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BOUNDEDNESS AND STABILITY IN THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH MULTIPLE DEVIATING ARGUMENTS

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Abstract. In this paper, we establish some new sufficient conditions which guarantee the stability and boundedness of solutions of certain nonlinear and non autonomous differential equations of third order with delay. By defining appropriate Lyapunov function, we obtain some new results on the subject. By this work, we extend and improve some stability and boundedness results in the literature.

1. Introduction

As is well known, the third-order differential equations are derived from many different areas of applied mathematics and physics, for instance, deflection of buckling beam with a fixed or variable cross-section, three-layer beam, electromagnetic waves, gravity-driven flows, etc; see [5, 10, 14, 32] for details. The nonlinear delay differential equations of third order have been the object of intensive research by numerous authors. In particular, there have been extensive results on the stability and boundedness of solutions of various nonlinear differential equations of third order in the literature. See for instance the papers of Ademola [1, 2], Afuwape and Omeike [3], Oudjedi et al. [15], Remili et al. [16, 17], Tunç [16–20], Zhu [34] and the references contained in these sources.

In the following, we provide some background details regarding the study of various classes of Delay differential equations.
In 2007, Zhang and Si [11] proved an asymptotic stability result for solutions to the following nonlinear third order scalar differential equation without delay:

$$x''' + g(x')x'' + f(x,x') + h(x) = 0\,.$$

At the same time, Tunç [21] investigated the stability of solutions of the differential equation

$$x''' + a_1x'' + f_2\big(x'(t-r(t))\big) + a_3x = p\big(t,x,x',x(t-r(t)),x'(t-r(t)),x''\big)\,.$$

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After, Tunç [23, 27] considered the equation

$$x''' + a(t)\psi(x')x'' + b(t)g(x') + c(t)f\big(x(t-r)\big) = p\big(t, x, x', x(t-r), x'(t-r), x''\big)\,,$$

and established some results on the qualitative behavior of solutions of the equation. Recently, in 2010 Afuwape and Omeike [4] considered third order non autonomous differential equation with delay

$$\begin{aligned} x'''(t) + h\big(x'(t)\big)x''(t) + g\big(x'(t-r(t))\big) + f\big(x(t-r(t))\big) \\ = p\big(t, x(t), x'(t), x(t-r(t)), x'(t-r(t)), x''(t)\big)\,. \end{aligned} \tag{1.1}$$

The authors established some sufficient condition under which all solutions of (1.1) are asymptotic stable for $p(\cdot) = 0$ and bounded for $p(\cdot) \neq 0$.

Finally, in 2013 Tunç and Gözen [30] discussed conditions for stability and uniform boundedness of solutions of equation

$$x'''(t) + a(t)x''(t) + nb(t)g\big(x'(t)\big) + c(t)\sum_{i=1}^{n} h_i\big(x(t-r_i)\big) = p(t)\,. \tag{1.2}$$

A primary purpose of this note is to study the uniform asymptotic stability of solutions of the following more general third order nonlinear multi-delay differential equation of the form

$$\begin{aligned} \big[h\big(x(t)\big)x''(t)\big]' + a(t)\psi\big(x'(t)\big)x''(t) + b(t)\sum_{i=1}^{n} g_i\big(x'(t-r_i(t))\big) \\ + c(t)\sum_{i=1}^{n} f_i\big(x(t-r_i(t))\big) = 0\,, \end{aligned} \tag{1.3}$$

and the boundedness of the following

$$\begin{aligned} \big[h\big(x(t)\big)x''(t)\big]' + a(t)\psi\big(x'(t)\big)x''(t) + b(t)\sum_{i=1}^{n} g_i\big(x'(t-r_i(t))\big) \\ + c(t)\sum_{i=1}^{n} f_i\big(x(t-r_i(t))\big) = p\big(t, x(t), X, x'(t), X', x''(t)\big)\,, \end{aligned} \tag{1.4}$$

where $0 \leq r_i(t) \leq \gamma$, $r_i'(t) \leq \omega_i$, $0 < \omega_i < 1$, ω_i and γ are some positive constants, γ will be determined later, $X = x(t-r_1(t)), \ldots, x(t-r_n(t))$ and $X' = x'(t-r_1(t)), \ldots, x'(t-r_n(t))$. The functions $a(t)$, $b(t)$, $c(t)$ are continuous on $[0, +\infty[$ and $h(x)$, $\psi(x')$, $g_i(x')$, $f_i(x)$ and $p(\cdot)$ are continuous in their respective arguments for all i, $(i = 1, 2, \ldots, n)$ with $(f_i(0) = g_i(0) = 0)$, and the primes in (1.3) and (1.4) denote differentiation with respect to t, $t \in \mathbb{R}^+$. Throughout the paper $x(t), y(t)$, and $z(t)$ are abbreviated as x, y, and z, respectively. Finally, the continuity of the functions h, g_i, ψ, f_i, p, a, b and c guarantee the existence of the solution of (1.3) and (1.4) (see [9]). It is assumed that the right-hand side of the equation (1.4) satisfies a Lipschitz condition in $x(t)$, $x'(t)$, X, X' and $x''(t)$. It is also supposed that the derivatives, $a'(t)$, $b'(t)$, $c'(t)$, $g_i'(y) = \frac{dg_i}{dy}$, $f_i'(y) = \frac{df_i}{dy}$, and $h' = \frac{dh}{dx}$ exist and are continuous.

The motivation for the present paper comes from the papers of [4], Omeike [13, 12], Sadek [19, 18], Swick [20], Tunç [23, 21, 30] and Zhu [34]. It follows that the equation (1.2) is a special case of (1.4). Our purpose is to extend and improve the result established by [4] and [23, 21, 30] for the asymptotic stability of the null solution and boundedness of all solutions, when $p = 0$ and $p \neq 0$ in (1.4). By defining an appropriate Lyapunov functional we show similar results for nonlinear equations (1.3) and (1.4).

2. Preliminaries

Before introducing our main results we will give some basic information for the general non-autonomous differential system with retarded argument. Consider the general non-autonomous differential system with a retarded argument:

$$x' = f(t, x_t)\,, \quad x_t(\theta) = x(t+\theta)\,, \quad -r \leq \theta \leq 0\,, \quad t \geq 0\,, \tag{2.1}$$

where $f\colon I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t,0) = 0$, $C_H := \{\phi \in C([-r,0], \mathbb{R}^n)\colon \|\phi\| \leq H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t,\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 2.1 ([8])**.** An element $\psi \in C$ is in the $\omega - limit$ set of ϕ, say $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,+\infty)$ and there is a sequence $\{t_n\}$, $t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)$ for $-r \leq \theta \leq 0$.

Definition 2.2 ([8])**.** A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t,0,\phi)$, is defined on $[0,\infty)$ and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.

Lemma 2.3 ([7])**.** *If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2.1) with $x_0(\phi) = \phi$ is defined on $[0,\infty)$ and $\|x_t(\phi)\| \leq H_1 < H$ for $t \in [0,\infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$\operatorname{dist}\big(x_t(\phi), \Omega(\phi)\big) \to 0 \quad as \quad t \to \infty\,.$$

Lemma 2.4 ([7])**.** *Let $V(t,\phi)\colon I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(t,0) = 0$, and such that:*

(i) $W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(|\phi(0)|) + W_3(\|\phi\|_2)$ *where* $\|\phi\|_2 = \big(\int_{t-r}^{t} \|\phi(s)\|^2 ds\big)^{\frac{1}{2}}$*;*

(ii) $\dot{V}_{(2.1)}(t,\phi) \leq -W_4(|\phi(0)|)$,

where W_i $(i = 1,2,3,4)$ are wedges. Then the zero solution of (2.1) is uniformly asymptotically stable.

3. Assumptions and main results

Let us introduce the temporary notation

$$\phi(t) = \frac{h'(x(t))}{h^2(x(t))} x'(t)\,.$$

Let $p(\cdot) = 0$. The first main problem of this paper is the following theorem.

Theorem 3.1. *In addition to the basic assumptions imposed on the functions $a(t)$, $b(t)$, $c(t)$, $\psi(x')$, $g_i(x')$, $h(x)$, $f_i(x)$ and p, let us assume that there exist positive constants such that the following conditions hold:*

(i) $f_i(0)=0,\ \ \frac{f_i(x)}{x} \geq \delta_i > 0\ (x\neq 0)$, *and* $|f_i'(x)| \leq \rho_i$ *for all* x,

(ii) $g_i(0)=0,\ \ \frac{g_i(y)}{y} \geq d_i > 0\ (y\neq 0)$, *and* $|g_i'(y)| \leq D_i$ *for all* y,

(iii) $1 \leq \psi(y) \leq \beta;\quad 0 < h_0 \leq h(x) \leq h_1$,

(iv) $0 < a \leq a(t) \leq A$, $0 < c \leq c(t) \leq b(t) \leq L$,

(v) $b'(t) \leq c'(t) \leq 0,\ \ \frac{1}{2}a'(t) \leq \delta_2 < \frac{c(\lambda d_i - \rho_i)}{\lambda\beta}$,

(vi) $\frac{d_i}{\rho_i} > \frac{1}{\lambda} > \frac{h_1}{a}$,

(vii) $\int_{-\infty}^{+\infty} |h'(u)|\, du < \infty$.

Then every solution of (1.1) *is uniformly asymptotically stable, provided that*

$$\gamma < \min\left\{\sum_{i=1}^{n}\frac{2h_0^2(a-\lambda h_1)(1-\omega)}{h_1^2[h_0^2 d_i(1+\lambda)+(\rho_i+D_i)(1-\omega)]}\ ,\sum_{i=1}^{n}\frac{2(c(\lambda d_i-\rho_i)-\lambda\beta\delta_2)(1-\omega)}{[\rho_i(1+\lambda)+\lambda(\rho_i+d_i)(1-\omega)]}\right\},$$

where γ is the bound on $r_i(t)$.

Proof. We write the equation (1.3) as the following equivalent system

$$\begin{aligned}
x' &= y\\
y' &= \frac{1}{h(x)}z\\
z' &= -\frac{a(t)}{h(x)}z\psi(y) - b(t)\sum_{i=1}^{n} g_i(y) - c(t)\sum_{i=1}^{n} f_i(x)\\
&\quad + b(t)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\frac{z(s)}{h(x(s))}g_i'\big(y(s)\big)\,ds\\
&\quad + c(t)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t} y(s)f_i'\big(x(s)\big)\,ds\,.
\end{aligned}\tag{3.1}$$

Our main tool in the proof of the theorem just stated above is a Lyapunov function $W = W(t, x_t, y_t, z_t)$ defined by

$$\begin{aligned}
W(t,x_t,y_t,z_t) &= \exp\Big(-\frac{\int_0^t |\phi(s)|\ ds}{\mu}\Big)\,V(t,x_t,y_t,z_t)\\
&= \exp\Big(-\frac{\int_0^t |\phi(s)|\ ds}{\mu}\Big)V\,,
\end{aligned}\tag{3.2}$$

where

$$V = \lambda c(t)F(x) + c(t)y\sum_{i=1}^{n} f_i(x) + b(t)G(y) + \lambda a(t)\int_0^y \psi(u)udu + \frac{1}{2h(x)}z^2 + \lambda yz + \sum_{i=1}^{n}\eta_i\int_{-r_i(t)}^{0}\int_{t+s}^{t} y^2(\xi)\,d\xi\,ds + \sum_{i=1}^{n}\chi_i\int_{-r_i(t)}^{0}\int_{t+s}^{t} z^2(\xi)\,d\xi\,ds\,, \tag{3.3}$$

such that $F(x) = \sum_{i=1}^{n}\int_0^x f_i(u)du$ and $G(y) = \sum_{i=1}^{n}\int_0^y g_i(u)du$, μ and η_i, χ_i are positive constants which will be specified later in the proof. From the definition of V in (3.3), we observe that the above functional can be rewritten as follows

$$V = c(t)\Big[\lambda F(x) + \frac{b(t)}{c(t)}G(y) + y\sum_{i=1}^{n} f_i(x)\Big] + \frac{1}{2h(x)}\big(z+\lambda h(x)y\big)^2 + \lambda a(t)\int_0^y \Big[\psi(u) - \frac{\lambda h(x)}{a(t)}\Big]u\,du + \sum_{i=1}^{n}\eta_i\int_{-r_i(t)}^{0}\int_{t+s}^{t} y^2(\xi)\,d\xi\,ds + \sum_{i=1}^{n}\chi_i\int_{-r_i(t)}^{0}\int_{t+s}^{t} z^2(\xi)\,d\xi\,ds\,.$$

The conditions (i)–(iv) and (vi) of the theorem show that $G(y) \geq \frac{1}{2}\sum_{i=1}^{n} d_i y^2$, then

$$V \geq \frac{c(t)}{2}\sum_{i=1}^{n} d_i\Big\{y + \frac{f_i(x)}{d_i}\Big\}^2 + \lambda c\sum_{i=1}^{n}\int_0^x \Big(1 - \frac{\rho_i}{\lambda d_i}\Big)f_i(s)\,ds + \frac{1}{2h(x)}(z+\lambda h(x)y)^2 + \lambda a\Big(1 - \frac{\lambda h_1}{a}\Big)\frac{y^2}{2} + \sum_{i=1}^{n}\eta_i\int_{-r_i(t)}^{0}\int_{t+s}^{t} y^2(\xi)\,d\xi\,ds + \sum_{i=1}^{n}\chi_i\int_{-r_i(t)}^{0}\int_{t+s}^{t} z^2(\xi)\,d\xi\,ds\,.$$

Since the integrals

$$\sum_{i=1}^{n}\eta_i\int_{-r_i(t)}^{0}\int_{t+s}^{t} y^2(\xi)\,d\xi\,ds \quad \text{and} \quad \sum_{i=1}^{n}\chi_i\int_{-r_i(t)}^{0}\int_{t+s}^{t} z^2(\xi)\,d\xi\,ds$$

are positive, then

$$V \geq \frac{c(t)}{2}\sum_{i=1}^{n} d_i\Big\{y + \frac{f_i(x)}{d_i}\Big\}^2 + \frac{\delta_3}{2}x^2 + \frac{1}{2h(x)}\big(z+\lambda h(x)y\big)^2 + \lambda a\Big(1 - \frac{\lambda h_1}{a}\Big)\frac{y^2}{2}\,,$$

where

$$\delta_3 = \sum_{i=1}^{n}\lambda c\Big(1 - \frac{\rho_i}{\lambda d_i}\Big)\delta_i > \sum_{i=1}^{n}\lambda c\Big(1 - \frac{\lambda}{\lambda}\Big)\delta_i = 0\,.$$

Thus, we can find a positive constant k, small enough such that

$$V \geq k(x^2 + y^2 + z^2)\,. \tag{3.4}$$

It is easy to check that by (iii) and (vii), we have

$$\int_0^t |\phi(s)|\, ds = \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|h'(u)|}{h^2(u)}\, du \le \frac{1}{h_0^2} \int_{-\infty}^{+\infty} |h'(u)|\, du \le N < \infty\,,$$

where $\alpha_1(t) = \min\{x(0), x(t)\}$, and $\alpha_2(t) = \max\{x(0), x(t)\}$. Therefore, we can find a continuous function $W_1(|\Phi(0)|)$ with

$$W_1(|\Phi(0)|) \ge 0 \quad \text{and} \quad W_1(|\Phi(0)|) \le W(t, \Phi)\,.$$

The existence of a continuous function $W_2(|\phi(0)|) + W_3(\|\phi\|_2)$ which satisfies the inequality $W(t, \phi) \le W_2(|\phi(0)|) + W_3(\|\phi\|_2)$, is easily verified.

For the time derivative of the functional $V(t, x_t, y_t, z_t)$, along the trajectories of the system (3.1), we have

$$\begin{aligned}
\frac{d}{dt}V &= \lambda c'(t)F(x) + c'(t)y\sum_{i=1}^n f_i(x) + b'(t)G(y) - \frac{z^2}{h(x)}\Big[\frac{a(t)}{h(x)}\Psi(y) - \lambda\Big] - \frac{1}{2}\phi(t)z^2 \\
&\quad + c(t)\sum_{i=1}^n f_i'(x)y^2 - \lambda b(t)y\sum_{i=1}^n g_i(y) + \lambda a'(t)\int_0^y \psi(u)u\, du + \sum_{i=1}^n \eta_i r_i(t)y^2 \\
&\quad + \Big(\lambda y + \frac{z}{h(x)}\Big)\sum_{i=1}^n \Big[c(t)\int_{t-r_i(t)}^t y(s)f_i'\big(x(s)\big)\, ds + b(t)\int_{t-r_i(t)}^t \frac{z(s)}{h(x(s))}g_i'\big(y(s)\big)\, ds\Big] \\
&\quad + \sum_{i=1}^n \chi_i r_i(t)z^2 - \sum_{i=1}^n \eta_i\big(1 - r_i'(t)\big)\int_{t-r_i(t)}^t y^2(\xi)\, d\xi \\
&\quad - \sum_{i=1}^n \chi_i\big(1 - r_i'(t)\big)\int_{t-r_i(t)}^t z^2(\xi)\, d\xi\,.
\end{aligned}$$

Consequently by the hypothesis (i)–(vi) we get

$$\begin{aligned}
\frac{d}{dt}V &\le \lambda c'(t)F(x) + c'(t)y\sum_{i=1}^n f_i(x) + \sum_{i=1}^n \frac{d_i}{2}b'(t)y^2 - \frac{a - \lambda h_1}{h_1^2}z^2 \\
&\quad + \Big[\lambda\beta\delta_2 - c\sum_{i=1}^n(\lambda d_i - \rho_i)\Big]y^2 + \sum_{i=1}^n \eta_i r_i(t)y^2 + \sum_{i=1}^n \chi_i r_i(t)z^2 + \frac{1}{2}|\phi(t)|z^2 \\
&\quad + \Big(\lambda y + \frac{z}{h(x)}\Big)\sum_{i=1}^n \Big[c(t)\int_{t-r_i(t)}^t y(s)f_i'\big(x(s)\big)\, ds + b(t)\int_{t-r_i(t)}^t \frac{z(s)}{h(x(s))}g_i'\big(y(s)\big)\, ds\Big] \\
&\quad - \sum_{i=1}^n \eta_i\big(1 - r_i'(t)\big)\int_{t-r_i(t)}^t y^2(\xi)\, d\xi - \sum_{i=1}^n \chi_i\big(1 - r_i'(t)\big)\int_{t-r_i(t)}^t z^2(\xi)\, d\xi\,.
\end{aligned}$$

Now consider the term

$$Q(t, x, y) = \lambda c'(t)F(x) + c'(t)y\sum_{i=1}^n f_i(x) + \sum_{i=1}^n \frac{d_i}{2}b'(t)y^2\,,$$

for all x, y and $t \geq 0$. There are two cases $c'(t) = 0$ or $c'(t) < 0$.
If $c'(t) = 0$, then $Q(t,x,y) = \sum_{i=1}^{n} \frac{d_i b'(t)}{2} y^2 \leq 0$. If $c'(t) < 0$, then

$$
\begin{aligned}
Q(t,x,y) &\leq \lambda c'(t)\Big[F(x) + \frac{1}{\lambda} y \sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n} \frac{d_i b'(t)}{2\lambda c'(t)} y^2\Big] \\
&\leq \lambda c'(t)\Big[F(x) + \sum_{i=1}^{n} \frac{d_i b'(t)}{2\lambda c'(t)} \Big\{y + \frac{c'(t) f_i(x)}{d_i b'(t)}\Big\}^2 - \sum_{i=1}^{n} \frac{c'(t) f_i^2(x)}{2\lambda d_i b'(t)}\Big].
\end{aligned}
$$

It is required that $\frac{c'(t)}{b'(t)} \leq 1$ by (v), then

$$
\begin{aligned}
Q(t,x,y) &\leq \lambda c'(t) \sum_{i=1}^{n} \int_0^x \Big(1 - \frac{\rho_i}{\lambda d_i}\Big) f_i(u)\, du \\
&\leq c'(t) \frac{\delta_3}{c\delta_i} F(x) \leq 0\,.
\end{aligned}
$$

In both cases, we have $Q(t,x,y) \leq 0$ for all $t \geq 0$, x and y. Using the inequality $|uv| \leq \frac{1}{2}(u^2 + v^2)$ and since $|f_i'(x)| \leq \rho_i$ and $|g_i'(y)| \leq D_i$, we obtain the following inequalities

$$
\begin{cases}
\lambda y \displaystyle\sum_{i=1}^{n} \int_{t-r_i(t)}^{t} y(s) f_i'\big(x(s)\big)\, ds \leq \sum_{i=1}^{n} \frac{\lambda \rho_i r_i(t)}{2} y^2 + \sum_{i=1}^{n} \frac{\lambda \rho_i}{2} \int_{t-r_i(t)}^{t} y^2(\xi)\, d\xi \\
\dfrac{z}{h(x)} \displaystyle\sum_{i=1}^{n} \int_{t-r_i(t)}^{t} y(s) f_i'\big(x(s)\big)\, ds \leq \sum_{i=1}^{n} \frac{\rho_i r_i(t)}{2h_0^2} z^2 + \sum_{i=1}^{n} \frac{\rho_i}{2} \int_{t-r_i(t)}^{t} y^2(\xi)\, d\xi\,,
\end{cases}
$$

and

$$
\begin{cases}
\lambda y \displaystyle\sum_{i=1}^{n} \int_{t-r_i(t)}^{t} z(s) g_i'\big(y(s)\big)\, ds \leq \sum_{i=1}^{n} \frac{\lambda D_i r_i(t)}{2} y^2 + \sum_{i=1}^{n} \frac{\lambda D_i}{2} \int_{t-r_i(t)}^{t} z^2(\xi)\, d\xi \\
\dfrac{z}{h(x)} \displaystyle\sum_{i=1}^{n} \int_{t-r_i(t)}^{t} z(s) g_i'\big(y(s)\big)\, ds \leq \sum_{i=1}^{n} \frac{D_i r_i(t)}{2h_0^2} z^2 + \sum_{i=1}^{n} \frac{D_i}{2} \int_{t-r_i(t)}^{t} z^2(\xi)\, d\xi\,.
\end{cases}
$$

We rearrange

$$
\begin{aligned}
\frac{d}{dt} V \leq &-\Big[c \sum_{i=1}^{n} (\lambda d_i - \rho_i) - \lambda\beta\delta_2 - \sum_{i=1}^{n} \Big(\eta_i + \frac{\lambda(\rho_i + D_i)}{2}\Big) r_i(t)\Big] y^2 \\
&- \Big[\frac{a - \lambda h_1}{h_1^2} - \sum_{i=1}^{n} \Big(\chi_i + \frac{\rho_i + D_i}{2h_0^2}\Big) r_i(t)\Big] z^2 + \frac{1}{2} |\phi(t)| z^2 \\
&+ \sum_{i=1}^{n} \frac{\rho_i}{2} \Big[1 + \lambda - \frac{2\eta_i}{\rho_i}(1 - \omega)\Big] \int_{t-r_i(t)}^{t} y^2(\xi)\, d\xi \\
&+ \sum_{i=1}^{n} \frac{D_i}{2} \Big[1 + \lambda - \frac{2\chi_i}{d_i}(1 - \omega)\Big] \int_{t-r_i(t)}^{t} z^2(\xi)\, d\xi\,.
\end{aligned}
$$

If we take $\dfrac{\rho_i(1+\lambda)}{2(1-\omega)} = \eta_i > 0$, $\dfrac{d_i(1+\lambda)}{2(1-\omega)} = \chi_i > 0$ and $r_i(t) \leq \gamma$, the last inequality becomes

$$\begin{aligned}\frac{d}{dt}V \leq &-\Big[c\sum_{i=1}^{n}(\lambda d_i-\rho_i)-\lambda\beta\delta_2-\gamma\sum_{i=1}^{n}\Big(\frac{\rho_i(1+\lambda)+\lambda(\rho_i+D_i)(1-\omega)}{2(1-\omega)}\Big)\Big]y^2\\ &-\Big[\frac{a-\lambda h_1}{h_1^2}-\gamma\sum_{i=1}^{n}\Big(\frac{h_0^2 d_i(1+\lambda)+(\rho_i+D_i)(1-\omega)}{2h_0^2(1-\omega)}\Big)\Big]z^2+|\phi(t)|z^2\,.\end{aligned}$$

Using (3.4), (3.2) and taking $\mu = k$ we obtain:

$$\begin{aligned}\frac{d}{dt}W &= \exp\Big(-\frac{\int_0^t|\phi(s)|\,ds}{k}\Big)\Big(\frac{d}{dt}V-\frac{|\phi(t)|}{k}V\Big)\\ &\leq \exp\Big(-\frac{\int_0^t|\phi(s)|\,ds}{k}\Big)\Big[-\Big[c\sum_{i=1}^{n}(\lambda d_i-\rho_i)-\lambda\beta\delta_2\\ &\qquad-\gamma\sum_{i=1}^{n}\Big(\frac{\rho_i(1+\lambda)+\lambda(\rho_i+D_i)(1-\omega)}{2(1-\omega)}\Big)\Big]y^2\\ &\qquad-\Big[\frac{a-\lambda h_1}{h_1^2}-\gamma\sum_{i=1}^{n}\Big(\frac{h_0^2 d_i(1+\lambda)+(\rho_i+D_i)(1-\omega)}{2h_0^2(1-\omega)}\Big)\Big]z^2\Big]\,.\end{aligned}\tag{3.5}$$

Therefore, if

$$\gamma < \min\left\{\sum_{i=1}^{n}\frac{2h_0^2(a-\lambda h_1)(1-\omega)}{h_1^2[h_0^2 d_i(1+\lambda)+(\rho_i+D_i)(1-\omega)]}\,,\sum_{i=1}^{n}\frac{2(c(\lambda d_i-\rho_i)-\lambda\beta\delta_2)(1-\omega)}{[\rho_i(1+\lambda)+\lambda(\rho_i+d_i)(1-\omega)]}\right\}$$

the inequality (3.5) becomes

$$\frac{d}{dt}W(t,x_t,y_t,z_t) \leq -W_4(x,y,z)$$

where $W_4(x,y,z) = N_1(y^2+z^2)$, for some $N_1 > 0$. It follows, by the conditions (i) and (iv), that $W_4(x,y,z) = 0$ if and only if $x = y = z = 0$ in the system (3.1), and $\dfrac{d}{dt}W(t,\phi) \leq -W_4(x,y,z) < 0$ for $\phi \neq 0$. Thus, all the conditions of Lemma 2.4 are satisfied. This shows that every solution of (1.3) is uniformly asymptotically stable. Hence the proof of Theorem 3.1 is complete. □

In the case $p(\cdot) \neq 0$. The second main result of this paper is the following theorem.

Theorem 3.2. *In addition to the assumptions of Theorem 3.1, if we assume that p is continuous, and*

$$|p(\cdot)| \leq q(t)\,,$$

where $q \in L^1(0,\infty)$, $L^1(0,\infty)$ is the space of Lebesgue integrable functions. Then all solutions of the perturbed equation (1.4) *are bounded.*

Proof. We consider the equivalent system of (1.4)

$$
\begin{aligned}
x' &= y\\
y' &= \frac{z}{h(x)}\\
z' &= -\frac{a(t)}{h(x)} z\psi(y) - b(t)\sum_{i=1}^{n} g_i(y) - c(t)\sum_{i=1}^{n} f_i(x)\\
&\quad + b(t)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t} \frac{z(s)}{h(x(s))} g_i'(y)\,ds\\
&\quad + c(t)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t} y f_i'(x(s))ds\\
&\quad + p\Big(t, x, \dots, x\big(t-r_n(t)\big), y, \dots, y\big(t-r_n(t)\big), \frac{z}{h(x)}\Big).
\end{aligned} \tag{3.6}
$$

An easy calculation from (3.6) and (3.2) yields that

$$\frac{d}{dt}U_{(3.6)} = \frac{d}{dt}U_{(3.1)} + \Big(\frac{z}{h(x)} + \lambda y\Big)p(\cdot).$$

Since $\frac{d}{dt}U_{(3.1)} \le 0$ and noting that $|x| \le 1 + x^2$, then

$$
\begin{aligned}
\frac{d}{dt}U_{(3.6)} &\le \Big(\frac{|z|}{h(x)} + \lambda|y|\Big)\, |q(t)| \le k_1(|z| + |y|)|q(t)|\\
&\le k_1(2 + z^2 + y^2)|q(t)| \le k_1\|X\|^2|q(t)| + 2k_1|q(t)|\\
&\le \frac{k_1}{\delta e^{-\frac{k_2}{\mu}}}|q(t)|U + 2k_1|q(t)|,
\end{aligned}
$$

where $k_1 = \max\Big\{\frac{1}{h_0}, \lambda\Big\}$, recalling that

$$\delta e^{-\frac{k_2}{\mu}}\|X\|^2 \le U(t, x_t, y_t, z_t).$$

Let $\kappa = \max\Big\{2k_1, \frac{k_1}{\delta e^{-\frac{k_2}{\mu}}}\Big\}$, then

$$\frac{d}{dt}U_{(3.6)} \le \kappa|q(t)| + \kappa|q(t)|U.$$

Multiplying each side of this inequality by the integrating factor $e^{-\kappa\int_0^t |q(s)|ds}$, we get

$$e^{-\kappa\int_0^t |q(s)|ds}\frac{d}{dt}U_{(3.6)} \le e^{-\kappa\int_0^t |q(s)|ds}\kappa|q(t)| + e^{-\kappa\int_0^t |q(s)|ds}\kappa|q(t)|U.$$

Integrating each side of this inequality from 0 to t, we get

$$e^{-\kappa\int_0^t |q(s)|ds}U - U(0, X_0) \le 1 - e^{-\kappa\int_0^t |q(s)|ds},$$

where $X_0 = (x(0), y(0), z(0))$. Since $\int_0^t |q(s)|ds \leq L$ for all $t \geq 0$, we have

$$U(t, x_t, y_t, z_t) \leq U(0, X_0)e^{\kappa L} + [e^{\kappa L} - 1] \quad \text{for} \quad t \geq 0.$$

Now, since the right-hand side is a constant, and since $U(t, x_t, y_t, z_t) \to \infty$ as $x^2 + y^2 + z^2 \to \infty$, it follows that there exists a $D > 0$ such that

$$|x(t)| \leq D\,, \quad |y(t)| \leq D\,, \quad |z(t)| \leq D \qquad \forall t \geq 0\,,$$

thus we can deduce

$$|x(t)| \leq C\,, \quad |x'(t)| \leq C\,, \quad |x''(t)| \leq C \quad \forall t \geq 0\,,$$

□

Example 3.3. Consider the equation

$$\begin{aligned}
&\Big(\Big(\frac{\cos x - 1}{1 + x^2} + 3\Big)x''\Big)' + \Big(\frac{21}{2} - \frac{1}{2}e^{-\frac{1}{2}t}\Big)\Big(\arctan x' + \frac{5\pi}{6}\Big)x'' \\
&\qquad + \Big(\frac{1}{1+t} + 1\Big)\sum_{i=1}^{n}\Big(2ix'(t - r_i(t)) + \frac{ix'(t - r_i(t))}{1 + ix'^2(t - r_i(t))}\Big) \\
&\qquad + \Big(\frac{1}{2(1+t)} + \frac{1}{2}\Big)\sum_{i=1}^{n}\Big[ix(t - r_i(t)) + \frac{ix(t - r_i(t))}{1 + |x(t - r_i(t))|}\Big] = 0\,.
\end{aligned} \tag{3.7}$$

Now, it is easy to see that

$$\begin{aligned}
&10 = a \leq a(t) = \frac{21}{2} - \frac{1}{2}e^{-\frac{1}{2}t} \leq \frac{21}{2}\,, \quad a'(t) = \frac{1}{4}e^{-\frac{1}{2}t} \leq \frac{1}{4}\,, \quad t \geq 0\,, \\
&c = \frac{1}{2} \leq c(t) = \frac{1}{2(1+t)} + \frac{1}{2} \leq b(t) = \frac{1}{1+t} + 1 \leq 2 = L\,, \quad t \geq 0\,, \\
&-1 \leq b'(t) \leq c'(t) \leq 0\,, \quad \forall t \geq 0\,, \\
&\delta_i = i \leq \frac{f_i(x)}{x} = \Big(i + \frac{i}{1 + |x|}\Big) \quad \text{with} \quad x \neq 0\,, \quad \text{and} \quad |f_i'(x)| \leq \rho_i = 2i\,, \\
&d_i = 2i \leq \frac{g_i(y)}{y} = 2i + \frac{i}{1 + iy^2} \quad \text{with} \quad y \neq 0\,, \quad \text{and} \quad |g_i'(y)| \leq D_i = 3i\,, \\
&1 \leq h(x) = \frac{\cos x - 1}{1 + x^2} + 3 \leq 3 = h_1\,, \\
&1 \leq \psi(y) = \arctan y + \frac{5\pi}{6} \leq \frac{4\pi}{3} = \beta\,, \\
&1 = \frac{\rho_i}{d_i} < \lambda < \frac{a}{h_1} = \frac{10}{3}\,, \\
&\frac{1}{2}a'(t) \leq \frac{1}{8} < \frac{c(\lambda d_i - \rho_i)}{\lambda\beta} < \frac{3i}{4\pi}\,.
\end{aligned}$$

An explicit calculation shows that

$$\begin{aligned}
\int_{-\infty}^{+\infty} |h'(u)|\, du &\leq \int_{-\infty}^{+\infty} \Big[\Big|\frac{-\sin u}{1 + u^2}\Big| + \Big|\frac{2u(\cos u - 1)}{(1 + u^2)^2}\Big|\Big]\, du \\
&\leq \pi + 8\,.
\end{aligned}$$

All the assumptions (i) through (vii) are satisfied, we can conclude using Theorem 3.1 that every solution of (3.7) is uniformly asymptotically stable.

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