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GRADIENT ESTIMATES OF LI YAU TYPE FOR A GENERAL HEAT EQUATION ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we consider gradient estimates on complete non-compact Riemannian manifolds (M,g) for the following general heat equation

$$u_t = \Delta_V u + au\log u + bu$$

where a is a constant and b is a differentiable function defined on $M\times[0,\infty)$. We suppose that the Bakry-Émery curvature and the N-dimensional Bakry-Émery curvature are bounded from below, respectively. Then we obtain the gradient estimate of Li-Yau type for the above general heat equation. Our results generalize the work of Huang-Ma ([4]) and Y. Li ([6]), recently.

1.. Introduction

Recently, the weighted Laplacian on smooth metric measure spaces has been attracted by many researchers. Recall that a triple $(M,g,e^{-f}dv)$ is called a smooth metric measure space if (M,g) is a Riemannian manifold, f is a smooth function on M and dv is the volume form with respect to g. On smooth metric measure spaces, the weighted Laplace operator is defined by

$$\Delta_f\cdot := \Delta\cdot - \langle\nabla f,\nabla\cdot\rangle$$

where Δ is the Laplace operator on M. On $(M,g,e^{-f}dv)$, the Bakry-Émery curvature Ric_f and the N-dimensional Bakry-Émery curvarute Ric_f^N are defined by

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}\, f,\quad \mathrm{Ric}_f^N := \mathrm{Ric}_f - \frac{1}{N}\nabla f\otimes\nabla f$$

where Ric, Hess f are the Ricci curvature and the Hessian of f on M, respectively.

An important generalization of the weighted Laplace operator on Riemannian manifolds is the following operator

$$\Delta_V\cdot := \Delta\cdot + \langle V,\nabla\cdot\rangle$$

where ∇ and Δ are respectively the Levi-Civita connection and the Laplace-Beltrami operator with respect to g, V is a smooth vector field on M. In [1] and [6], the

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authors introduced two curvatures

$$\mathrm{Ric}_V := \mathrm{Ric} - \frac{1}{2}\mathcal{L}_V g, \mathrm{Ric}_V^N := \mathrm{Ric}_V - \frac{1}{N} V \otimes V$$

where $N \in \mathbb{N}$ is a positive constant and $\mathcal{L}_V$ is the Lie derivative associated to the vector field V. When $V = -\nabla f$ then two curvatures Ric_V, Ric_V^N become the Bakry-Émery curvature and the N-dimensional Bakry-Émery curvature, respectively.

In this paper, let (M, g) be a Riemannian manifold and V be a smooth vector field on M. We consider the following general heat equation

$$u_t = \Delta_V u + au \log u + bu \tag{1.1}$$

where a is a constant and b is a function defined on $M \times [0, \infty)$ which is differentiable on $M \times [0, +\infty)$. When M is a compact manifold and $b = 0$, Li ([6]) studied gradient estimates of Li-Yau type for equation (1.1). His results can be considered as a generalization of the famous work of Li and Yau ([5]). Moreover, Li also studied gradient estimates of Hamilton type for the equation (1.1) when $a = b = 0$ on complete noncompact manifolds. In the general case, when a, b are constants and M is a complete noncompact manifold, Huang and Ma introduced a gradient estimate of Li-Yau type which is independent of K. Here $K > 0$ such that $-K$ is the lower bound of the N-dimensional Bakry-Émery curvature. Then, they derived the Gaussian lower bound of the heat kernel for the equation $u_t = \Delta_V u$. Recently, Dung and the author investigated gradient estimates of Hamilton-Souplet-Zhang type. Our work is a generalization of the results of Huang-Ma, Y. Li and other mathematicians, see [3, 5, 6] for further discussion and the references there in.

Motivated by the above result, it is very natural for us to look for gradient estimates of Li-Yau type for the general heat equation (1.1). In this paper, under some natural conditions on the curvatures, we are able to extend the work of Huang-Ma and Li to complete noncompact manifolds. Our main theorem is as follows.

Theorem 1.1. *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with Ric_V bounded from below by the constant $-K := -K(2R)$, where $R > 0$, $K(2R) > 0$ in the geodesic ball $B(p, 2R)$ centered at some fixed point $p \in M$ and V be a smooth vector field on M such that $|V| \leq L$ for some positive constant $L \in \mathbb{R}$. Suppose that a is a real constant, b is a differentiable function defined on $M \times [0, +\infty)$ and the general heat equation*

$$\frac{\partial u}{\partial t} = \Delta_V u + au \log u + bu$$

has a positive solution u on $M \times [0, \infty)$. Then, for all $x \in B(p, R)$, $t \in (0, \infty)$, we have

(1) *If $a \le 0$, then*

$$\beta\frac{|\nabla u|^2}{u^2} + a\log u - \frac{u_t}{u} \le \frac{n}{2(1-\delta)\beta}\Big\{\frac{nc_1^2}{16\delta\beta(1-\beta)R^2} + A + \frac{1}{t} + \frac{6\beta\theta}{n} + \frac{\beta L^2}{(1-\beta)N} - \frac{a}{2} + \frac{\theta\beta}{2(1-\beta)} + \sqrt{\frac{\theta\beta(1+\beta-a)}{n}}\Big\};$$

(2) *If $a \ge 0$, then*

$$\beta\frac{|\nabla u|^2}{u^2} + a\log u - \frac{u_t}{u} \le \frac{n}{2(1-\delta)\beta}\Big\{\frac{nc_1^2}{16\delta\beta(1-\beta)R^2} + A + \frac{1}{t} + \frac{6\beta\theta}{n} + \frac{\beta L^2}{(1-\beta)N} + a + \frac{\theta\beta}{2(1-\beta)} + \sqrt{\frac{\theta\beta(1+\beta+a)}{n}}\Big\},$$

where c_1 and c_2 are positive constants, $\beta = e^{-2Kt}$, $0<\delta<1, \theta := \max\{|b|, |b_t|, |\nabla b|\} \in \mathbb{R}$ and A is defined by

$$A = \frac{(n-1+\sqrt{(n-1)K}R + LR)c_1 + c_2 + 2c_1^2}{R^2}.$$

The paper is organized as follows. In the section 2, we give a proof of Theorem 1.1. In section 3, we point out that we can recover the main theorem in [4] by using Theorem 1.1. Moreover, we also show some applications to give gradient estimate s of solution of some general heat equations and prove a Harnack inequality for such a solution. This is an extension of the work of Huang-Ma and Li.

2.. Gradient estimate of Li Yau type

To begin with, let us recall the following Laplacian comparison theorem in [1].

Theorem 2.1 ([1])**.** *Let (M,g) be a complete noncompact Riemannian manifold with Ric_V bounded from below by the constant $-K := -K(2R)$, where $R>0$, $K(2R)>0$ in the geodesic ball $B(p,2R)$ with radius $2R$ around $p\in M$. Suppose that V is a smooth vector field on M satisfying $\langle V, \nabla\rho\rangle \le v(\rho)$ for some nondecreasing function $v(\cdot)$, where $\rho(x)$ is the distance from a fixed point p to the considered point x. Then*

$$\Delta_V\rho \le \sqrt{(n-1)K} + \frac{n-1}{\rho} + v(\rho).$$

Noting that if $v(\cdot)$ is bounded by a positive constant L then we have

$$\Delta_V\rho \le \sqrt{(n-1)K} + \frac{n-1}{\rho} + L. \tag{2.2}$$

To prove the Theorem 1.1, we first derive the following important lemma.

Lemma 2.2. *Let (M,g) be a complete noncompact Riemannian manifold with Ric_V bounded from below by the constant $-K := -K(2R)$, where $R>0$, $K(2R)>0$ in the geodesic ball $B(p,2R)$ with radius $2R$ around $p\in M$ and V is a smooth* *vector field on M such that $|V|$ is bounded by a positive constant L. For the smooth function $w = \log u$, where u be a positive solution to* (1.1) *then*

$$\begin{aligned}\Delta_V F - F_t \geq t\Big\{\frac{2\beta}{n}(\Delta_V w)^2 + \Big(\frac{-2\beta L^2}{N} - a(\beta-1)\Big)|\nabla w|^2 - 2\beta\langle \nabla w, \nabla b\rangle + b_t - ab\Big\} \\ - 2\langle \nabla w, \nabla F\rangle - aF - \frac{F}{t},\end{aligned}$$

where $F = t(\beta|\nabla w|^2 + aw - w_t)$.

Proof. Let $w = \log u$ with u be the positive solution to (1.1) then

$$w_t = |\nabla w|^2 + \Delta_V w + aw + b\,.$$

Hence,

$$\Delta_V w_t = -2\langle \nabla w, \nabla w_t\rangle - aw_t + w_{tt} - b_t\,. \tag{2.3}$$

and

$$\Delta_V w = (\beta-1)|\nabla w|^2 - \frac{F}{t} - b \tag{2.4}$$

$$= \Big(1 - \frac{1}{\beta}\Big)(-aw + w_t) - \frac{F}{t\beta} - b\,. \tag{2.5}$$

Since $\mathrm{Ric}_V \geq -K$, $|V| \leq L$ and *V-Bochner-Weitzenböck* formula (see [6]) implies

$$\Delta_V|\nabla w|^2 \geq \frac{2}{n}(\Delta_V w)^2 - 2\Big(K + \frac{L^2}{N}\Big)|\nabla w|^2 + 2\langle \nabla w, \nabla \Delta_V w\rangle\,. \tag{2.6}$$

By the definition F, it is easy to show that

$$\begin{aligned}F_t &= \frac{F}{t} + t\big(-2K\beta|\nabla w|^2 + 2\beta\langle \nabla w, \nabla w_t\rangle + aw_t - w_{tt}\big)\\ \Delta_V F &= t\big(\beta\Delta_V(|\nabla w|^2) + a\Delta_V w - \Delta_V w_t\big)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta_V F - F_t = t\big(\beta\Delta_V(|\nabla w|^2) + a\Delta_V w - \Delta_V w_t\big) - \frac{F}{t}\\ - t\big(-2K\beta|\nabla w|^2 + 2\beta\langle \nabla w, \nabla w_t\rangle + aw_t - w_{tt}\big)\,.\end{aligned} \tag{2.7}$$

Combining (2.3), (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned}\Delta_V F - F_t \geq t\Big\{\frac{2\beta}{n}(\Delta_V w)^2 + \Big(\frac{-2\beta L^2}{N} - 2\beta a\Big(1 - \frac{1}{\beta}\Big)\Big)|\nabla w|^2 - 2\beta\langle \nabla w, \nabla b\rangle + \\ - a^2\Big(1 - \frac{1}{\beta}\Big)w + a\Big(1 - \frac{1}{\beta}\Big)w_t - ab + b_t\Big\}\\ - 2\langle \nabla w, \nabla F\rangle + \Big(\frac{-a}{\beta} - \frac{1}{t}\Big)F\,.\end{aligned} \tag{2.8}$$

On the other hand, by direct computation, we have

$$-a^2\Big(1 - \frac{1}{\beta}\Big)w + a\Big(1 - \frac{1}{\beta}\Big)w_t = -\frac{aF}{t} + \frac{aF}{t\beta} + a(\beta-1)|\nabla w|^2\,. \tag{2.9}$$

Substituting (2.9) into (2.8), we get

$$\Delta_V F - F_t \geq t\Big\{\frac{2\beta}{n}(\Delta_V w)^2 + \Big(\frac{-2\beta L^2}{N} - a(\beta-1)\Big)|\nabla w|^2 - 2\beta\langle\nabla w, \nabla b\rangle + b_t - ab\Big\} - 2\langle\nabla w, \nabla F\rangle - aF - \frac{F}{t}.$$

The proof is complete. □

Now, we prove the Theorem 1.1.

Proof of Theorem 1.1. Let $\xi(r)$ be a cut-off function such that $\xi(r) = 1$ for $r \leq 1$, $\xi(r) = 0$ for $r \geq 2$, $0 \leq \xi(r) \leq 1$, and

$$0 \geq \xi^{\frac{-1}{2}}(r)\xi^{'}(r) \geq -c_1\,,$$
$$\xi^{''}(r) \geq -c_2$$

for positive constants c_1 and c_2.

Put $\varphi(x) = \xi\big(\frac{\rho(x)}{R}\big)$, it is easy to see that

$$\frac{|\nabla\varphi|^2}{\varphi} = \frac{|\nabla\xi|^2}{\xi} = \frac{1}{\xi(r)}\frac{\big(\xi(r)^{'}\big)^2}{R^2}|\nabla\rho(x)|^2 \leq \frac{(-c_1)^2}{R^2} = \frac{c_1^2}{R^2}\,. \tag{2.10}$$

Hence, by the inequality (2.2), we have

$$\begin{aligned}\Delta_V\varphi &= \frac{\xi(r)^{''}|\nabla\rho|^2}{R^2} + \frac{\xi(r)^{'}\Delta_V\rho}{R}\\ &\geq \frac{-c_2}{R^2} + \frac{(-c_1)}{R}\Big[\sqrt{(n-1)K} + \frac{n-1}{\rho} + L\Big]\\ &= -\frac{R\Big[\sqrt{(n-1)K} + \frac{n-1}{\rho} + L\Big]c_1 + c_2}{R^2}\\ &\geq -\frac{\big(n-1+\sqrt{(n-1)K}R + LR\big)c_1 + c_2}{R^2}\,. \end{aligned}\tag{2.11}$$

For $T \geq 0$, let (x,t) be a point in $B_{2R}(p)\times[0,T]$ at which φF attains its maximum. At the point (x,t), we have

$$\begin{cases}\nabla(\varphi F) = 0\\ \Delta_V(\varphi F) \leq 0\\ F_t \geq 0\end{cases}.$$

Since $\nabla(\varphi F) = \varphi\nabla F + F\nabla\varphi = 0$, this implies $\nabla F = -F\varphi^{-1}\nabla\varphi$. It follows that

$$\Delta_V(\varphi F) = \varphi\Delta_V F + F\Delta_V\varphi - 2F\varphi^{-1}|\nabla\varphi|^2 \leq 0.$$

Substituting (2.10) and (2.11) into the above inequality, we obtain

$$\begin{aligned} \varphi\Delta_V F &\le F\left(\frac{2|\nabla\varphi|^2}{\varphi} - \Delta_V\varphi\right) \\ &\le F\left(\frac{\left(n-1+\sqrt{(n-1)K}R+LR\right)c_1 + c_2 + 2c_1^2}{R^2}\right) = FA \end{aligned} \tag{2.12}$$

where $A = \dfrac{\left(n-1+\sqrt{(n-1)K}R+LR\right)c_1 + c_2 + 2c_1^2}{R^2}$.

Combining Lemma 2.2 and (2.12), we infer

$$\begin{aligned} FA &\ge \varphi\Delta_V F \ge \varphi\Delta_V F - F_t \\ &\ge t\varphi\left\{\frac{2\beta}{n}(\Delta_V w)^2 + \Big(\frac{-2\beta L^2}{N} - a(\beta-1)\Big)|\nabla w|^2 - 2\beta\left\langle\nabla w, \nabla b\right\rangle + b_t - ab\right\} \\ &\quad + \varphi\Big\{-2\left\langle\nabla w, \nabla F\right\rangle - aF - \frac{F}{t}\Big\}. \end{aligned} \tag{2.13}$$

Here we used $F_t \le 0$. Since $0 = \nabla(\varphi F) = \varphi\nabla F + F\nabla\varphi$, we have

$$-2\varphi\left\langle\nabla w, \nabla F\right\rangle = 2F\left\langle\nabla w, \nabla\varphi\right\rangle \ge -2F|\nabla w|\,|\nabla\varphi| \ge -2\frac{c_1}{R}\varphi^{\frac12}F|\nabla w|\,. \tag{2.14}$$

By (2.4), we yield

$$(\Delta_V w)^2 \ge \Big[(\beta-1)|\nabla w|^2 - \frac{F}{t}\Big]^2 + 2\Big[(\beta-1)|\nabla w|^2 - \frac{F}{t}\Big](-b)\,. \tag{2.15}$$

Plugging (2.14) and (2.15) into (2.13), we obtain

$$\begin{aligned} FA \ge \varphi t\Big\{&\frac{2\beta}{n}\Big(\Big((\beta-1)|\nabla w|^2 - \frac{F}{t}\Big)^2 + 2\Big((\beta-1)|\nabla w|^2 - \frac{F}{t}\Big)(-b)\Big) \\ &+ \Big(\frac{-2\beta L^2}{N} - a(\beta-1)\Big)|\nabla w|^2 - 2\beta\left\langle\nabla w, \nabla b\right\rangle + b_t - ab\Big\} \\ &+ \varphi\Big\{-aF - \frac{F}{t}\Big\} - 2\frac{c_1}{R}\varphi^{\frac12}F|\nabla w|\,. \end{aligned} \tag{2.16}$$

By the similar argument as *Davies* [2] or as *Negrin* [7], we put $\mu = \frac{|\nabla w|^2}{F}$. Then (2.16) can be read as

$$\begin{aligned} \frac{2\varphi t\beta}{n}\frac{[(\beta-1)\mu tF - F]^2}{t^2} \le AF &+ \frac{4\varphi t\beta}{n}\frac{[(\beta-1)\mu tF - F]b}{t} \\ &+ \varphi Ft\mu\Big(\frac{2\beta L^2}{N} + a(\beta-1)\Big) + 2\beta\varphi t\left\langle\nabla w, \nabla b\right\rangle \\ &+ \varphi t(ab - b_t) + 2\frac{c_1}{R}\mu^{\frac12}\varphi^{\frac12}F^{\frac32} + a\varphi F + \frac{\varphi F}{t}\,. \end{aligned}$$

Multiplying both sides of the above inequality by φt we arrive at

$$\begin{aligned}\frac{2\beta[(\beta-1)t\mu-1]^2}{n}(\varphi F)^2 &\le 2\frac{c_1}{R}t\mu^{\frac{1}{2}}\varphi^{\frac{3}{2}}F^{\frac{3}{2}}+(At+1)\varphi F\\ &\quad+\Big\{\frac{4\beta[(\beta-1)t\mu-1]b}{n}+t\mu\Big(\frac{2\beta L^2}{N}+a(\beta-1)\Big)+a\Big\}t\varphi^2F\\ &\quad+2\beta\varphi^2t^2\left\langle\nabla w,\nabla b\right\rangle+\varphi^2t^2(ab-b_t)\,.\end{aligned}\tag{2.17}$$

Now we want to estimate the right hand side of (2.17). The first term of the right-hand side of (2.17) can be estimated as follows.

$$2\frac{c_1}{R}t\mu^{\frac{1}{2}}(\varphi F)^{\frac{3}{2}}\le\frac{2\delta\beta[(\beta-1)t\mu-1]^2}{n}(\varphi F)^2+\frac{nc_1^2t^2\mu}{2\delta\beta[(\beta-1)t\mu-1]^2R^2}(\varphi F)\tag{2.18}$$

with $0<\delta<1$, and the third term of the right-hand side of (2.17) is evaluated as below.

$$2\varphi^2t^2\beta\left\langle\nabla w,\nabla b\right\rangle\le 2\varphi^2t^2\beta|\nabla b|(\mu F)^{\frac{1}{2}}\le t^2\beta|\nabla b|(\mu\varphi F+1)\,.\tag{2.19}$$

By the definition of θ, it is easy to see that

$$B:=t^2\beta|\nabla b|\le\theta t^2\beta\quad\text{and}\quad C:=t^2\beta|\nabla b|+\varphi^2t^2(ab-b_t)\le\theta t^2\beta+\varphi^2t^2(|a|+1)\theta\,.$$

Plugging these above estimates and (2.18), (2.19) into (2.17), we obtain

$$\begin{aligned}\frac{2\beta[(\beta-1)t\mu-1]^2(\varphi F)^2}{n}&\le\frac{2\delta\beta[(\beta-1)t\mu-1]^2}{n}(\varphi F)^2+\frac{nc_1^2t^2\mu}{2\delta\beta[(\beta-1)t\mu-1]^2R^2}(\varphi F)\\ &\quad+\Big\{\frac{4\beta[(\beta-1)t\mu-1]b}{n}+t\mu\Big(\frac{2\beta L^2}{N}+a(\beta-1)\Big)+a\Big\}t\varphi^2F\\ &\quad+(At+1)\varphi F+\mu B\varphi F+C\,.\end{aligned}\tag{2.20}$$

Now, we have two cases.

1. If $a\le 0$ then $at\varphi^2F\le 0$, $|a|=-a$, and

$$\frac{4t\beta[(\beta-1)t\mu-1]b}{n}\le-\frac{4t\beta[(\beta-1)t\mu-1]\theta}{n}\,.$$

By (2.20), we have

$$\begin{aligned}(\varphi F)^2&\le\frac{n}{2(1-\delta)\beta[(\beta-1)t\mu-1]^2}\Big\{\frac{nc_1^2t^2\mu}{2\delta\beta[(\beta-1)t\mu-1]^2R^2}+At+1\\ &\quad+\Big(a+\frac{\theta\beta}{\beta-1}-\frac{4\beta\theta}{n}\Big)t^2\mu(\beta-1)+\frac{4t\beta\theta}{n}+2t^2\mu\frac{\beta L^2}{N}\Big\}\varphi F\\ &\quad+\frac{n}{2(1-\delta)\beta[(\beta-1)t\mu-1]^2}\big(\theta t^2\beta+\varphi^2t^2(1-a)\theta\big)\,.\end{aligned}$$

Using the fact that if a, $b \geq 0$ satisfying $x^2 \leq ax + b$ then $x \leq a + \sqrt{b}$, the above inequality implies

$$\begin{aligned}\varphi F \leq & \frac{n}{2(1-\delta)\beta[(\beta-1)t\mu-1]^2}\Big\{\frac{nc_1^2t^2\mu}{2\delta\beta[(\beta-1)t\mu-1]^2R^2}+At+1\\ &+\Big(a+\frac{\theta\beta}{\beta-1}-\frac{4\beta\theta}{n}\Big)t^2\mu(\beta-1)+\frac{4t\beta\theta}{n}+2t^2\mu\frac{\beta L^2}{N}\Big\}\\ &+\sqrt{\frac{n(\theta t^2\beta+\varphi^2t^2(1-a)\theta)}{2(1-\delta)\beta[(\beta-1)t\mu-1]^2}}\,. \end{aligned}\tag{2.21}$$

Since $\big((\beta-1)\mu t-1\big)^2 \geq 2(1-\beta)\mu t+1 \geq 1$, we have

$$\frac{1}{2(1-\delta)\beta\big((\beta-1)\mu t-1\big)^2} \leq \frac{1}{2(1-\delta)\beta}\,.$$

Therefore,

$$\begin{aligned}&\frac{1}{2(1-\delta)\beta\big((\beta-1)\mu t-1\big)^2}\frac{nc_1^2t^2\mu}{2\delta\beta[(\beta-1)\mu t-1]^2R^2}\\ &\qquad\leq \frac{n}{2(1-\delta)\beta}\frac{c_1^2t}{16\delta\beta(1-\beta)R^2},\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}&\frac{1}{2(1-\delta)\beta\big((\beta-1)\mu t-1\big)^2}\Big(At+1+\frac{4t\beta\theta}{n}\Big)\\ &\qquad\leq \frac{1}{2(1-\delta)\beta}\Big(At+1+\frac{4t\beta\theta}{n}\Big),\end{aligned}\tag{2.23}$$

where in (2.22), we used

$$\big((1-\beta)t\mu+1\big)^2 \geq 2(1-\beta)t\mu\,.$$

Since $\big((\beta-1)t\mu-1\big)^2 \geq 2(1-\beta)t\mu$, we have

$$\begin{aligned}&\frac{1}{2(1-\delta)\beta[(\beta-1)\mu t-1]^2}\Big(\Big(a+\frac{\theta\beta}{\beta-1}-\frac{4\beta\theta}{n}\Big)t^2\mu(\beta-1)+2t^2\mu\frac{\beta L^2}{N}\Big)\\ &\qquad\leq \frac{1}{2(1-\delta)\beta}\frac{-1}{2}\Big(\Big(a+\frac{\theta\beta}{\beta-1}-\frac{4\beta\theta}{n}\Big)t+\frac{t\beta L^2}{(1-\beta)N}\Big)\,.\end{aligned}\tag{2.24}$$

Moreover, since $\varphi^2 \leq 1$ and $0<\delta<1$, we infer

$$\begin{aligned}\sqrt{\frac{n\big(\theta t^2\beta+\varphi^2t^2(1-a)\theta\big)}{2(1-\delta)\beta[(\beta-1)t\mu-1]^2}} &\leq \sqrt{\frac{n\big(\theta t^2\beta+\varphi^2t^2(1-a)\theta\big)}{2(1-\delta)\beta}}\\ &\leq \frac{nt}{2(1-\theta)\beta}\sqrt{\frac{2\theta\beta(1+\beta-a)}{n}}\,.\end{aligned}\tag{2.25}$$

Plugging (2.22), (2.24), (2.23) and (2.25) into (2.21), we obtain

$$\begin{aligned}\varphi F \le& \frac{n}{2(1-\delta)\beta}\Big\{\frac{tnc_1^2}{16\delta\beta(1-\beta)R^2}+At+1+\frac{4t\beta\theta}{n}+\frac{t\beta L^2}{(1-\beta)N}-\frac{at}{2}\\&+\frac{\theta t\beta}{2(1-\beta)}+\frac{4t\beta\theta}{2n}\Big\}+\frac{nt}{2(1-\theta)\beta}\sqrt{\frac{2\theta\beta(1+\beta-a)}{n}}\\=&\frac{n}{2(1-\delta)\beta}\Big\{\frac{tnc_1^2}{16\delta\beta(1-\beta)R^2}+At+1+\frac{6t\beta\theta}{n}+\frac{t\beta L^2}{(1-\beta)N}-\frac{at}{2}\\&+\frac{\theta t\beta}{2(1-\beta)}\Big\}+\frac{nt}{2(1-\theta)\beta}\sqrt{\frac{2\theta\beta(1+\beta-a)}{n}}.\end{aligned}$$

In particular, at $(x_0,T)\in B(p,R)\times[0,T]$, we have

$$\begin{aligned}\beta\frac{|\nabla u|^2}{u^2}+a\log u-\frac{u_T}{u}\le&\frac{n}{2(1-\delta)\beta}\Big\{\frac{nc_1^2}{16\delta\beta(1-\beta)R^2}+A+\frac{1}{T}+\frac{6\beta\theta}{n}\\&+\frac{\beta L^2}{(1-\beta)N}-\frac{a}{2}+\frac{\theta\beta}{2(1-\beta)}+\sqrt{\frac{2\theta\beta(1+\beta-a)}{n}}\Big\}.\end{aligned}$$

Hence, we complete the proof of the part (1).

2. If $a\ge 0$ then $a(\beta-1)t^2\varphi^2\mu F\le 0,\ |a|=a$ and

$$\frac{4t\beta[(\beta-1)t\mu-1]b}{n}\le-\frac{4t\beta[(\beta-1)t\mu-1]\theta}{n}.$$

The inequality (2.20) implies

$$\begin{aligned}(\varphi F)^2\le&\frac{n}{2(1-\delta)\beta[(\beta-1)t\mu-1]^2}\Big\{\frac{nc_1^2t^2\mu}{2\delta\beta[(\beta-1)t\mu-1]^2R^2}+At+1\\&+\Big(\frac{\theta\beta}{\beta-1}-\frac{4\beta\theta}{n}\Big)t^2\mu(\beta-1)+\frac{4t\beta\theta}{n}+at+2t^2\mu\frac{\beta L^2}{N}\Big\}\varphi F\\&+\frac{n}{2(1-\delta)\beta[(\beta-1)t\mu-1]^2}\big(\theta t^2\beta+\varphi^2t^2(1+a)\theta\big).\end{aligned}$$

By the same argument as in the proof of the part (1), we conclude that

$$\begin{aligned}\varphi F\le&\frac{n}{2(1-\delta)\beta[(\beta-1)t\mu-1]^2}\Big\{\frac{nc_1^2t^2\mu}{2\delta\beta[(\beta-1)t\mu-1]^2R^2}+At+1\\&+\Big(\frac{\theta\beta}{\beta-1}-\frac{4\beta\theta}{n}\Big)t^2\mu(\beta-1)+\frac{4t\beta\theta}{n}+at+2t^2\mu\frac{\beta L^2}{N}\Big\}\\&+\sqrt{\frac{n\big(\theta t^2\beta+\varphi^2t^2(1+a)\theta\big)}{2(1-\delta)\beta[(\beta-1)t\mu-1]^2}}.\end{aligned}\tag{2.26}$$

Since $\big((\beta-1)ut-1\big)^2\ge 2(1-\beta)\mu t$, we have

$$\begin{aligned}&\frac{1}{2(1-\delta)\beta[(\beta-1)\mu t-1]^2}\Big(\Big(\frac{\theta\beta}{\beta-1}-\frac{4\beta\theta}{n}\Big)t^2\mu(\beta-1)+2t^2\mu\frac{\beta L^2}{N}\Big)\\&\le\frac{1}{2(1-\delta)\beta}\Big(\frac{-t}{2}\Big(\frac{\theta\beta}{\beta-1}-\frac{4\beta\theta}{n}\Big)+\frac{t\beta L^2}{(1-\beta)N}\Big).\end{aligned}\tag{2.27}$$

Moreover, since $\big((\beta-1)ut-1\big)^2 \geq 1$, $\varphi^2 \leq 1$ and $0<\delta<1$, we infer

$$\begin{aligned}&\frac{1}{2(1-\delta)\beta[(\beta-1)\mu t-1]^2}\Big(At+1+\frac{4t\beta\theta}{n}+at\Big)\\ &\qquad\leq \frac{1}{2(1-\delta)\beta}\Big(At+1+\frac{4t\beta\theta}{n}+at\Big)\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\sqrt{\frac{n\big(\theta t^2\beta+\varphi^2t^2(1-a)\theta\big)}{2(1-\delta)\beta[(\beta-1)t\mu-1]^2}} &\leq \sqrt{\frac{n\big(\theta t^2\beta+\varphi^2t^2(1+a)\theta\big)}{2(1-\delta)\beta}}\\ &\leq \frac{nt}{2(1-\theta)\beta}\sqrt{\frac{2\theta\beta(1+\beta+a)}{n}}.\end{aligned} \tag{2.29}$$

Combining (2.27), (2.28), (2.29) and (2.26), we conclude that

$$\begin{aligned}\varphi F \leq& \frac{n}{2(1-\delta)\beta}\Big\{\frac{tnc_1^2}{16\delta\beta(1-\beta)R^2}+At+1+\frac{4t\beta\theta}{n}+\frac{t\beta L^2}{(1-\beta)N}\\ &+\frac{\theta t\beta}{2(1-\beta)}+\frac{4t\beta\theta}{2n}+at\Big\}+\sqrt{\frac{n\big(\theta t^2\beta+\varphi^2t^2(1+a)\theta\big)}{2(1-\delta)\beta}}\\ =&\frac{n}{2(1-\delta)\beta}\Big\{\frac{tnc_1^2}{16\delta\beta(1-\beta)R^2}+At+1+\frac{6t\beta\theta}{n}+\frac{t\beta L^2}{(1-\beta)N}\\ &+\frac{\theta t\beta}{2(1-\beta)}+at\Big\}+\frac{nt}{2(1-\theta)\beta}\sqrt{\frac{2\theta\beta(1+\beta+a)}{n}}.\end{aligned}$$

Therefore, for all $(x_0,T)\in B(p,R)\times[0,T]$, we have

$$\begin{aligned}\beta\frac{|\nabla u|^2}{u^2}+a\log u-\frac{u_t}{u}\leq& \frac{n}{2(1-\delta)\beta}\left\{\frac{nc_1^2}{16\delta\beta(1-\beta)R^2}+A+\frac{1}{T}+\frac{6\beta\theta}{n}\right.\\ &\left.+\frac{\beta L^2}{(1-\beta)N}+a+\frac{\theta\beta}{2(1-\beta)}+\sqrt{\frac{2\theta\beta(1+\beta+a)}{n}}\right\}.\end{aligned}$$

The proof of the part (2) is complete. □

3.. Applications

Theorem 3.1. *Let (M,g) be a noncompact n-dimensional Riemannian manifold with Ric_V^N bounded from below by the constant $-K:=-K(2R)$, where $R>0$, $K(2R)>0$ in the geodesic ball $B(p,2R)$ with radius $2R$ around $p\in M$ and V is a smooth vector field on M. Let a be a constant and the equation*

$$\frac{\partial u}{\partial t}=\Delta_V u+au\log u$$

has a positive solution u on $M\times[0,\infty)$. Then

1. *If $a\leq 0$, we have*

$$\beta\frac{|\nabla u|^2}{u^2}+a\log u-\frac{u_t}{u}\leq\frac{N+n}{2(1-\delta)\beta}\Big(\frac{(N+n)c_1^2}{16\delta\beta(1-\beta)R^2}+A+\frac{1}{t}-\frac{a}{2}\Big);$$

2. *If $a \geq 0$, we have*

$$\beta\frac{|\nabla u|^2}{u^2} + a\log u - \frac{u_t}{u} \leq \frac{N+n}{2(1-\delta)\beta}\Big(\frac{(N+n)c_1^2}{16\delta\beta(1-\beta)R^2} + A + \frac{1}{t} + a\Big),$$

where c_1 and c_2 are positive constants, $0<\delta<1$, $\beta = e^{-2Kt}$ and A is defined by

$$A = \frac{\big(n-1+\sqrt{nK}R\big)c_1 + c_2 + 2c_1^2}{R^2}\,.$$

Proof. Note that if $\mathrm{Ric}_V^N \geq -K$ then the Laplacian comparison can be read as follows (see [6])

$$\Delta_V\rho \leq \sqrt{(n-1)K}\coth\Big(\sqrt{\frac{K}{n-1}}\rho\Big) \leq \sqrt{(n-1)K} + \frac{n-1}{\rho}\,.$$

Moreover, (2.18) can be estimate by

$$2\frac{c_1}{R}t\mu^{\frac{1}{2}}(\varphi F)^{\frac{3}{2}} \leq \frac{2\delta\beta[(\beta-1)t\mu-1]^2}{N+n}(\varphi F)^2 + \frac{(N+n)c_1^2t^2\mu}{2\delta\beta[(\beta-1)-1]^2R^2}(\varphi F)\,.$$

Now, let

$$A = \frac{\big(n-1+\sqrt{(n-1)K}R\big)c_1 + c_2 + 2c_1^2}{R^2}$$

and using the same argument as in the proof of Theorem 1.1, we complete the proof of Theorem 3.1. □

In particular, if V is $-\nabla f$ where f is a smooth function on M, we recover the result of Huang-Ma in [4]. Hence, our result is a generalization of Huang-Ma's work. Moreover, let $R\to\infty$ in Theorem 3.1, we obtain the following global gradient estimate of a general heat equation.

Theorem 3.2. *Let (M,g) be a noncompact n-dimensional Riemannian manifold with Ric_V^N bounded from below by the constant $-K$, where $K>0$ and V is a smooth vector field on M. Let a be a constant and the equation*

$$\frac{\partial u}{\partial t} = \Delta_V u + au\log u$$

has a positive solution u on $M\times[0,\infty)$. Then

1. *If $a \leq 0$, we have*

$$\beta\frac{|\nabla u|^2}{u^2} + a\log u - \frac{u_t}{u} \leq \frac{N+n}{2(1-\delta)\beta}\Big(\frac{1}{t} - \frac{a}{2}\Big);$$

2. *If $a \geq 0$, we have*

$$\beta\frac{|\nabla u|^2}{u^2} + a\log u - \frac{u_t}{u} \leq \frac{N+n}{2(1-\delta)\beta}\Big(\frac{1}{t} + a\Big),$$

where $\beta = e^{-2Kt}$ and $0<\delta<1$.

Now, similarly to [4], we show a Harnack type inequality.

Theorem 3.3. *Let (M, g) be a noncompact n-dimensional Riemannian manifold with Ric_V^N bounded from below by the constant $-K$, where $K > 0$ and V is the smooth vector field on M. Suppose that the equation*

$$\frac{\partial u}{\partial t} = \Delta_V u$$

has a positive solution u on $M \times [0, \infty)$. Then

1. *The solution u satisfies*

$$\frac{u_t}{u} - e^{-2Kt}\frac{|\nabla u|^2}{u^2} + e^{2Kt}\frac{N+n}{2t} \geq 0 \tag{3.30}$$

2. *For any points (x_1, t_1) and (x_2, t_2) in $M \times [0, +\infty)$ with $0 < t_1 < t_2$, we have the following Harnack inequality*

$$u(x_1, t_1) \leq u(x_2, t_2)\Big(\frac{t_2}{t_1}\Big)^{\frac{N+n}{2}} e^{\phi(x_1,x_2,t_1,t_2)+B}.$$

Here

$$\phi(x_1, x_2, t_1, t_2) = \inf_\gamma \int_0^t \frac{1}{4}e^{2Kt}|\dot\gamma|^2 dt\,, \quad B = \frac{N+n}{2}\big(e^{2Kt_2} - e^{2Kt_1}\big)$$

where γ is a parameterized curve with $\gamma(t_1) = x_1$, $\gamma(t_2) = x_2$.

Proof. 1. Applying Theorem 3.2 with $a = 0$, we have

$$\beta\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{N+n}{2(1-\delta)\beta t}\,. \tag{3.31}$$

Letting $\delta \to 0$ and $\beta = e^{-2Kt}$ into the inequality (3.31) we obtain

$$\frac{u_t}{u} - e^{-2Kt}\frac{|\nabla u|^2}{u^2} + e^{2Kt}\frac{N+n}{2t} \geq 0\,.$$

The proof is complete.

2. The proof can be followed by using (3.30) and the argument in [4]. We omit the details. □

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