2\beta_2|z|^2\,. \end{aligned} \tag{29}$$

Remark 4. If we consider $\operatorname{Re}\frac{\partial}{\partial z}\left[q(z,\bar{z})f(z,\bar{z})\right] = 0$ from Dulac criterion, Theorem 4, we get the equation

$$\frac{1}{|z|^2}\left(2\beta_2|z|^4 + \beta_1|z|^2 + \frac{\lambda \operatorname{Im} z^2}{|z|^2}\right) = 0\,. \tag{30}$$

The nonzero solutions of this equation (30) are

$$|z|^2 = \frac{-\beta_1 - \sqrt{\beta_1^2 - 8\beta_2\lambda\frac{\operatorname{Im} z^2}{|z|^2}}}{4\beta_2} \quad \text{and} \quad |z|^2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 8\beta_2\lambda\frac{\operatorname{Im} z^2}{|z|^2}}}{4\beta_2} \tag{31}$$

for $\beta_2 \neq 0$ and $\beta_1^2 - 8\beta_2\lambda\frac{\operatorname{Im} z^2}{|z|^2} \geq 0$. Using the polar coordinates ρ, φ, we obtain $\frac{\operatorname{Im} z^2}{|z|^2} = \sin(2\varphi)$ and the equation

$$2\beta_2\rho^4 + \beta_1\rho^2 + \lambda\sin(2\varphi) = 0\,. \tag{32}$$

Therefore the set of points that satisfies the equation (32) can be written as

$$\begin{aligned} M_- &= \left\{(\rho,\varphi) \in \mathbb{R}^2 : \rho^2 = \frac{-\beta_1 - \sqrt{\beta_1^2 - 8\beta_2\lambda\sin(2\varphi)}}{4\beta_2}; \varphi \in \langle 0, 2\pi)\right\}, \\ M_+ &= \left\{(\rho,\varphi) \in \mathbb{R}^2 : \rho^2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 8\beta_2\lambda\sin(2\varphi)}}{4\beta_2}; \varphi \in \langle 0, 2\pi)\right\}, \end{aligned} \tag{33}$$

where $\beta_2 \neq 0$ and $\beta_1^2 - 8\beta_2\lambda\sin(2\varphi) \geq 0$.

The following lemmas will be useful for the theorem concerning the existence of closed trajectories. Figure 2 shows sets M_1 and M_2 from these lemmas. The following considerations assume $\beta_2 < 0$. It will be helpful to use $\beta_2 = -|\beta_2|$.

Lemma 4. *Let* $\beta_1 > 0$, $\beta_2 < 0$ *and* $\frac{\beta_1^2}{8\beta_2\lambda} < -1$. *Then the set*

$$M_1 = \left\{(\rho,\varphi) \in \mathbb{R}^2 : 2\beta_2\rho^4 + \beta_1\rho^2 + \lambda\sin(2\varphi) > 0\right\} \tag{34}$$

is a doubly connected region and it holds that

$$M_1 = \left\{(\rho,\varphi) \in \mathbb{R}^2 : 0 \le \rho_1(\varphi) < \rho \le \rho_2(\varphi), \varphi \in \langle 0, 2\pi)\right\}\,, \tag{35}$$

where

$$\rho_1(\varphi) = \begin{cases} \sqrt{\frac{\beta_1 - \sqrt{\beta_1^2 + 8|\beta_2|\lambda\sin(2\varphi)}}{4|\beta_2|}} & \text{for} \quad \varphi \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) \\ 0 & \text{for} \quad \varphi \in \left\langle 0, \frac{\pi}{2}\right\rangle \cup \left\langle \pi, \frac{3\pi}{2}\right\rangle \end{cases},$$

$$\rho_2(\varphi) = \sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 + 8|\beta_2|\lambda\sin(2\varphi)}}{4|\beta_2|}} \quad \text{for} \quad \varphi \in \langle 0, 2\pi)\,.$$

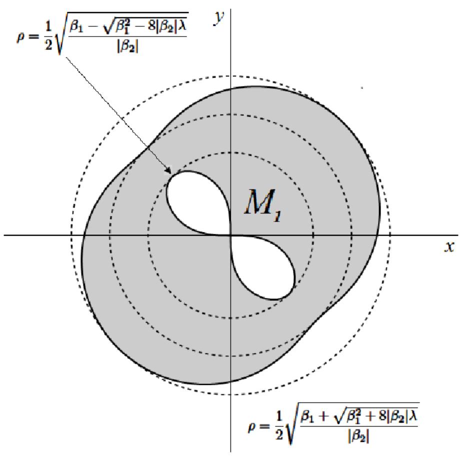
Moreover, it holds that

$$0 < \rho_1(\varphi) \le \frac{1}{2}\sqrt{\frac{\beta_1 - \sqrt{\beta_1^2 - 8|\beta_2|\lambda}}{|\beta_2|}} < \frac{1}{2}\sqrt{\frac{\beta_1}{|\beta_2|}} \tag{36}$$

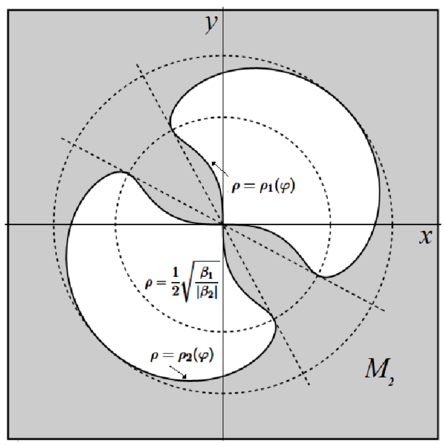
and

$$\frac{1}{2}\sqrt{\frac{\beta_1}{|\beta_2|}} < \frac{1}{2}\sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 - 8|\beta_2|\lambda}}{|\beta_2|}} \le \rho_2(\varphi) \le \frac{1}{2}\sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 + 8|\beta_2|\lambda}}{|\beta_2|}} \tag{37}$$

for $\varphi \in \langle 0, 2\pi)$.



(a) The doubly connected region from Lemma 4.



(b) The doubly connected region from Lemma 5.

FIG. 2: The sets M_1 and M_2 from Lemma 4 and Lemma 5.

Proof. Suppose $\beta_1 > 0$ and $\beta_2 < 0$. We will use $\beta_2 = -|\beta_2|$.
The condition $\frac{\beta_1^2}{8\beta_2\lambda} < -1$ implies that the discriminant D of (32) satisfies

$$D = \beta_1^2 - 8\beta_2\lambda\sin(2\varphi) \geq \beta_1^2 + 8\beta_2\lambda > 0.$$

Providing that $\sqrt{\frac{\beta_1 - \sqrt{\beta_1^2 + 8|\beta_2|\lambda\sin(2\varphi)}}{4|\beta_2|}} > 0$ iff $\sin(2\varphi) < 0$, i.e. iff $\varphi \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$, we can easily see that the equation (32) has two positive solutions $\rho_1(\varphi)$, $\rho_2(\varphi)$ for $\varphi \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ and one positive solution $\rho_2(\varphi)$ for $\varphi \in \left\langle 0, \frac{\pi}{2}\right\rangle \cup \left\langle \pi, \frac{3\pi}{2}\right\rangle$.

Because of

$$2\beta_2\rho^4 + \beta_1\rho^2 + \lambda\sin(2\varphi) = 2\beta_2(\rho - \rho_1(\varphi)) \cdot (\rho - \rho_2(\varphi)) \cdot (\rho + \rho_1(\varphi)) \cdot (\rho + \rho_2(\varphi))$$

for $\varphi \in \langle\frac{\pi}{2}, \pi\rangle \cup \langle\frac{3\pi}{2}, 2\pi\rangle$ it is obvious that the set M_1 can be written in the form (35). Clearly, the set M_1 is a doubly connected region and the inequalities (36), (37) are fulfilled. See Figure 2 (a).
These facts prove the lemma. $\square$

Remark 5. Let us comment the inequalities (36) and (37). We get the radius $\frac{1}{2}\sqrt{\frac{\beta_1 - \sqrt{\beta_1^2 - 8|\beta_2|\lambda}}{|\beta_2|}}$ from $\rho_1(\varphi)$ for $\varphi = \frac{3\pi}{4}, \frac{7\pi}{4}$. The radius $\rho_2(\varphi)$ is equal to value $\frac{1}{2}\sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 - 8|\beta_2|\lambda}}{|\beta_2|}}$ for $\varphi = \frac{3\pi}{4}, \frac{7\pi}{4}$ and $\frac{1}{2}\sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 + 8|\beta_2|\lambda}}{|\beta_2|}}$ for $\varphi = \frac{\pi}{4}, \frac{5\pi}{4}$. It holds that

$$\frac{1}{2}\sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 - 8|\beta_2|\lambda}}{|\beta_2|}} < \sqrt{\frac{\beta_1}{2|\beta_2|}} < \frac{1}{2}\sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 + 8|\beta_2|\lambda}}{|\beta_2|}}.$$

The radius $\rho_2(\varphi)$ for $\varphi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ gives the value $\sqrt{\frac{\beta_1}{2|\beta_2|}}$.

Lemma 5. *Let $\beta_1 > 0$, $\beta_2 < 0$ and $\frac{\beta_1^2}{8\beta_2\lambda} \in (-1, 0)$. Then the set*

$$M_2 = \left\{(\rho, \varphi) \in \mathbb{R}^2 : 2\beta_2\rho^4 + \beta_1\rho^2 + \lambda\sin(2\varphi) < 0\right\} \tag{38}$$

is a doubly connected region and it contains the subset of point

$$\tilde{M} = \left\{(\rho, \varphi) \in \mathbb{R}^2 : \rho > \frac{1}{2}\sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 + 8|\beta_2|\lambda}}{|\beta_2|}}\ ;\ \varphi \in \langle 0, 2\pi)\right\}. \tag{39}$$

The set M_2 can be written in the form

$$
\begin{aligned}
M_2 =&\Big\{(\rho,\varphi)\in\mathbb{R}^2 : \rho>\rho_2(\varphi)\ ;\ \varphi\in\Big\langle 0,\frac{\pi}{2}\Big\rangle\cup\Big\langle \pi,\frac{3\pi}{2}\Big\rangle\Big\}\\
&\cup\Big\{(\rho,\varphi)\in\mathbb{R}^2 : \rho\in(0,\rho_1(\varphi))\cup(\rho_2(\varphi),\infty)\ ;\ \varphi\in\Big(\frac{\pi}{2},\frac{\pi}{2}+\varphi_0\Big\rangle\\
&\cup\langle\pi-\varphi_0,\pi)\cup\Big(\frac{3\pi}{2},\frac{3\pi}{2}+\varphi_0\Big\rangle\cup\langle 2\pi-\varphi_0,2\pi)\Big\}\\
&\cup\Big\{(\rho,\varphi)\in\mathbb{R}^2 : \rho>0\ ;\ \varphi\in\Big(\frac{\pi}{2}+\varphi_0,\pi-\varphi_0\Big)\\
&\cup\Big(\frac{3\pi}{2}+\varphi_0,2\pi-\varphi_0\Big)\Big\},
\end{aligned}
\tag{40}
$$

where

$$\rho_1(\varphi)=\sqrt{\frac{\beta_1-\sqrt{\beta_1^2+8|\beta_2|\lambda\sin(2\varphi)}}{4|\beta_2|}}$$

for $\varphi\in\Big(\frac{\pi}{2},\frac{\pi}{2}+\varphi_0\Big\rangle\cup\langle\pi-\varphi_0,\pi)\cup\Big(\frac{3\pi}{2},\frac{3\pi}{2}+\varphi_0\Big\rangle\cup\langle 2\pi-\varphi_0,2\pi)$,

$$\rho_2(\varphi)=\sqrt{\frac{\beta_1+\sqrt{\beta_1^2+8|\beta_2|\lambda\sin(2\varphi)}}{4|\beta_2|}}$$

for $\varphi\in\big\langle-\varphi_0,\frac{\pi}{2}+\varphi_0\big\rangle\cup\Big\langle\pi-\varphi_0,\frac{3\pi}{2}+\varphi_0\Big\rangle$,

$$\varphi_0=\frac{1}{2}\arcsin\frac{\beta_1^2}{8|\beta_2|\lambda}\,.$$

Proof. The discriminant of (32) is nonnegative iff $\beta_1^2\ge 8\beta_2\lambda\sin(2\varphi)$, i.e. iff $\sin(2\varphi)\ge\frac{\beta_1^2}{8\beta_2\lambda}$. The last inequality is satisfied for $\varphi\in\big\langle-\varphi_0,\frac{\pi}{2}+\varphi_0\big\rangle\cup\big\langle\pi-\varphi_0,\frac{3\pi}{2}+\varphi_0\big\rangle$. Therefore the equation (32) has no positive solutions for $\varphi\in\big(\frac{\pi}{2}+\varphi_0,\pi-\varphi_0\big)\cup\big(\frac{3\pi}{2}+\varphi_0,2\pi-\varphi_0\big)$. Taking into consideration the conditions for the positivity of $\frac{\beta_1-\sqrt{\beta_1^2+8|\beta_2|\lambda\sin(2\varphi)}}{4|\beta_2|}$ derived in the proof of Lemma 4 we observe that the equation (32) has two positive solutions $\rho_1(\varphi),\rho_2(\varphi)$ for $\varphi\in\big(\frac{\pi}{2},\frac{\pi}{2}+\varphi_0\big)\cup(\pi-\varphi_0,\pi)\cup\big(\frac{3\pi}{2},\frac{3\pi}{2}+\varphi_0\big)\cup(2\pi-\varphi_0,2\pi)$ and one positive solution $\rho_2(\varphi)$ for $\varphi\in\big\{\frac{\pi}{2}+\varphi_0\big\}\cup\{\pi-\varphi_0\}\cup\big\{\frac{3\pi}{2}+\varphi_0\big\}\cup\{2\pi-\varphi_0\}\cup\big\langle 0,\frac{\pi}{2}\big\rangle\cup\big\langle\pi,\frac{3\pi}{2}\big\rangle$. The set M_2 is a doubly connected region and the statement (40) holds. See Figure 2 (b).

In view of the fact that $0<\rho_2(\varphi)\le\frac{1}{2}\sqrt{\frac{\beta_1+\sqrt{\beta_1^2+8|\beta_2|\lambda}}{|\beta_2|}}$, the region M_2 contains the set $\tilde{M}$ as a subset.
Lemma has been proved. □

Lemma 6. *Consider the autonomous system* (21) *for* $\beta_3=0$ *and* $\beta_2<0$. *The following assertions are valid:*

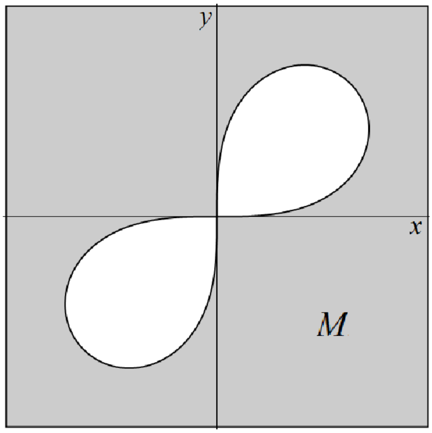


FIG. 3: The set M from Lemma 6.

(1) *Let* $\beta_1 < 0$. *In the doubly connected region* $M = \{(\rho, \varphi) \in \mathbb{R}^2 : \rho^2 > -\frac{\lambda}{\beta_1}\sin(2\varphi)\}$ *there is at most one closed trajectory of* (21) *lying entirely in set* M.

(2) *Let* $\beta_1 > 0$ *and* $\frac{\beta_1^2}{8\beta_2\lambda} < -1$. *In the doubly connected region*

$$M_1 = \left\{(\rho, \varphi) \in \mathbb{R}^2 : 0 \le \rho_1(\varphi) < \rho \le \rho_2(\varphi), \varphi \in \langle 0, 2\pi)\right\} ,$$

there is at most one closed trajectory of (21) *lying entirely in set* M_1. *The functions* $\rho_1(\varphi)$, $\rho_2(\varphi)$ *are defined in Lemma 4.*

(3) *Let* $\beta_1 > 0$ *and* $\frac{\beta_1^2}{8\beta_2\lambda} \in (-1, 0)$. *In the doubly connected region*

$$M_2 = \left\{(\rho, \varphi) \in \mathbb{R}^2 : 2\beta_2\rho^4 + \beta_1\rho^2 + \lambda\sin(2\varphi) < 0\right\}$$

there is at most one closed trajectory of (21) *lying entirely in set* M_2.

Proof. We shall apply Dulac criteria, Theorem 4, to the equation (21) and Lemmas 4, 5. We have already derived that

$$\operatorname{Re}\frac{\partial}{\partial z}\left[q(z,\bar z)f(z,\bar z)\right] = \frac{\lambda \operatorname{Im} z^2}{|z|^4} + \beta_1 + 2\beta_2|z|^2 .$$

Case (1). The set M is a doubly connected annular region containing the second and the fourth quadrant, see Figure 3. The inequality $\rho^2 > -\frac{\lambda}{\beta_1}\sin(2\varphi)$ can be expressed as $\frac{\lambda \operatorname{Im} z^2}{|z|^4} < -\beta_1$. It holds that

$$\operatorname{Re}\frac{\partial}{\partial z}\left[q(z,\bar z)f(z,\bar z)\right] = \frac{\lambda \operatorname{Im} z^2}{|z|^4} + \beta_1 + 2\beta_2|z|^2 < 0$$

for $z \in M$ and $\beta_2 < 0$. The existence of at most one closed trajectory of (21) in M now follows from Dulac criteria, Theorem 4.

Similarly the assertions (2) and (3) follow from Dulac criteria, Theorem 4, and Lemmas 4, 5. $\square$

Remark 6. Suppose $\beta_1 > 0$ and $\frac{\beta_1^2}{8\beta_2\lambda} < -1$. Lemma 4 and Remark 4 imply that it holds

$$\mathrm{Re}\,\frac{\partial}{\partial z}\left[q(z,\bar{z})f(z,\bar{z})\right] = \frac{\lambda\,\mathrm{Im}\,z^2}{|z|^4} + \beta_1 + 2\beta_2|z|^2 < 0$$

for the set

$$M_3 = \left\{(\rho,\varphi) \in \mathbb{R}^2 : \rho > \rho_2(\varphi)\ ;\ \varphi \in \langle 0, 2\pi)\right\}\,. \tag{41}$$

From the proof of Lemma 4 it can be seen that the set M_3 is a subset of the complement of the set M_1 in $\mathbb{R}^2$. If we apply Dulac criteria, Theorem 4, to the equation (21) in the set M_3, we get that there is at most one closed trajectory of (21) lying entirely in the set M_3.

For our following considerations it seems useful to use the system (22) in polar coordinates ρ, φ. The first equation in (22) describes the rate of change of distance from an origin and second equation describes the angular velocity. The trivial solution $\rho = 0$ of the first equation $\rho' = 0$ corresponds to the equilibrium $(0,0)$. It holds that $h(\rho,\varphi) = 0$ in (22) for the solutions ρ of the equation

$$\alpha - \lambda\sin(2\varphi) + \beta_1\rho^2 + \beta_2\rho^4 = 0\,. \tag{42}$$

This equation can have zero, one, or two positive solutions for $\beta_2 \neq 0$. The following sets describe the nonnegative solutions of (42):

$$\begin{aligned}\rho_+ &= \left\{\rho \in \mathbb{R} : \rho = \sqrt{\frac{-\beta_1 + \sqrt{\beta_1^2 - 4\beta_2(\alpha - \lambda\sin(2\varphi))}}{2\beta_2}}\ ; \varphi \in \langle 0, 2\pi)\right\},\\ \rho_- &= \left\{\rho \in \mathbb{R} : \rho = \sqrt{\frac{-\beta_1 - \sqrt{\beta_1^2 - 4\beta_2(\alpha - \lambda\sin(2\varphi))}}{2\beta_2}}\ ; \varphi \in \langle 0, 2\pi)\right\}.\end{aligned} \tag{43}$$

The number of positive solutions depends on the sign of discriminant $D = \beta_1^2 - 4\beta_2(\alpha - \lambda\sin 2\varphi)$.

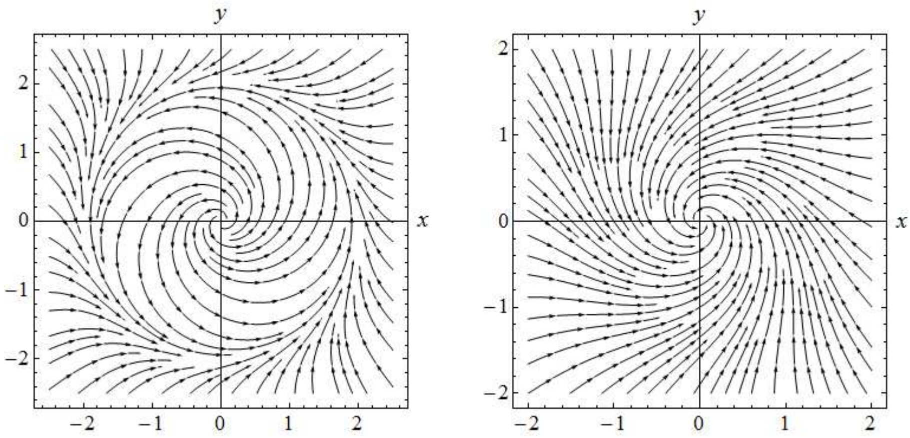
Theorem 6. *Consider the autonomous system* (20) *for* $\beta_3 = 0$ *and* $\beta_2 < 0$, $|d| > \lambda$. *The following assertions are valid:*

(1) *The system* (20) *has at least two limit cycles around the origin for parameter values satisfying* $-\lambda > \alpha > \frac{\beta_1^2}{4\beta_2} + \lambda$ *and* $\beta_1 > 0$.

(2) *The system* (20) *has at least one limit cycle around the origin for parameter values satisfying* $\beta_1 \in \mathbb{R}$ *and* $\alpha > \lambda$.

(3) *The system* (20) *has no limit cycle for parameter values satisfying* $-\lambda > \alpha > \frac{\beta_1^2}{4\beta_2} + \lambda$, $\beta_1 < 0$ *or* $\alpha < \frac{\beta_1^2}{4\beta_2} - \lambda$, $\beta_1 \in \mathbb{R}$.

Proof. Let $\beta_2 < 0$. The system (20) has the focus at the origin for $|d| > \lambda$, which is unique equilibrium of (20), and exhibits Bautin bifurcation at the origin for sufficiently small $||\boldsymbol{\alpha}||$ according to the Theorem 5. Therefore this system can have zero, one, or two limit cycles near the origin. For the nonnegative solutions of (42) it holds $h(\rho,\varphi) = 0$ in (22).



(a) The case (2) of Theorem 6 for $\alpha = 0.075$, $\beta_1 = 0.0125$, $\beta_2 = -0.009375$, $\beta_3 = 0$, $d = 0.15$, $\lambda = 0.005$.

(b) The case (3) of Theorem 6 for $\alpha = -0.075$, $\beta_1 = -0.0125$, $\beta_2 = -0.009375$, $\beta_3 = 0$, $d = 0.15$, $\lambda = 0.005$.

FIG. 4: Direction field of the system (23).

Case (1). Suppose $-\lambda > \alpha > \frac{\beta_1^2}{4\beta_2} + \lambda$ and $\beta_1 > 0$. The equation (42) has discriminant $D = \beta_1^2 - 4\beta_2(\alpha - \lambda \sin(2\varphi))$. It holds that

$$D = \beta_1^2 - 4\beta_2(\alpha - \lambda \sin(2\varphi)) \geq \beta_1^2 - 4\beta_2(\alpha - \lambda) > 0 \quad \text{for} \quad \alpha > \frac{\beta_1^2}{4\beta_2} + \lambda \,. \tag{44}$$

Therefore the equation (42) has two real roots which the sets (43) describe. The set of the points ρ_-, for which it holds that $h(\rho, \varphi) = 0$, is always a nonempty set of real numbers for every polar angle φ and $\beta_1 > 0$. The set ρ_+ contains only positive real numbers because it is true that

$$-\beta_1 + \sqrt{\beta_1^2 - 4\beta_2(\alpha - \lambda \sin(2\varphi))} \leq -\beta_1 + \sqrt{\beta_1^2 - 4\beta_2(\alpha + \lambda)} < 0 \quad \text{for} \quad \alpha < -\lambda \,. \tag{45}$$

It holds:

$$\rho^* < \rho_1^* \leq \rho_- \leq \rho_2^* \,, \tag{46}$$

where

$$\begin{aligned} \rho^* &= \sqrt{\frac{\beta_1}{2|\beta_2|}} \,, \ \rho_1^* = \sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 + 4|\beta_2|(\alpha - \lambda)}}{2|\beta_2|}} \,, \\ \rho_- &= \sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 + 4|\beta_2|(\alpha - \lambda \sin(2\varphi))}}{2|\beta_2|}} \,, \\ \rho_2^* &= \sqrt{\frac{\beta_1 + \sqrt{\beta_1^2 + 4|\beta_2|(\alpha + \lambda)}}{2|\beta_2|}} \,, \end{aligned} \tag{47}$$

and

$$\rho_3^* \le \rho_+ \le \rho_4^* < \rho^* \,, \tag{48}$$

where

$$\begin{aligned}
\rho_3^* &= \sqrt{\frac{\beta_1 - \sqrt{\beta_1^2 + 4|\beta_2|(\alpha+\lambda)}}{2|\beta_2|}}\,,\ \rho_+ = \sqrt{\frac{\beta_1 - \sqrt{\beta_1^2 + 4|\beta_2|(\alpha-\lambda\sin(2\varphi))}}{2|\beta_2|}}\,,\\
\rho_4^* &= \sqrt{\frac{\beta_1 - \sqrt{\beta_1^2 + 4|\beta_2|(\alpha-\lambda)}}{2|\beta_2|}}\,,\ \rho^* = \sqrt{\frac{\beta_1}{2|\beta_2|}}\,,
\end{aligned} \tag{49}$$

for all $\varphi \in \mathbb{R}$. The set of points ρ_- is located in an annulus which is bounded by the circular trajectories ρ_1^* and ρ_2^*. This annulus will be denoted as $A(\rho_1^*, \rho_2^*)$. Similarly, the set of points ρ_+ is located in an annulus $A(\rho_3^*, \rho_4^*)$. It holds that $\rho' < 0$ for $\rho \in (0, \rho_+) \cup (\rho_-, \infty)$ and $\rho' > 0$ for $\rho \in (\rho_+, \rho_-)$ and every polar angle φ. These facts are shown in Figure 5.

The direction field is directed towards the annulus $A(\rho_1^*, \rho_2^*)$ and outward the annulus $A(\rho_3^*, \rho_4^*)$. The assumptions of the Poincaré-Bendixson theorem are satisfied and the Poincaré-Bendixson theorem implies the existence of at least two limit cycles around the stable focus at the origin. Thus the proof of assertion (1) is complete.

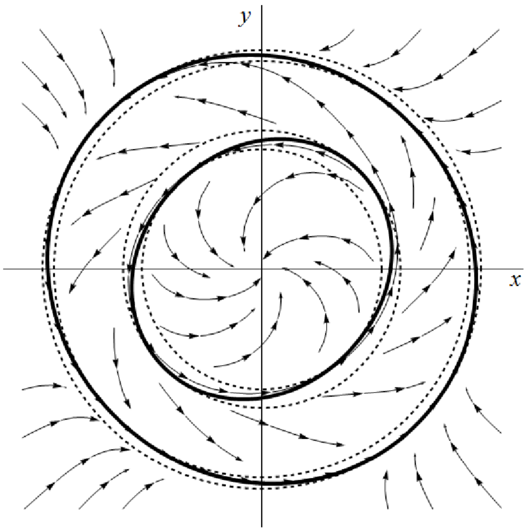


FIG. 5: Directional field of (23) with the annulus $A(\rho_1^*, \rho_2^*)$ for the set of points ρ_- and the annulus $A(\rho_3^*, \rho_4^*)$ for ρ_+.

Case (2). Let $\beta_1 \geq 0$, $\alpha > \lambda$. The equation (42) has $D = \beta_1^2 - 4\beta_2(\alpha - \lambda\sin(2\varphi)) > 0$ (see case (1)). It holds that

$$-\beta_1 + \sqrt{\beta_1^2 - 4\beta_2(\alpha - \lambda\sin 2\varphi)} \geq -\beta_1 + \sqrt{\beta_1^2 - 4\beta_2(\alpha - \lambda)} > 0 \tag{50}$$

for $\alpha > \lambda$, therefore the set of points ρ_+ is empty. This fact implies that the equation (42) has only one positive real root – the set ρ_- is nonempty.
Let $\beta_1 < 0$ and $\alpha > \lambda$. The set ρ_+ is empty and the set ρ_- is nonempty because for $\alpha > \lambda$ it is true that

$$-\beta_1 - \sqrt{\beta_1^2 - 4\beta_2(\alpha - \lambda\sin(2\varphi))} \leq -\beta_1 - \sqrt{\beta_1^2 - 4\beta_2(\alpha - \lambda)} < 0\,. \tag{51}$$

The set of points with radius ρ_- is located in the annulus $A(\rho_1^*, \rho_2^*)$ for $\beta_1 \in \mathbb{R}$. It holds that $\rho' > 0$ for $\rho \in (0, \rho_-)$ and $\rho' < 0$ for $\rho \in (\rho_-, \infty)$. This fact implies that all trajectories go to the annulus $A(\rho_1^*, \rho_2^*)$.
The origin is an unstable focus for $\alpha > \lambda$ and the Poincaré-Bendixson theorem implies the existence of at least one limit cycle. The assertion (2) has been proved.

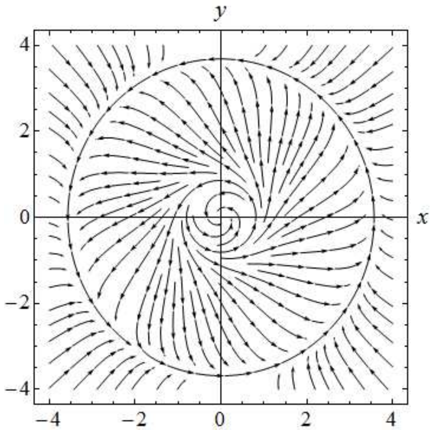
Case (3). Let $-\lambda > \alpha > \frac{\beta_1^2}{4\beta_2} + \lambda$. This inequality implies that the equation (42) has the discriminant $D = \beta_1^2 - 4\beta_2(\alpha - \lambda\sin(2\varphi)) > 0$. The set of points ρ_+ is empty for $\beta_1 < 0$. For $\alpha < -\lambda$ it holds that

$$-\beta_1 - \sqrt{\beta_1^2 - 4\beta_2(\alpha - \lambda\sin(2\varphi))} \geq -\beta_1 - \sqrt{\beta_1^2 - 4\beta_2(\alpha + \lambda)} > 0 \tag{52}$$

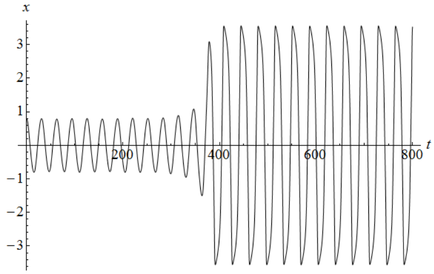
and therefore the set ρ_- is empty too.
The inequality $\alpha < \frac{\beta_1^2}{4\beta_2} - \lambda$ implies that the equation (42) has discriminant $D = \beta_1^2 - 4\beta_2(\alpha - \lambda\sin(2\varphi)) < 0$ and therefore the set of points, for which it is true that $h(\rho, \varphi) = 0$, does not exist.
The derivation ρ' is always negative for a positive real number ρ according to the condition for α. The origin is a stable focus in this case and every trajectory goes to the origin. Thus the proof is complete. □

Remark 7. Figure 4 shows the existence of the stable focus and one limit cycle from Theorem 6. The existence of two limit cycles can be seen in Figure 6. Figure 6b shows the periodic trajectory $x(t)$ of a numerical solution of 2nd order differential equation (3), whose amplitude corresponds to the unstable limit cycle and later to the stable limit cycle.

The stability of the cycles is detectable from the first equation of (22) and from the eigenvalues of the focus at the origin. The equilibrium is stable for $\alpha < 0$ and unstable for $\alpha > 0$. The first Lyapunov quantity $l_1 = \frac{2\pi\beta_1 d}{\sqrt{d^2 - \lambda^2}(d+\lambda)}$. Therefore, the Bautin bifurcation point $\alpha = \beta_1 = 0$ separates two branches corresponding to a Hopf bifurcation with negative and positive Lyapunov quantity. Theorem 6 implies that the behaviour of the system (23) corresponds to the behaviour of Bautin bifurcation.



(a) The case (1) of Theorem 6 for $\alpha = -0.075$, $\beta_1 = 0.125$, $\beta_2 = -0.009375$, $\beta_3 = 0$, $d = 0.15$, $\lambda = 0.005$.



(b) The numerical solution of the equation (3) for $\alpha = -0.075$, $\beta_1 = 0.125$, $\beta_2 = -0.009375$, $\beta_3 = 0$, $d = 0.15$, $\lambda = 0.005$.

Fig. 6: Directional field of (23) and numerical solution of (3) that show unstable and stable limit cycles.

6. Bautin bifurcation of the modified generalized Van der Pol-Mathieu system for $\beta_3 \neq 0$

Now we suppose the system (23) for $\beta_3 \neq 0$. This system has the focus at the origin for $|d| > \lambda$.

For the application of Theorem 6 we transform the system (23) to the complex variable $z = x + iy$ and we shall use the relations $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$:

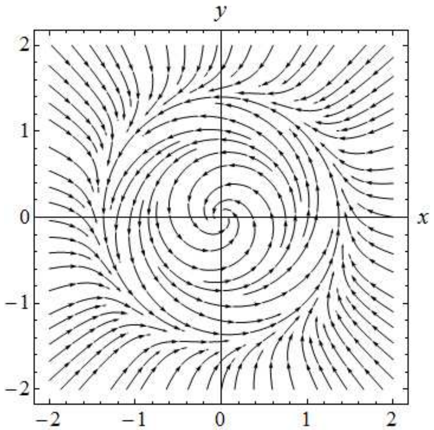
$$\begin{aligned}\frac{dz}{dt} &= \Big(\frac{\alpha}{\sqrt{d^2-\lambda^2}} + i\Big)z + \frac{\beta_1}{2\sqrt{d^2-\lambda^2}(d+\lambda)} \cdot z\left(\lambda z^2 + 2dz\bar{z} + \lambda\bar{z}^2\right)\\ &\quad + \frac{\beta_2}{4\sqrt{d^2-\lambda^2}(d+\lambda)^2} \cdot z\left[\lambda^2 z^4 + 4\lambda d z^3\bar{z} + 2(\lambda^2+2d^2)z^2\bar{z}^2 + 4\lambda d z\bar{z}^3\right.\\ &\quad \left.+\lambda^2\bar{z}^4\right] - \frac{\beta_3}{2^n\sqrt{d^2-\lambda^2}(d+\lambda)^n} \cdot z\left(\lambda z^2 + 2dz\bar{z} + \lambda\bar{z}^2\right)^n .\end{aligned} \tag{53}$$

The following theorem describes the existence of limit cycles.

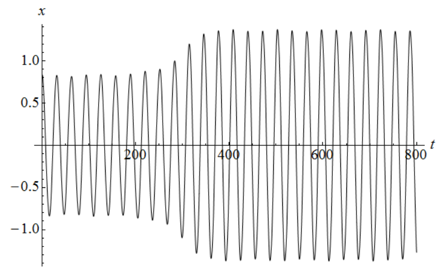
Theorem 7. *Suppose $\beta_3 \neq 0$, $\beta_2 \neq 0$ and $|d| > \lambda$. The modified generalized Van der Pol-Mathieu autonomous system* (23) *exhibits Bautin bifurcation at the equilibrium* $(0,0)$ *for $\boldsymbol{\alpha} = (\alpha, \beta_1)$ near the origin.*

Proof. We shall use Lemmas 2 and 3.

Consider $\beta_3 \neq 0$ and $|d| > \lambda$. The autonomous system (23) can be transformed to the form (53) which is the same form as (9) in Lemma 2. It holds that $\mu(\boldsymbol{\alpha}) = \frac{\alpha}{\sqrt{d^2-\lambda^2}}$ equals zero for $\alpha = 0$ and $\omega(0) = 1 > 0$. Therefore the conditions in Lemma 2 has



(a) Two limit cycles for $\alpha = -0.075$, $\beta_1 = 0.125$, $\beta_2 = -0.009375$, $\beta_3 = 0.0195313$, $d = 0.15$, $\lambda = 0.005$, $n = 3$.



(b) The numerical solution of the equation (3) for $\alpha = -0.075$, $\beta_1 = 0.125$, $\beta_2 = -0.009375$, $\beta_3 = 0.0195313$, $d = 0.15$, $\lambda = 0.005$, $n = 3$.

FIG. 7: Directional field of (23) and numerical solution of (3) that show unstable and stable limit cycles with a stable focus at the origin.

been fulfilled.

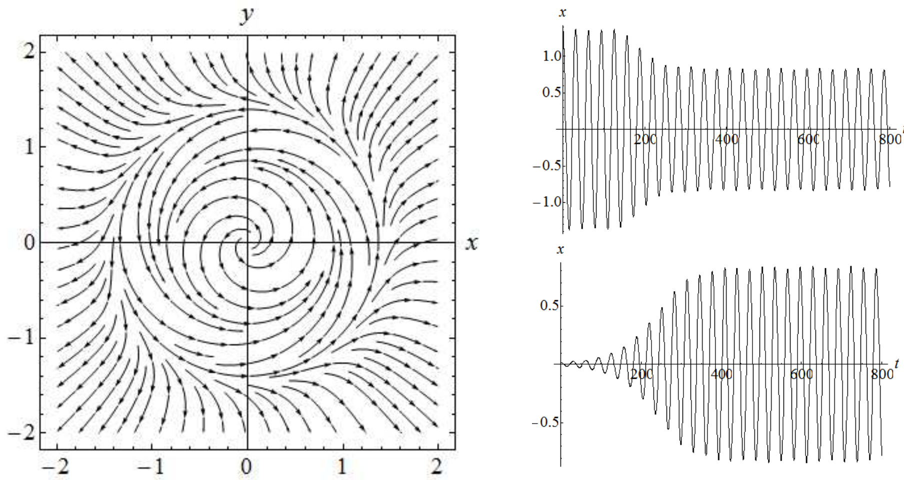
Lemmas 2 and 3 imply that the system (23) is locally topologically equivalent near the equilibrium $(0, 0)$ to the system (23) for $\beta_3 = 0$. Theorem 5 proves this theorem for $\beta_2 \neq 0$. □

Remark 8. Figure 7 shows an unstable limit cycle inside a stable limit cycle. This fact can be seen from the numerical solution in the neighborhood of the unstable limit cycle in Figure 7b.
Figure 8 shows the system (23) for $\beta_3 < 0$. This system has the unstable focus at the origin and the stable limit cycle inside the unstable limit cycle.

The modified generalized Van der Pol-Mathieu autonomous system (23) for $\beta_3 \neq 0$ is locally topologically equivalent near $(0, 0)$ to the system (23) for $\beta_3 = 0$ in view of the proof of Theorem 7. Remark 8 shows the existence of two limit cycles.

7. Consequences for the modified generalized Van der Pol-Mathieu equation

This section shows the theorem about the quasi-periodic behaviour of the modified generalized Van der Pol-Mathieu equation (3). For parameters of the



(a) Two limit cycles for $\alpha = 0.075$, $\beta_1 = -0.125$, $\beta_2 = 0.009375$, $\beta_3 = -0.0195313$, $d = 0.15$, $\lambda = 0.005$, $n = 3$.

(b) The numerical solution of the equation 3 for $\alpha = 0.075$, $\beta_1 = -0.125$, $\beta_2 = 0.009375$, $\beta_3 = -0.0195313$, $d = 0.15$, $\lambda = 0.005$, $n = 3$.

FIG. 8: Directional field of (23) and numerical solution of (3) that show the unstable and stable limit cycles with an unstable focus at the origin.

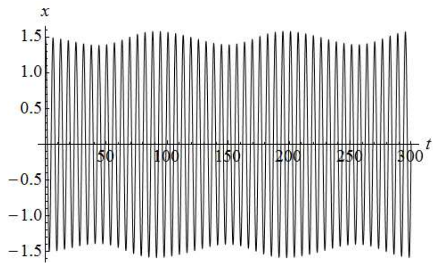
equation (3) it holds that

$$\alpha_0 \frac{2\alpha\omega_0}{\varepsilon}, \quad d_0 = \frac{2d\omega_0}{\varepsilon}, \quad h_0 = \frac{4\lambda}{\varepsilon},$$
$$\beta_{01} = \frac{8\beta_1\omega_0}{\varepsilon}, \quad \beta_{02} = \frac{8\beta_2\omega_0}{\varepsilon}, \beta_{03} = \frac{\beta_3\omega_0 2^{n+1}(n+1)!}{\varepsilon(2n-1)!!}.$$

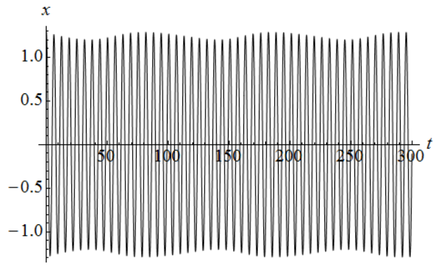
The condition $|d| > \lambda$ for the existence of the focus at the origin in the autonomous system (23) is equivalent to $|d_0| > \frac{h_0\omega_0}{4}$. Using Theorem 1, Theorem 6, Theorem 7 together with the substitution $\tau = (\omega_0 + d_0\varepsilon)t$ and the estimated solution of (3) $x(\tau) = a(\tau)\cos\tau + b(\tau)\sin\tau$ we obtain the following statement:

Theorem 8. *Suppose the modified generalized Van der Pol-Mathieu equation* (3) *for* $\beta_{02} < 0$ *and* $|d_0| > \frac{h_0\omega_0}{4}$. *If* $\beta_{01} \in \mathbb{R}$, $\alpha_0 > \frac{h_0\omega_0}{2}$ *or* $\beta_{01} > 0$, $-\frac{h_0\omega_0}{2} > \alpha_0 > \frac{\beta_{01}^2}{8\beta_{02}} + \frac{h_0\omega_0}{2}$, *then, for small* $\varepsilon > 0$, *the autonomous system* (23) *has a stable periodic solution and the equation* (3) *exhibits quasi-periodic behaviour.*

Remark 9. Figure 9 shows the quasi-periodic behaviour of the solution of (3) (drawn for $\alpha_0 = -0.3$, $\beta_{01} = 2$, $\beta_{02} = -0.3$, $\beta_{03} = 1$, $d_0 = 0.3$, $h_0 = 0.2$, $\varepsilon = 0.1$, $\omega_0 = 1$).



(a) The quasi-periodic solution for $n = 2$ and $x'(0) = 0$, $x(0) = 1.5$.



(b) The quasi-periodic solution for $n = 5$ and $x'(0) = 0$, $x(0) = 1.3$.

Fig. 9: The solution of (3) with quasi-periodic behaviour.

8. Conclusion

The autonomous system (23) associated to the modified generalized Van der Pol-Mathieu equation has been studied. It has been proved that the modified generalized Van der Pol-Mathieu autonomous system (23) exhibits Bautin bifurcation at the equilibrium $(0, 0)$ for $\beta_3 = 0$; see Theorems 5 and 6. These results are generalized for the modified generalized Van der Pol-Mathieu autonomous system (23) for $\beta_3 \neq 0$; see Theorem 7, and for the modified generalized Van der Pol-Mathieu equation (3), that exhibits the quasi-periodic behaviour; see Theorem 8. The numerical results show the existence of limit cycles and complement the theoretical results.

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