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INITIAL VALUE PROBLEMS FOR FRACTIONAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH HADAMARD TYPE DERIVATIVE

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Abstract. We establish sufficient conditions for the existence of solutions of a class of fractional functional differential inclusions involving the Hadamard fractional derivative with order $\alpha \in (0,1]$. Both cases of convex and nonconvex valued right hand side are considered.

1. Introduction

This paper deals with the existence of solutions for initial value problems (IVP for short), for Hadamard fractional order differential functional inclusions. We consider the initial value problem

$$ {}^{H}D^{\alpha}y(t) \in F(t,y_t)\,, \quad \text{for almost all} \quad t \in J=[1,T], 0<\alpha\le 1, \tag{1}$$

$$ y(t)=\varphi(t) \quad t\in[1-r,1]\,, \tag{2}$$

where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative, $F\colon [1-r,T]\times C([1-r,T],\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\varphi \in C([1-r,1],\mathbb{R})$ with $\varphi(1)=0$. For any continuous function y defined on $[1-r,T]$ and any $t\in J$, we denote by y_t the element of $C([1-r,1],\mathbb{R})$ which is defined by

$$ y_t(\theta)=y(t+\theta)\,, \quad \theta\in[1-r,1]\,.$$

Hence $y_t(\cdot)$ represents the history of the state from times $t-r$ up to the present time t.

Differential equations of fractional order are valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. There has been a significant development in fractional differential equations in recent years; see the monographs of Hilfer [23], Kilbas et al. [26], Podlubny [29], Momani et al. [28], and the papers by Agarwal et al. [1] and Benchohra et al. [7, 6, 5].

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Applied problems require the definitions of fractional derivatives allowing the utilization of physically interpretable initial data, which contain $y(0)$, $y'(0)$, and so on. Caputo's fractional derivative satisfies these demands. For more details concerning geometric and physical interpretation of fractional derivatives of Riemann-Liouville type and Caputo type, see [29].

However, the literature on Hadamard-type fractional differential equations has not undergone as much development; see [2, 31]. The fractional derivative that Hadamard [20] introduced in 1892, differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of arbitrary exponent. Detailed descriptions of the Hadamard fractional derivative and integral can be found in [10, 11, 12].

In this paper, we shall present two existence results for the problem (1)–(2), when the right hand side is convex as well as nonconvex valued. The first result relies on the nonlinear alternative of Leray-Schauder type, while the second result is based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [14]. These results extend to the multivalued case some previous results in the literature, and constitute a contribution for this emerging field.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 1 \le t \le T\}\,,$$

and we let $L^1(J,\mathbb{R})$ denote the Banach space of functions $y\colon J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_1^T |y(t)|dt\,.$$

$AC(J,\mathbb{R})$ is the space of functions $y\colon J \to \mathbb{R}$, which are absolutely continuous. Also $C([1-r,1],\mathbb{R})$ is endowed with the norm

$$\|\varphi\|_C = \sup\{|\varphi(\theta)| : 1-r \le \theta \le 1\}\,.$$

Let $(X, \|\cdot\|)$ be a Banach space. Let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multivalued map $G\colon X \to \mathcal{P}(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{X\subset B} G(X)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\big\{\sup\{|y| : y \in G(x)\}\big\}$.

G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$,

$y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$. A multivalued map $G\colon J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function,

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\},$$

is measurable.

Definition 2.1. A multivalued map $F\colon J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:

(1) $t \to F(t, u)$ is measurable for each $u \in \mathbb{R}$.
(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in AC(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1([1,T], \mathbb{R}) : v(t) \in F(t, y_t) \ \text{ a.e. } \ t \in [1,T]\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by:

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}\,.$$

Definition 2.2. A multivalued operator $N\colon X \to P_{\mathrm{cl}}(X)$ is called:

(1) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y)\,, \quad \text{for each} \quad x, y \in X\,.$$

(2) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 2.3 ([14])**.** *Let (X, d) be a complete metric space. If $N\colon X \to P_{\mathrm{cl}}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \varnothing$.

For more details on multivalued maps see the books of Aubin and Cellina [3], Aubin and Frankowska [4], Deimling [15] and Castaing and Valadier[13].

Definition 2.4 ([26])**.** The Hadamard fractional integral of order r for a function $h\colon [1, +\infty) \to \mathbb{R}$ is defined as

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \Big(\log \frac{t}{s}\Big)^{r-1} \frac{h(s)}{s} ds\,, \quad r > 0\,,$$

provided the integral exists.

Definition 2.5 ([26])**.** For a function h given on the interval $[1, +\infty)$, the r Hadamard fractional-order derivative of h, is defined by

$$({}^H D^r h)(t) = \frac{1}{\Gamma(n - r)} \Big(t \frac{d}{dt}\Big)^n \int_1^t \Big(\log \frac{t}{s}\Big)^{n-r-1} \frac{h(s)}{s} ds\,,$$
$$n - 1 < r < n, n = [r] + 1\,.$$

Here $[r]$ denotes the integer part of r and $\log(\cdot) = \log_e(\cdot)$.

3. Main results

Definition 3.1. A function $y \in C([1-r,T],\mathbb{R}) \cap AC([1,T],\mathbb{R})$ is said to be a solution of (1)–(2), if there exists a function $v \in L^1([1,T],\mathbb{R})$ with $v(t) \in F(t,y_t)$, for a.e. $t \in [1,T]$, such that

$$ {}^H D^\alpha y(t) = v(t)\,, \quad \text{a.e. } t \in [1,T],\ 0 < \alpha \leq 1\,, $$

and the function y satisfies condition (2).

Theorem 3.2. *Assume the following hypotheses hold:*

(H1) $F\colon [1,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a Carathéodory multi-valued map.*

(H2) *There exist* $p \in C([1,T],\mathbb{R}^+)$ *and* $\psi\colon [0,\infty) \to (0,\infty)$ *continuous and nondecreasing such that*

$$ \|F(t,u)\|_{\mathcal{P}} = \sup\{|v| \,:\, v(t) \in F(t,y_t)\} \leq p(t)\psi(\|u\|_C) $$
$$ \text{for} \quad t \in [1,T] \quad \text{and each} \quad u \in C([1-r,1],\mathbb{R})\,. $$

(H3) *There exists* $l \in L^1([1,T],\mathbb{R}^+)$, *with* $I^\alpha l < \infty$, *such that*

$$ H_d\big(F(t,u),F(t,\bar{u})\big) \leq l(t)|u-\bar{u}| \quad \text{for every} \quad u,\bar{u} \in \mathbb{R} $$

and

$$ d(0,F(t,0)) \leq l(t)\,, \quad a.e. \quad t \in [1,T]\,. $$

(H4) *There exists a number* $M > 0$ *such that*

$$ \frac{M}{\dfrac{\psi(M)\|p\|_\infty}{\Gamma(\alpha+1)}(\log T)^\alpha} > 1\,. \tag{3} $$

Then the IVP (1))*–*(2) *has at least one solution on* $[1-r,T]$.

Proof. Transform the problem (1)–(2) into a fixed point problem. Consider the multivalued operator,

$$ N(y)(t) = \left\{ h \in C([1-r,T]), h(t) = \begin{cases} \varphi(t), & \text{if } t \in [1-r,1], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, v \in S_{F,y}, & \\ & \text{if } t \in [1,T] \end{cases} \right\}. $$

We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder. The proof will be given in several steps.

Step 1: $N(y)$ *is convex for each* $y \in C([1-r,T],\mathbb{R})$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist v_1, $v_2 \in S_{F,y}$ such that for each $t \in [1,T]$, we have

$$ h_i(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{v_i(s)}{s} ds\,, $$

for $i = 1, 2$. Let $0 \leq \gamma \leq 1$. Then, for each $t \in [1, T]$, we have

$$(\gamma h_1 + (1-\gamma)h_2)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} \frac{1}{s}[\gamma v_1(s) + (1-\gamma)v_2(s)]\, ds\,.$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$\gamma h_1 + (1-\gamma)h_2 \in N(y)\,.$$

Step 2: *N maps bounded sets into bounded sets in $C([1-r, T], \mathbb{R})$.*

Let $B_{\mu_*} = \{y \in C([1-r, T], \mathbb{R}) : \|y\|_\infty \leq \mu_*\}$ be a bounded sets in $C([1-r, T], \mathbb{R})$ and $y \in B_{\mu_*}$. Then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that, for each $t \in [1, T]$, we have

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} \frac{v(s)}{s}\, ds\,.$$

By (H2), we have, for each $t \in J$,

$$\begin{aligned}
|h(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} \frac{|v(s)|}{s} ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} \frac{p(s)\psi(\|y(s)\|_C)}{s} ds \\
&\leq \frac{\psi(\mu^*)\|p\|_\infty}{\Gamma(\alpha+1)} (\log T)^\alpha\,.
\end{aligned}$$

Thus

$$\|h\|_\infty \leq \frac{\psi(\mu^*)\|p\|_\infty}{\Gamma(\alpha+1)} (\log T)^\alpha := \ell\,.$$

Step 3: *N maps bounded sets into equicontinuous sets of $C([1-r, T], \mathbb{R})$.*

Let t_1, $t_2 \in [1, b]$, $t_1 < t_2$, and B_{μ^*} be a bounded set of $C([1-r, T], \mathbb{R})$ as in Step 2. Let $y \in B_{\mu^*}$ and $h \in N(y)$. Then

$$\begin{aligned}
|h(t_2) - h(t_1)| &= \Big|\frac{1}{\Gamma(\alpha)} \int_1^{t_1} \Big[\Big(\log \frac{t_2}{s}\Big)^{\alpha-1} - \Big(\log \frac{t_1}{s}\Big)^{\alpha-1}\Big] \frac{v(s)}{s} ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \Big(\log \frac{t_2}{s}\Big)^{\alpha-1} \frac{v(s)}{s} ds\Big| \\
&\leq \frac{\|p\|_\infty \psi(\mu^*)}{\Gamma(\alpha)} \int_1^{t_1} \Big[\Big(\log \frac{t_2}{s}\Big)^{\alpha-1} - \Big(\log \frac{t_1}{s}\Big)^{\alpha-1}\Big] \frac{ds}{s} \\
&\quad + \frac{\|p\|_\infty \psi(\mu^*)}{\Gamma(\alpha)} \int_{t_1}^{t_2} \Big(\log \frac{t_2}{s}\Big)^{\alpha-1} \frac{ds}{s}\,.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Step 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $N\colon C([1-r, T], \mathbb{R}) \to \mathcal{P}(C([1-r, T], \mathbb{R}))$ is completely continuous.

Step 4: *N has a closed graph.*

Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y}$, such that, for each $t \in [1,T]$

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_n(s)}{s}\, ds\,.$$

We must show that there exists $v_* \in S_{F,y}$ such that, for each $t \in [1,T]$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s}\, ds\,.$$

Since $F(t,\cdot)$ is upper semi-continuous, then for every $\epsilon > 0$, there exist $n_0(\epsilon) \geq 0$ such that, for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \epsilon B(0,1)\,, \quad \text{a.e.} \quad t \in [1,T]\,.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v_*(\cdot) \quad \text{as} \quad m \to \infty$$

and

$$v_*(t) \in F(t, y_*(t)), \ \text{ a.e. } t \in [1,T]\,.$$

For every $w \in F(t, y_*(t))$, we have

$$|v_{n_m}(t) - v_*(t)| \leq |v_{n_m}(t) - w| + |w - v_*(t)|\,.$$

Then

$$|v_{n_m}(t) - v_*(t)| \leq d(v_{n_m}(t), F(t, y_*(t))\,.$$

By an analogous relation, obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$|v_{n_m}(t) - v_*(t)| \leq H_d(F(t, y_n(t)), F(t, y_*(t))) \leq l(t)\|y_n - y_*\|_\infty.$$

Then

$$\begin{aligned} |h_n(t) - h_*(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v_n(s) - v_*(s)|\, ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} l(s) ds \|y_{n_m} - y_*\|_\infty\,. \end{aligned}$$

Hence

$$\|h_n(t) - h_*(t)\|_\infty \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} l(s) ds \|y_{n_m} - y_*\|_\infty \to 0$$

as $m \to \infty$.

Step 5: *A priori bounds on solutions.*

Let y be such that $y \in \lambda N(y)$ with $\lambda \in (0,1]$. Then there exists $v \in S_{F,y}$ such that, for each $t \in [1,T]$,

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s}\, ds\,.$$

This implies by (H2) that, for each $t \in [1,T]$, we have

$$\begin{aligned}|y(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{|v(s)|}{s}ds\\ &\leq \frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{p(s)\psi(\|y_s\|_C)}{s}\,ds\\ &\leq \frac{\psi(\|y\|_{[1-r,T]})\|p\|_\infty}{\Gamma(\alpha+1)}(\log T)^\alpha\,.\end{aligned}$$

Thus

$$\frac{\|y\|_{[1-r,T]}}{\psi(\|p\|_\infty\|y\|_{[1-r,T]})\dfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}} < 1\,.$$

Then by condition (3), there exist $M > 0$ such that $\|y\|_\infty \neq M$. Let $U = \{y \in C(J,\mathbb{R}) : \|y\|_\infty < M\}$. The operator $N\colon \bar{U} \to \mathcal{P}(C([1-r,T],\mathbb{R}))$ is upper semi continuous and completely continuous. From the choice of U, there is no $y \in \delta U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1]$. As a consequence of the nonlinear alternative of Leray-Shauder , we deduce that N has a fixed point $y \in \bar{U}$ which is a solution of the problem (1)–(2). This completes the proof. □

We present now a result for the problem (1)–(2) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler, that is Lemma 2.3.

Theorem 3.3. *Assume* (H3) *and the following hypothesis holds:*

(H5) $F\colon [1,T]\times\mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *has the property that* $F(\cdot,u)\colon [1,T] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $u \in \mathbb{R}$.
If

$$\frac{\|l\|_\infty(\log T)^\alpha}{\Gamma(\alpha+1)} < 1 \tag{4}$$

then the IVP (1)–(2) *has at least one solution on* $[1-r,T]$.

Proof. We shall show that N, as defined in the proof of Theorem 3.2, satisfies the assumptions of Lemma 2.3. The proof will be given in two steps.

Step 1: $N(y) \in \mathcal{P}_{cl}(C([1-r,T],\mathbb{R}))$ *for each* $y \in C([1-r,T],\mathbb{R})$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \to \bar{y}$ in $C([1-r,T],\mathbb{R})$. Then, $\bar{y} \in C([1-r,T],\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in [1,T]$,

$$y_n(t) = \frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{v_n(s)}{s}\,ds\,.$$

Using the fact that F has compact values and from (H3), we may pass to a subsequence if necessary to get that (v_n) converges weakly to some v in $L^1_w([1,T],\mathbb{R})$ (the space endowed with the weak topology). An application of Mazur's theorem implies that (v_n) converges strongly to v and hence $v \in S_{F,y}$. Then for each $t \in [1,T]$,

$$y_n(t) \to \bar{y}(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} \frac{v(s)}{s}\, ds\,.$$

So, $\bar{y} \in N(y)$.

Step 2: *There exist $\gamma < 1$ such that $H_d(N(y), N(\bar{y})) \le \gamma \|y - \bar{y}\|_{[1-r,T]}$ for each y, $\bar{y} \in C([1-r,T],\mathbb{R})$.*

Let $y, \bar{y} \in C([1-r,T],\mathbb{R})$ and $h_1 \in N(y)$. Then, there exists $v_1 \in F(t, y_t)$ such that for each $t \in [1,T]$

$$y_1(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} \frac{v_1(s)}{s}\, ds\,.$$

From (H3) it follows that

$$H_d\big(F(t, y(t)), F\big(t, \bar{y})(t)\big) \le l(t)|y(t) - \bar{y}(t)|.$$

Hence, there exists $w \in F(t, \bar{y}(t))$ such that

$$|v_1(t) - w| \le l(t)|y(t) - \bar{y}(t)|, \quad t \in [1,T].$$

Consider $U\colon [1,T] \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le l(t)|y(t) - \bar{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t))$ is measurable, there exists a function $v_2(t)$ which is measurable selection for V. So, $v_2 \in F(t, \bar{y}_t)$, and for each $t \in [1,T]$

$$|v_1(t) - v_2(t)| \le l(t)|y(t) - \bar{y}(t)|\,, \quad t \in [1,T].$$

Let us define for each $t \in [1,T]$,

$$y_2(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} \frac{v_2(s)}{s}\, ds\,.$$

Then for each $t \in [1,T]$,

$$\begin{aligned}
|h_1(t) - h_2(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} |v_1(s) - v_2(s)|\, ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} |l(s)||y_s - \bar{y_s}|\, ds \\
&\le \frac{\|l\|_\infty (\log T)^\alpha}{\Gamma(\alpha+1)} \|y - \bar{y}\|_\infty\,.
\end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \le \frac{\|l\|_\infty (\log T)^\alpha}{\Gamma(\alpha+1)} \|y - \bar{y}\|_\infty.$$

For an analogous relation, obtained by interchanging the roles of y and $\bar{y}$ it follows that

$$H_d\big(N(y), N(\bar{y})\big) \le \frac{\|l\|_\infty (\log T)^\alpha}{\Gamma(\alpha+1)} \|y - \bar{y}\|_{[1-r,T]}.$$

So by (4), N is a contraction and thus, by Lemma 2.3, N has a fixed point y which is a solution to (1)–(2). The proof is complete. $\square$

4. An example

We apply Theorem 3.2 to the following fractional differential inclusion,

$$ {}^{H}D^{\alpha}y(t) \in F(t, y_t)\,, \quad \text{for almost all} \quad t \in J = [1, T],\ 0 < \alpha \leq 1\,, \tag{5} $$

$$ y(t) = \varphi(t)\,, \quad t \in [1-r, 1]\,, \tag{6} $$

where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative, $\varphi \in C([1-r, 1], \mathbb{R})$ with $\varphi(1) = 0$, and $F\colon [1-r, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is the multivalued map,

$$ F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\}\,, $$

and where f_1, $f_2\colon [1-r, T] \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [1-r, T]$, $f_1(t, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [1-r, T]$, $f_2(t, \cdot)$ is upper semi-continuous (i.e., the set the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in C([1-r, T], \mathbb{R}^{+})$ and $\psi\colon [0, \infty) \mapsto (0, \infty)$ continuous and nondecreasing such that

$$ \max(|f_1(t, y)|, |f_2(t, y)| \leq p(t)\psi(|y|)\,, \quad t \in [1-r, T]\,, \qquad \text{and all} \quad y \in \mathbb{R}. $$

It is clear that F is compact and convex valued, and also upper semi-continuous. Since all the conditions of Theorem 3.2 are satisfied, problem (5)–(6) has at least one solution y on $[1-r, T]$.

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