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SOME FURTHER RESULTS ON LIFTS OF LINEAR VECTOR FIELDS RELATED TO PRODUCT PRESERVING GAUGE BUNDLE FUNCTORS ON VECTOR BUNDLES

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Abstract. We present some lifts (associated to a product preserving gauge bundle functor on vector bundles) of sections of the dual bundle of a vector bundle, some derivations and linear connections on vector bundles.

1. Introduction

Weil functors (product preserving bundle functors on manifolds) were classified by [1], [8] and [3]. Indeed the results of these papers say in particular that the set of equivalence classes of such functors are in bijection with the set of equivalence classes of Weil algebras. These functors were used by many authors (ex. [2], [5], [6], [7]) to present some lifts of various geometric objects (smooth functions, tensor fields, linear connections on manifolds,...).

Product preserving gauge bundle functor on vector bundles (an example of bundle functors on local categories) were classified in [10]: The set of equivalence classes of such functors are in bijection with the set of equivalence classes of pairs (A, V), where A is a Weil algebra and V a A-module such that $\dim_{\mathbb{R}}(V) < \infty$. Similarly to what is done for Weil functors some authors (ex. [11], [14]) present some lifts of some geometric objects related to product preserving gauge bundle functor on vector bundles.

In this paper, we continue what we began in [14] by presenting some lifts (associated to a product preserving gauge bundle functor on vector bundles) of sections of the dual bundle of a vector bundle, some derivations and linear connections on vector bundles.

2. Algebraic description of Weil functors

2.1. Weil algebra.

A *Weil algebra* is a real commutative unital algebra such that $A = \mathbb{R}\cdot 1_A \bigoplus N$,

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where N is a finite dimensional ideal of nilpotent elements. For other equivalent definitions of Weil algebras and examples, one can refer to [7].

2.2. **The Weil functor $T^A\colon \mathcal{M}f \to \mathcal{FM}$.**
We write $\mathcal{M}f$ for the category of finite dimensional differential manifolds and mappings of class C^∞; furthermore, $\mathcal{FM}$ is the category of fibered manifolds and fibered manifolds morphisms.

Let us recall this construction of Weil functors based on [16]. For a Weil algebra $A = \mathbb{R}\cdot 1_A \bigoplus N$ and any point x of a manifold M, let $C^\infty_x(M,\mathbb{R})$ and $\mathrm{Hom}(C^\infty_x(M,\mathbb{R}),A)$ be the algebra of germs on x of smooth functions and the set of algebra homomorphisms from $C^\infty_x(M,\mathbb{R})$ into A respectively; If $\mathcal{E}ns$ denotes the category of sets and mappings, one defines a functor $T^A\colon \mathcal{M}f \to \mathcal{E}ns$ by:

$$T^A M := \bigcup_{x\in M} \mathrm{Hom}(C^\infty_x(M,\mathbb{R}),A) \quad \text{and} \quad \left(T^A f\right)_x(\varphi_x) := \varphi_x \circ f^*_x\,,$$

for a manifold M and $f \in C^\infty(M,M')$, where $f^*_x \in \mathrm{Hom}(C^\infty_{f(x)}(M',\mathbb{R}), C^\infty_x(M,\mathbb{R}))$ is the pull-back algebra homomorphism defined by $f^*\big(\mathrm{germ}_{f(x)}(h)\big) = \mathrm{germ}_x(h\circ f)$.

Now, let $q_{A,M}\colon T^AM \to M$, $\left(T^AM\right)_x \ni \varphi \mapsto x$; hence $\left(T^AM, M, q_{A,M}\right)$ is a well-defined fibered manifold. Indeed let $c = (U,u^i)$, $1\le i\le m$ be a chart of M; then the map

$$\begin{aligned}\phi_c\colon (q_{A,M})^{-1}(U) &\longrightarrow U\times N^m\\ \varphi_x &\longmapsto \left(x, \varphi_x\left(\mathrm{germ}_x(u^i - u^i(x))\right)\right)\end{aligned}$$

is a local trivialization of T^AM. Given another manifold M' and a smooth map $f\colon M\to M'$, T^Af is a fibered map. Indeed for charts $c = (U,u,m)$, $c' = (W,w,m')$ of M, M' such that $f(U)\subset W$, $\phi_{c'}\circ T^Af\circ\phi_c^{-1}$ is the map

$$\begin{aligned}U\times N^m &\to W\times N^{m'}\\ (x,n_i) &\mapsto \left(f(x), n'_j\right)\end{aligned}$$

where $n'_j = \sum\limits_{\alpha\in\mathbb{N}^m\setminus\{0\}} \frac{1}{\alpha!} D_\alpha(w^j\circ f\circ u^{-1})(u(x)) n_1^{\alpha_1}\dots n_m^{\alpha_m}$, $1\le j\le m'$ with $D_\alpha F^j = \frac{\partial^{|\alpha|}F^j}{(\partial x^1)^{\alpha_1}\dots(\partial x^m)^{\alpha_m}}$.

$T^A\colon \mathcal{M}f\to\mathcal{FM}$ is a product preserving bundle functor called the *Weil functor* associated to A.

Let $c = (U,u)$ be a chart of M; in all the paper, we'll use fibered charts $\left(q^{-1}_{A,M}(U), u^{i,\alpha}\right)$, $1\le i\le m$, $0\le\alpha\le K$ $(=\dim N)$ of T^AM associated to the fibered isomorphism T^Au and defined by $u^{i,\alpha} = e^*_\alpha\circ T^A\left(u^i\right)$, where (e^*_α) is the dual basis of a fixed basis $(e_\alpha)_{0\le\alpha\le K}$ of A such that $e_0 = 1_A$.

3. Product preserving gauge bundle functor on vector bundles

3.1. **Product preserving gauge bundle functor on $\mathcal{VB}$.**
Let $F\colon \mathcal{VB}\to\mathcal{FM}$ be a covariant functor from the category $\mathcal{VB}$ of all vector bundles and their vector bundle homomorphisms into the category $\mathcal{FM}$ of fibered

manifolds and their fibered maps. Let $B_{\mathcal{VB}}\colon \mathcal{VB} \to \mathcal{M}f$ and $B_{\mathcal{FM}}\colon \mathcal{FM} \to \mathcal{M}f$ be the respective base functors.

Definition 3.1. F is a gauge bundle *functor on* $\mathcal{VB}$ when the following conditions are satisfied:

• **Prolongation.** $B_{\mathcal{FM}} \circ F = B_{\mathcal{VB}}$ i.e. F transforms a vector bundle $E \xrightarrow{q} M$ in a fibered manifold $FE \xrightarrow{p_E} M$ and a vector bundle morphism $E \xrightarrow{f} G$ over $M \xrightarrow{\overline{f}} N$ in a fibered map $FE \xrightarrow{Ff} FG$ over $\overline{f}$.

• **Localization.** For any vector bundle $E \xrightarrow{q} M$ and any inclusion of an open vector subbundle $i\colon q^{-1}(U) \hookrightarrow E$, the fibered map $Fq^{-1}(U) \to p_E^{-1}(U)$ over id_U induced by Fi is an isomorphism then the map Fi can be identified to the inclusion $p_E^{-1}(U) \hookrightarrow FE$.

Given two gauge bundle functors F_1, F_2 on $\mathcal{VB}$, by *a natural transformation* $\tau\colon F_1 \to F_2$ we shall mean a system of base preserving fibered maps $\tau_E\colon F_1E \to F_2E$ for every vector bundle E satisfying $F_2 f \circ \tau_E = \tau_G \circ F_1 f$ for every vector bundle morphism $f\colon E \to G$.

A gauge bundle functor F on $\mathcal{VB}$ is *product preserving* if for any product projections $E_1 \xleftarrow{\mathrm{pr}_1} E_1 \times E_2 \xrightarrow{\mathrm{pr}_2} E_2$ in the category $\mathcal{VB}$, $FE_1 \xleftarrow{F\,\mathrm{pr}_1} F(E_1 \times E_2) \xrightarrow{F\,\mathrm{pr}_2} FE_2$ are product projections in the category $\mathcal{FM}$. In other words, the map $(F\,\mathrm{pr}_1, F\,\mathrm{pr}_2) : F(E_1 \times E_2) \to F(E_1) \times F(E_2)$ is a fibered isomorphism over $\mathrm{id}_{M_1 \times M_2}$.

Example 3.1. Let $A = \mathbb{R}{\cdot}1_A \bigoplus N$ be a Weil algebra.

(a) Each Weil functor T^A induces a product preserving gauge bundle functor $T^A\colon \mathcal{VB} \to \mathcal{FM}$ in a natural way.

(b) **The A-vertical bundle functor** $V_A\colon \mathcal{VB} \to \mathcal{FM}$: For a vector bundle (E, M, q), let $V_AE = \bigcup_{x\in M} T^A(E_x)$ and q_E^A the restriction to V_AE of the bundle projection $q_{A,E}\colon T^AE \to E$. Let $c = (q^{-1}(U), x^i, y^j)$, $1 \le i \le m$, $1 \le j \le n$ be a fibered chart of E; then $\varphi_u \in V_AE \cap q_{A,E}^{-1}(q^{-1}(U)))$ iff $\varphi_u\left(\mathrm{germ}_u(x^i - x^i(q(u))\right) = 0$, $1 \le i \le m$. This shows that V_AE is a submanifold of T^AE and (V_AE, E, q_E^A) is a fibered manifold. For a vector bundle $f\colon E \to G$ over $\overline{f}\colon M \to N$, one defines a fibered morphism $V_Af\colon V_AE \to V_AG$ over f by $(V_Af)_x = T^A(f_x)$. The gauge bundle *functor* $V_A\colon \mathcal{VB} \to \mathcal{FM}$ is given by:

$$\begin{cases} V_A(E, M, q) = (V_AE, M, p_E^A) \\ \text{and} \\ V_A(\overline{f}, f) = (\overline{f}, V_Af)\,, \end{cases}$$

with $p_E^A = q \circ q_E^A$. Moreover V_A is product preserving. Indeed let (E_i, M_i, q_i), $i = 1, 2$ vector bundles and $p_{ri}\colon E_1 \times E_2 \to E_i$, $i = 1, 2$ the induced projections. Then the isomorphisms $(T^Ap_{r1}, T^Ap_{r2})\colon T^A(E_1 \times E_2) \to T^AE_1 \times T^AE_2$ sends $V_A(E_1 \times E_2)$

on $V_AE_1 \times V_AE_2$. The local trivialization of V_AE associated to c is the map

$$\begin{aligned}\phi_c\colon (p_E^A)^{-1}(U) &\longrightarrow U \times A^n\\ \varphi_u &\longmapsto \big(q(u), \varphi_u(\mathrm{germ}_u(y^j))\big)\,.\end{aligned}$$

(c) **The gauge bundle functor** $T^{A,V}\colon \mathcal{VB} \to \mathcal{FM}$: Let V be a A-module such that $\dim_{\mathbb{R}}(V) < \infty$. For a vector bundle (E, M, q) and $x \in M$, let

$$T_x^{A,V}E = \big\{(\varphi_x, \psi_x)/\varphi_x \in \mathrm{Hom}(C_x^\infty(M,\mathbb{R}), A) \text{ and } \ \psi_x \in \mathrm{Hom}_{\varphi_x}(C_x^{\infty, f\cdot l}(E), V)\big\}$$

where $\mathrm{Hom}(C_x^\infty(M,\mathbb{R}), A)$ is the set of algebra homomorphisms φ_x from the algebra $C_x^\infty(M,\mathbb{R}) = \{\mathrm{germ}_x(g)\ /\ g \in C^\infty(M,\mathbb{R})\}$ into A and $\mathrm{Hom}_{\varphi_x}(C_x^{\infty,f\cdot l}(E), V)$ is the set of module homomorphisms ψ_x over φ_x from the $C_x^\infty(M,\mathbb{R})$-module $C_x^{\infty,f\cdot l}(E,\mathbb{R}) = \{\mathrm{germ}_x(h)\ /\ h\colon E \to \mathbb{R} \text{ is fiberwise linear}\}$ into V. Let $T^{A,V}E = \bigcup_{x\in M} T_x^{A,V}E$ and $p_E^{A,V}\colon T^{A,V}E \to M, T_x^{A,V}E \ni (\varphi,\psi) \mapsto x$. $\big(T^{A,V}E, M, p_E^{A,V}\big)$ is a well-defined fibered manifold. Indeed let $c = (q^{-1}(U), x^i = u^i \circ q, y^j)$, $1 \le i \le m$, $1 \le j \le n$ be a fibered chart of E; then the map

$$\begin{aligned}\phi_c\colon (p_E^{A,V})^{-1}(U) &\longrightarrow U \times N^m \times V^n\\ (\varphi_x, \psi_x) &\longmapsto \big(x, \varphi_x\big(\mathrm{germ}_x(u^i - u^i(x))\big), \psi_x(\mathrm{germ}_x(y^j))\big)\,;\end{aligned}$$

is a local trivialization of $T^{A,V}E$. Given another vector bundle (G, M', q') and a vector bundle homomorphism $f\colon E \to G$ over $\overline{f}\colon M \to M'$, let

$$\begin{aligned}T^{A,V}f\colon T^{A,V}E &\longrightarrow T^{A,V}G\\ (\varphi_x, \psi_x) &\longmapsto (\varphi_x \circ \overline{f}_x^*, \psi_x \circ f_x^*)\,,\end{aligned}$$

where $\overline{f}_x^*\colon C_{\overline{f}(x)}^\infty(N) \to C_x^\infty(M)$ and $f_x^*\colon C_{\overline{f}(x)}^{\infty,f\cdot l\cdot}(G) \to C_x^{\infty,f\cdot l\cdot}(E)$ are given by the pull-back with respect to $\overline{f}$ and f. Then $T^{A,V}f$ is a fibered map over $\overline{f}$. $T^{A,V}\colon \mathcal{VB} \to \mathcal{FM}$ is a product preserving gauge bundle functor (see [10]).

Remark 3.1. Let $F\colon \mathcal{VB} \to \mathcal{FM}$ be a product preserving gauge bundle functor.

(a) F associates the pair $\left(A^F, V^F\right)$ where $A^F = F\left(\mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}\right)$ is a Weil algebra and $V^F = F\left(\mathbb{R} \to pt\right)$ is a A^F-module such that $\dim_{\mathbb{R}}\left(V^F\right) < \infty$. Moreover there is a natural isomorphism $\Theta\colon F \to T^{A^F,V^F}$ and equivalence classes of functors F are in bijection with equivalence classes of pairs $\left(A^F, V^F\right)$. In particular, the product preserving gauge bundle functors T^A and V_A are equivalent to $T^{A,A}$and $T^{\mathbb{R},A}$ respectively.

(b) Let $c = (q^{-1}(U), x^i, y^j)$, $1 \le i \le m$, $1 \le j \le n$ be a fibered chart of a vector bundle (E, M, q) i.e. $\overline{\varphi} := \left(x^i, y^j\right) : q^{-1}(U) \to u\left(U\right) \times \mathbb{R}^n$ is a chart of E such that (U, u) is a chart of M and the map $\varphi := \left(u^{-1} \times \mathrm{id}_{\mathbb{R}^n}\right) \circ \overline{\varphi}$ is a local trivialization of E. For such a chart, we'll associate the local frame $\left(\zeta_j\right)_{1\le j\le n}$ given on U by $\zeta_j\left(x\right) = \varphi^{-1}\left(x, e_j\right)$, where $\left(e_j\right)$ is the canonical basis of $\mathbb{R}^n$. If $\ell\colon \Gamma\left(E^*\right) \to C_{\mathrm{lin}}^\infty(E)$ denotes the canonical module isomorphism over $C^\infty(M)$

between the module of smooth sections of the dual bundle E^* and the module of fiberwise linear smooth functions $E \to \mathbb{R}$, the dual local frame $\left(\zeta^j\right)_{1\leq j\leq n}$ of $(\zeta_j)_{1\leq j\leq n}$ is defined by $\zeta^j = \ell^{-1}\left(y^j\right)$, $1 \leq j \leq n$.

(c) Let $1+K = \dim A^F$ and $L = \dim V^F$; let $c = (q^{-1}(U), x^i, y^j)$, $1 \leq i \leq m$, $1 \leq j \leq n$ a fibered chart of E, the map $\overline{\varphi} := \left(x^i, y^j\right) : q^{-1}(U) \to u\left(U\right) \times \mathbb{R}^n$ is a vector bundle isomorphism over $U \to u\left(U\right) \times \mathrm{pt}$, hence $F\overline{\varphi}\colon p_E^{-1}(U) \to Fu\left(U\right) \times \left(V^F\right)^n$ induces the fibered chart $\left(p_E^{-1}\left(U\right), x^{i,\alpha}, y^{j,\beta}\right)$, $0 \leq \alpha \leq K$, $1 \leq \beta \leq L$ of FE given by

$$(3.1)\qquad \begin{cases} x^{i,\alpha} = e_\alpha^* \circ F\left(x^i\right) \\ y^{j,\beta} = \varepsilon_\beta^* \circ F\left(y^j\right) \end{cases}$$

where (ε_β^*) is the dual basis of a fixed basis $(\varepsilon_\beta)_{1\leq\beta\leq L}$ of V^F.

4. On lifts of sections

Let $F\colon \mathcal{VB} \to \mathcal{FM}$ be a product preserving gauge bundle functor and a vector bundle (E, M, q).

4.1. Natural transformations $\overline{\overline{Q}}\left(a\right) : F \to F$.

Similarly to what is done in [14], let us denote $\overline{\mu}_E\colon \mathbb{R} \times E \to E$, $\mathbb{R} \times E_x \ni (\alpha, u) \mapsto \alpha \cdot u \in E_x$, the fibered multiplication. This is a vector bundle morphism over the projection $\mathbb{R} \times M \to M$, hence for any $a \in A^F$, we have a natural transformation $\overline{\overline{Q}}\left(a\right) : F \to F$ given by the partial maps $F\overline{\mu}_E\left(a, \cdot\right) : FE \to FE$.

Assume that $F = T^A$ the canonical product preserving gauge bundle functor deduced from a Weil bundle, then the maps $\overline{Q}\left(a\right)_M := \kappa_M \circ \overline{\overline{Q}}\left(a\right)_{TM} \circ \kappa_M^{-1}$ define the natural affinor $\overline{Q}\left(a\right) : TT^A \to TT^A$ associated to $a \in A$ (see [7]).

4.2. Lifts of sections.

Any smooth section $s\colon M \to E$ *of* E is a vector bundle morphism over itself, hence $Fs\colon T^{A^F}M \to T^{A^F}E$ by Remark 3.1 (a). To have some lifts of s using previous natural transformations, it is necessary to consider the case $F = T^A$.

Definition 4.1. For a smooth section $s\colon M \to E$ *of* (E, M, q), its a-lift $(a \in A)$ related to F is given by $s^{(a)} = \overline{\overline{Q}}\left(a\right)_E \circ Fs$.

In particular let $E = TM$ and $\overline{X} \in \mathfrak{X}(M)$; hence $\overline{X}^{(a)} := \overline{Q}\left(a\right)_M \circ \mathcal{F}\overline{X}$ is the a-lift of [2].

Definition 4.2. For any $\varphi \in \Gamma\left(E^*\right)$, its μ-lift ($\mu\colon V^F \to \mathbb{R}$ linear) related to F is given by $\varphi^{(\mu)} = \widetilde{\ell}^{-1}\left(\ell_\varphi^{(\mu)}\right)$, where $\widetilde{\ell}\colon \Gamma\left(\left(FE\right)^*\right) \to C^\infty_{\mathrm{lin}}(FE)$ is the canonical module isomorphism.

Clearly, $\left(h^*\left(\varphi\right)\right)^{(\mu)} = \left(Fh\right)^*\left(\varphi^{(\mu)}\right)$, for $h\colon G \to E$ a vector bundle morphism over id_M and $h^*\colon \Gamma\left(E^*\right) \to \Gamma\left(G^*\right)$ the pull-back map. Moreover $\left(\varphi_1 + \varphi_2\right)^{(\mu)} = \varphi_1^{(\mu)} + \varphi_2^{(\mu)}$, for all $\varphi_1, \varphi_2 \in \Gamma\left(E^*\right)$.

Let $(\zeta_{j\beta})$ be the local frame of FE associated to the fibered chart $\left(p_E^{-1}(U), x^{i,\alpha}, y^{j,\beta}\right)$, Remark 3.1 (b); the dual local frame $\left(\zeta^{j\beta}\right)$ satisfies

$$\zeta^{j\beta} = \widetilde{\ell}^{-1}\left(y^{j,\beta}\right) = \widetilde{\ell}^{-1}\left(y^{j(\varepsilon_\beta^*)}\right) = \zeta^{j(\varepsilon_\beta^*)},$$

hence

Proposition 4.1. *The $\varphi^{(\mu)}$, $\varphi \in \Gamma(E^*)$ and $\mu \colon V^F \to \mathbb{R}$ linear, generate the module $\Gamma\left((FE)^*\right)$ over $C^\infty(FM)$.*

5. On lifts of derivations

5.1. **The Lie algebroid of derivations on a vector bundle.**

Let M be a differential manifold.

Definition 5.1.

(1) A Lie algebroid on M is a vector bundle (E, M, q) on which the module $\Gamma(E)$ of smooth sections of E is endowed with a Lie algebra structure and there is a vector bundle morphism $\rho \colon E \to TM$ over id_M called the *anchor* of E, such that:

(a) $\forall s_1, s_2 \in \Gamma(E)$, $\forall f \in C^\infty(M)$,$[s_1, f \cdot s_2] = f[s_1, s_2] + (\rho(s_1) \cdot f)s_2$,

(b) The map $\rho \colon \Gamma(E) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.

(2) A *derivation* on a vector bundle (E, M, q) is a $\mathbb{R}$-linear endomorphism $D \colon \Gamma(E) \to \Gamma(E)$ such that

$$D(fs) = fD(s) + \alpha(D)(f)\, s, \text{ for } f \in C^\infty(M) \text{ and } s \in \Gamma(E)$$

where $\alpha(D)$ is a vector field on M.

Example 5.1.

(1) The tangent bundle (TM, M, q_M) is a Lie algebroid with the anchor id_{TM}, the composition law on $\Gamma(TM) = \mathfrak{X}(M)$ is just the usual bracket on vector fields.

(2) The covariant derivative of a linear connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ induces a derivation $\nabla_X \colon \Gamma(E) \to \Gamma(E)$, $S \mapsto \nabla_X S$, for a fixed vector field X on M.

Let us denote $\mathcal{D}(E)$ the set of all derivations on E; this is a $C^\infty(M)$-module.

A derivation D on E is obviously a local operator and the value $D(s)(x)$ depends only on the 1-jet $j_x^1 s$ of s at x. Indeed let (ε_j) be a local frame of E over a neighborhood U of x, let $s\mid_U = s^j \varepsilon_j$; consider $\lambda \in C^\infty(M)$ such that $\lambda = 1$ in a neighborhood W of x and $\operatorname{Supp}\lambda \subset U$. Define

$$\widetilde{s^j} = \begin{cases} \lambda s^j & \text{on } U \\ 0 & \text{on } M - U \end{cases} \quad \text{and} \quad \widetilde{\varepsilon_j} = \begin{cases} \lambda \varepsilon_j & \text{on } U \\ 0 & \text{on } M - U \, ; \end{cases}$$

hence $\widetilde{s} = \sum_j \widetilde{s^j}\widetilde{\varepsilon_j} \in \Gamma(E)$ coincide with s on W. So if $j_x^1 s = 0$, $D(s)(x) = D(\widetilde{s})(x) = \widetilde{s^j}(x)\, D(\widetilde{\varepsilon_j})(x) + \alpha(D)\left(\widetilde{s^j}\right)(x)\, \widetilde{\varepsilon_j}(x) = 0.$

There is a vector bundle morphism

$$\varphi_D \colon J^1E \to E,\ j^1_x s \mapsto D(s)(x)$$

and the map $\mathcal{D}(E) \to \{\varphi_D, D \in \mathcal{D}(E)\}$, $D \mapsto \varphi_D$ is a module isomorphism. The set $\mathrm{Diff}^1(E) := \{(\varphi_D)_x, D \in \mathcal{D}(E),\ x \in M\} \subset \mathrm{Hom}(J^1E, E)$ is a sub bundle, see [9], hence $\mathcal{D}(E)$ corresponds to the module $\Gamma\big(\mathrm{Diff}^1(E)\big) \subset \Gamma\big(\mathrm{Hom}(J^1E, E)\big)$ of smooth sections of $\mathrm{Diff}^1(E)$.

For derivations $D_1, D_2 \in \mathcal{D}(E)$, the bracket

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \in \mathcal{D}(E)$$

and $\alpha([D_1, D_2]) = [\alpha(D_1), \alpha(D_2)]$; moreover for $f \in C^\infty(M)$,

$$[D_1, fD_2] = f[D_1, D_2] + \alpha(D_1)(f)\, D_2\,,$$

hence identifying $\mathcal{D}(E)$ with $\Gamma\left(\mathrm{Diff}^1(E)\right)$, $\mathrm{Diff}^1(E) \to M$ is a Lie algebroid with anchor map $\alpha\colon \mathrm{Diff}^1(E) \to TM$.

5.2. **Further results on lifts of linear vector fields.**
A *linear vector field* on a vector bundle (E, M, q) is a vector bundle morphism

$$\begin{tikzcd} E \arrow[r, "X"] \arrow[d, "q"'] & TE \arrow[d, "T(q)"] \\ M \arrow[r, "\bar{X}"] & TM \end{tikzcd}$$

where $X \in \mathfrak{X}(E)$ is a vector field on E and $\overline{X} \in \mathfrak{X}(M)$ is a vector field on M, i.e. a *projectable vector field* on E that is a vector bundle morphism.

Let us denote $\mathfrak{X}^{\mathrm{lin}}(E)$ the set of linear vector fields on E. The following result clarifies the concept of linear vector field.

Proposition 3.4.2 [9]. *Let $X \in \mathfrak{X}(E)$, $\overline{X} \in \mathfrak{X}(M)$. The following assertions are equivalent:*

(1) $\big(\overline{X}, X\big) \in \mathfrak{X}^{\mathrm{lin}}(E)$;

(2) *Viewed as a derivation, $X\colon C^\infty(E) \to C^\infty(E)$ sends $C^\infty_{\mathrm{lin}}(E)$ into $C^\infty_{\mathrm{lin}}(E)$ and sends $q^*C^\infty(M) = \{f \circ q;\ f \in C^\infty(M)\}$ into $q^*C^\infty(M)$;*

(3) *If $\mathrm{Fl}^{\overline{X}}$ and Fl^X denote the flows of $\overline{X}$ and X respectively, then Fl^X_t is a vector bundle morphism over $\mathrm{Fl}^{\overline{X}}_t$ when defined.*

According to the previous result, locally each linear vector field $\big(\overline{X}, X\big)$ can be write $X = \overline{X}^i \circ q \frac{\partial}{\partial x^i} + X^j \frac{\partial}{\partial y^j}$, where the $X^j\colon \left(x^k, y^l\right) \mapsto X^j\left(x^k, y^l\right)$ are linear on $\left(y^l\right)$; the bracket $\left[\big(\overline{X}, X\big), \big(\overline{Y}, Y\big)\right]$ belongs obviously to $\mathfrak{X}^{\mathrm{lin}}(E)$.

Let $\big(\overline{X}, X\big) \in \mathfrak{X}^{\mathrm{lin}}(E)$ and $\ell\colon \Gamma(E^*) \to C^\infty_{\mathrm{lin}}(E)$ the canonical module isomorphism ; according to the previous result, there is a map

$$D^{(*)}_X\colon \Gamma(E^*) \to \Gamma(E^*)$$

defined by $D_X^{(*)}(\varphi) = \ell^{-1}[X(\ell_\varphi)]$. $D_X^{(*)}$ is additive and for $f \in C^\infty(M$, $D_X^{(*)}(f\varphi) = f\, D_X^{(*)}(\varphi) + \overline{X}(f)\,\varphi$, hence $D_X^{(*)} \in \mathcal{D}(E^*)$, and we have a map

$$\begin{aligned} D^{(*)} \colon \mathfrak{X}^{\text{lin}}(E) &\longrightarrow \mathcal{D}(E^*) \\ X &\longmapsto D_X^{(*)}, \end{aligned}$$

which is linear over $C^\infty(M)$ and bracket preserving. By Proposition 3.4.4 [9], $D^{(*)}$ is the induced map on sections of an isomorphism of Lie algebroids.

In [14], we defined the *a-lift* of $X \in \mathfrak{X}^{\text{lin}}(E)$ $(a \in A^F)$ as follows :

$$X^{(a)} := Q(a)_E \circ (\mathcal{F}_E)\, X \in \mathfrak{X}^{\text{lin}}(FE)\,,$$

where $\mathcal{F}\colon T \rightsquigarrow TF$ is the flow operator of F and $Q(a) : TF \to TF$ a natural transformation induced by the fibered multiplication $\mu_E \colon \mathbb{R} \times TE \to TE$.

Proposition 5.1. *The maps $D_{X^{(a)}}^{(*)}$ are the only derivations on $\Gamma\big((FE)^*\big)$ such that*

$$D_{X^{(a)}}^{(*)}\big(\varphi^{(\mu)}\big) = \big(D_X^{(*)}(\varphi)\big)^{(\mu_a)},$$

for $\varphi \in \Gamma(E^*)$ and $\mu \colon V^F \to \mathbb{R}$ linear.

Proof. $D_{X^{(a)}}^{(*)}$ is unique by Proposition 4.1. Moreover,

$$\begin{aligned} D_{X^{(a)}}^{(*)}\big(\varphi^{(\mu)}\big) &= \widetilde{\ell}^{-1}\big(X^{(a)}\big(\widetilde{\ell}_{\varphi^{(\mu)}}\big)\big) \\ &= \widetilde{\ell}^{-1}\big(X^{(a)}\big(\ell_\varphi^{(\mu)}\big)\big) \\ &= \widetilde{\ell}^{-1}\big(X(\ell_\varphi)^{(\mu_a)}\big), \quad \text{by Theorem 5.1 [14]} \\ &= \widetilde{\ell}^{-1}\big(\big(\ell_{D_X^{(*)}(\varphi)}\big)^{(\mu_a)}\big) \\ &= \widetilde{\ell}^{-1}\big(\widetilde{\ell}\big(D_X^{(*)}(\varphi)^{(\mu_a)}\big)\big) \\ &= D_X^{(*)}(\varphi)^{(\mu_a)}\,. \end{aligned}$$

□

6. The linear connection $\mathcal{F}\Phi$

For a product preserving gauge bundle functor $F \colon \mathcal{VB} \to \mathcal{FM}$ on vector bundles, let us denote $\kappa \colon F \circ T \to T \circ F$ the natural isomorphism (Corollary 3 [10]) associated to the isomorphism of pairs $(A^{F\circ T}, V^{F\circ T}) \overset{\cong}{\to} (A^{T\circ F}, V^{T\circ F})$.

Let $\Phi \colon TE \to TE$ be the vertical projection of a linear connection on a vector bundle (E, M, q) i.e. Φ is a vector bundle morphism over id_E and id_{TM} such that $\Phi \circ \Phi = \Phi$ and $\text{Im}\Phi = VE = \bigcup\limits_{u \in E} (Tq)^{-1}\left(\{0_{T_{q(u)}M}\}\right)$ the vertical bundle of (E, M, q).

Proposition 6.1. *Then the map $\mathcal{F}\Phi = \kappa_E \circ F\Phi \circ \kappa_E^{-1}$ is the vertical projection of a linear connection on (FE, FM, Fq).*

Proof. $\mathcal{F}\Phi$ is clearly a vector bundle morphism over FE and TFM such that $\mathcal{F}\Phi \circ \mathcal{F}\Phi = \mathcal{F}\Phi$. To see that $\operatorname{Im} \mathcal{F}\Phi = VFE$ it is sufficient to take $F = T^{A,V}$. $\square$

Example 6.1. (a) Let $F = T^{A,A}$; $\mathcal{F}\Phi = \mathcal{T}^A\Phi$ is just the vertical projection of the Slovák's connection $\mathcal{T}^A\Gamma$ (see [15] or [7]).

(b) Let $F = V_A = T^{\mathbb{R},A}$; $\mathcal{F}\Phi = \mathcal{V}^A\Phi$ is just the vertical projection of the vertical lift $\mathcal{V}^A\Gamma$ of Γ (see [7]).

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