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$\mathfrak{g}$-QUASI-FROBENIUS LIE ALGEBRAS

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Abstract. A Lie version of Turaev's $\overline{G}$-Frobenius algebras from 2-dimensional homotopy quantum field theory is proposed. The foundation for this Lie version is a structure we call a $\mathfrak{g}$-*quasi-Frobenius Lie algebra* for $\mathfrak{g}$ a finite dimensional Lie algebra. The latter consists of a quasi-Frobenius Lie algebra $(\mathfrak{q},\beta)$ together with a left $\mathfrak{g}$-module structure which acts on $\mathfrak{q}$ via derivations and for which β is $\mathfrak{g}$-invariant. Geometrically, $\mathfrak{g}$-quasi-Frobenius Lie algebras are the Lie algebra structures associated to symplectic Lie groups with an action by a Lie group G which acts via symplectic Lie group automorphisms. In addition to geometry, $\mathfrak{g}$-quasi-Frobenius Lie algebras can also be motivated from the point of view of category theory. Specifically, $\mathfrak{g}$-quasi Frobenius Lie algebras correspond to *quasi Frobenius Lie objects* in $\mathbf{Rep}(\mathfrak{g})$. If $\mathfrak{g}$ is now equipped with a Lie bialgebra structure, then the categorical formulation of $\overline{G}$-Frobenius algebras given in [16] suggests that the Lie version of a $\overline{G}$-Frobenius algebra is a quasi-Frobenius Lie object in $\mathbf{Rep}(D(\mathfrak{g}))$, where $D(\mathfrak{g})$ is the associated (semiclassical) Drinfeld double. We show that if $\mathfrak{g}$ is a quasitriangular Lie bialgebra, then every $\mathfrak{g}$-quasi-Frobenius Lie algebra has an induced $D(\mathfrak{g})$-action which gives it the structure of a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra.

1. Introduction

Renewed interest in Frobenius algebras arose shortly after Witten's introduction of *Topological Qunatum Field Theory* (TQFT) in [28]. Shortly afterwards, Atiyah proposed a set of axioms for TQFT [3], thus making Witten's work more accessible to the mathematical community. Working from Atiyah's axioms, L. Abrams showed that 2-dimensional TQFTs are classified by commutative Frobenius algebras [1]. Hence, in the 2-dimensional case, the algebraic structure of a TQFT is that of a Frobenius algebra.

The notion of a $(d+1)$-dimensional TQFT was generalized to a $(d+1)$-dimensional *Homotopy Quantum Field Theory* (HQFT) by V. Turaev in [25] by equipping closed d-manifolds and $(d+1)$-dimensional cobordisms with homotopy classes of maps into a target space X. In the special case when X is a $K(\overline{G},1)$-space for $\overline{G}$ a finite group,

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one finds that the 2-dimensional HQFTs are classified by Frobenius algebras with a $\overline{G}$-grading and a $\overline{G}$-action which satisfies a number of conditions [15, 25]. These Frobenius algebras came to be called *$\overline{G}$-Frobenius algebras* (or *crossed $\overline{G}$-algebras*).

In [16], a categorical formulation of $\overline{G}$-Frobenius algebras was presented where $\overline{G}$-Frobenius algebras were shown to correspond to certain types of Frobenius objects in $\mathbf{Rep}(D(k[\overline{G}]))$, the braided monoidal category of finite dimensional left $D(k[\overline{G}])$-modules, where $D(k[\overline{G}])$ is the Drinfeld double of the group ring $k[\overline{G}]$ with its usual Hopf structure. Now the semiclassical analogue of $D(k[\overline{G}])$ (or more generally $D(H)$ for H a finite dimensional Hopf algebra) is $D(\mathfrak{g})$, the Drinfeld double of a finite dimensional Lie bialgbera $(\mathfrak{g},\gamma)$ [7, 9, 10, 18]. The relationship between $\overline{G}$-Frobenius algebras and $D(k[\overline{G}])$ in [16] motivates the following question:

With $(\mathfrak{g},\gamma)$ fixed, what structure plays the role of a $\overline{G}$-Frobenius algebra for $D(\mathfrak{g})$?

Since $D(\mathfrak{g})$ is the Lie version of $D(k[\overline{G}])$, the structure in question should be the Lie version of a $\overline{G}$-Frobenius algebra. To answer the aforementioned question, we introduce the notion of *$\mathfrak{g}$-quasi-Frobenius Lie algebras* for $\mathfrak{g}$ a finite dimensional Lie algebra. A $\mathfrak{g}$-quasi-Frobenius Lie algebra consists of a quasi-Frobenius Lie algebra $(\mathfrak{q},\beta)$ together with a left $\mathfrak{g}$-module structure which acts on $\mathfrak{q}$ via derivations and for which β is $\mathfrak{g}$-invariant. Geometrically, $\mathfrak{g}$-quasi-Frobenius Lie algebras are the Lie algebra structures of symplectic Lie groups with an action by a Lie group G which acts via symplectic Lie group automorphisms. We call the aforementioned structures *G-symplectic Lie groups.*

Interestingly, $\mathfrak{g}$-quasi-Frobenius Lie algebras have a categorical formulation. To obtain this formulation, we introduce the notion of a *quasi-Frobenius Lie object* for any additive symmetric monoidal category. The work of Goyvaerts and Vercuysse on the categorification of Lie algebras [12] provides the foundation for defining quasi-Frobenius Lie objects. The latter then yields an alternate (yet equivalent) definition of a $\mathfrak{g}$-quasi-Frobenius Lie algebra: *a $\mathfrak{g}$-quasi Frobenius Lie algebra is simply a quasi Frobenius Lie object in* $\mathbf{Rep}(\mathfrak{g})$, where $\mathbf{Rep}(\mathfrak{g})$ is the category of finite dimensional representations of $\mathfrak{g}$. Using the categorical formulation of [16] as motivation, we obtain the Lie version of a $\overline{G}$-Frobenius algebra: for a fixed finite dimensional Lie bialgebra $(\mathfrak{g},\gamma)$, the Lie version of a $\overline{G}$-Frobenius algebra is a quasi-Frobenius Lie object in $\mathbf{Rep}(D(\mathfrak{g}))$. In other words, with respect to $(\mathfrak{g},\gamma)$, a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra is the Lie version of a $\overline{G}$-Frobenius algebra. The definition of $D(\mathfrak{g})$ implies that a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra is equivalent to a quasi-Frobenius Lie algebra $(\mathfrak{q},\beta)$ which is both a $\mathfrak{g}$ and $\mathfrak{g}^*$-quasi-Frobenius Lie algebra where the $\mathfrak{g}$ and $\mathfrak{g}^*$ actions satisfy a certain compatibility condition.

The rest of the paper is organized as follows. In Section 2, we give a brief review of quasi-Frobenius Lie algebras, symplectic Lie groups, Lie bialgebras, and the Drinfeld double. In Section 3, we formally define $\mathfrak{g}$-quasi-Frobenius Lie algebras and prove a general result for their construction. We conclude the section with the categorical formulation of these structures. In Section 4, G-symplectic Lie groups are introduced. We show that $\mathfrak{g}$-quasi-Frobenius Lie algebras are the Lie algebra structures of G-symplectic Lie groups. In addition, we show that the category of finite dimensional $\mathfrak{g}$-quasi-Frobenius Lie algebras is equivalent to the category of simply connected G-symplectic Lie groups where G is also simply connected. In Section 5, we focus our attention on $D(\mathfrak{g})$-quasi-Frobenius Lie algebras. We show that if $\mathfrak{g}$ is a quasitriangular Lie bialgebra, then every $\mathfrak{g}$-quasi-Frobenius Lie algebra has an induced $D(\mathfrak{g})$-action which extends the original $\mathfrak{g}$-action and gives the underlying quasi-Frobenius Lie algebra the structure of a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra. In particular, for any finite dimensional Lie algebra $\mathfrak{g}$ (viewed as a Lie bialgebra with co-bracket $\gamma \equiv 0$), every $\mathfrak{g}$-quasi-Frobenius Lie algebra is a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra, where $D(\mathfrak{g})$ is the Drinfeld double of $(\mathfrak{g}, 0)$.

2. Preliminaries

In this section, we briefly review some of the relevant background for the current paper. Throughout this section, k is a field of characteristic zero.

2.1. **Quasi-Frobenius Lie Algebras.** The definition of a *Frobenius Lie algebra* [22, 23] is modeled after the definition of a Frobenius algebra. Formally, a Frobenius Lie algebra is defined as follows:

Definition 2.1. A *Frobenius Lie algebra* over k is a pair $(\mathfrak{g}, \alpha)$ where $\mathfrak{g}$ is a Lie algebra and $\alpha\colon \mathfrak{g} \to k$ is a linear map with the property that the skew-symmetric bilinear form β on $\mathfrak{g}$ defined by

$$\beta(x, y) := \alpha([x, y]) \quad \forall\, x, y \in \mathfrak{g}$$

is nondegenerate.

As a consequence of the Jacobi identity, the skew-symmetric bilinear form β in Definition 2.1 satisfies the following identity:

$$\beta([x, y], z) + \beta([y, z], x) + \beta([z, x], y) = 0\,, \quad \forall\, x, y, z \in \mathfrak{g}\,. \tag{2.1}$$

Equation (2.1) is equivalent to the statement that β is a 2-cocycle in the Lie algebra cohomology of $\mathfrak{g}$ with values in k (where $\mathfrak{g}$ acts trivially on k). This motivates the following generalization of Definition 2.1:

Definition 2.2. A *quasi-Frobenius Lie algebra* over k is a pair $(\mathfrak{g}, \beta)$ where $\mathfrak{g}$ is a Lie algebra over k and β is a nondegenerate 2-cocycle in the Lie algebra cohomology of $\mathfrak{g}$ with values in k (where $\mathfrak{g}$ acts trivially on k).

Remark 2.3. A quasi-Frobenius Lie algebra $(\mathfrak{g}, \beta)$ is a Frobenius Lie algebra iff β is exact, i.e., $\beta(x, y) = (-\delta\alpha)(x, y) := \alpha([x, y])$ for some linear map $\alpha\colon \mathfrak{g} \to k$.

Proposition 2.4. *Every 2-dimensional non-abelian Lie algebra admits the structure of a Frobenius Lie algebra. In particular, every 2-dimensional non-abelian quasi-Frobenius Lie algebra is Frobenius.*

Proof. Let $\mathfrak{g}$ be a 2-dimensional non-abelian Lie algebra. Then $\mathfrak{g}$ admits a basis u_1, u_2 such that $[u_1, u_2] = u_2$. Let $\alpha\colon \mathfrak{g} \to k$ be the linear map defined by $\alpha(u_1) = 0$ and $\alpha(u_2) = 1$. Then $(\mathfrak{g}, \alpha)$ is a Frobenius Lie algebra.

If $(\mathfrak{g}, \beta)$ is a quasi-Frobenius Lie algebra, set $\alpha(u_1) = 0$ and $\alpha(u_2) = \beta(u_1, u_2)$. Then it's easy to see that $\beta(x, y) = \alpha([x, y])$ for all $x, y \in \mathfrak{g}$. Hence, $(\mathfrak{g}, \beta)$ is Frobenius. □

Remark 2.5. Since every finite dimensional quasi-Frobenius Lie algebra $(\mathfrak{g}, \beta)$ is also a symplectic vector space, it follows that the dimension of $\mathfrak{g}$ is necessarily even.

Proposition 2.6. *Let $\mathfrak{g}$ be a Lie algebra of dimension n over k and let $e_1, e_2, \ldots, e_n$ be a basis of $\mathfrak{g}$. Then the following statements are equivalent:*

(1) *There exists $\alpha \in \mathfrak{g}^*$ such that $(\mathfrak{g}, \alpha)$ is a Frobenius Lie algebra.*
(2) *There exists $\alpha \in \mathfrak{g}^*$ such that* $\det(\alpha([e_i, e_j])) \neq 0$
(3) $\det([e_i, e_j]) \neq 0$*, where $[e_i, e_j] \in \mathfrak{g}$ are viewed as elements of the symmetric algebra $S(\mathfrak{g})$.*

Proof. (1) $\Leftrightarrow$ (2) Immediate.

(2) $\Rightarrow$ (3) Recall that $S(\mathfrak{g})$ is naturally isomorphic to the polynomial ring in n-variables where the variables are taken to be the basis $e_1, e_2, \ldots, e_n$. Extend the linear map $\alpha\colon \mathfrak{g} \to k$ to a unit preserving algebra map $\alpha\colon S(\mathfrak{g}) \to k$ via

$$\alpha(v_1 v_2 \cdots v_r) := \alpha(v_1)\alpha(v_2) \cdots \alpha(v_r)$$

for $v_1, \ldots, v_r \in \mathfrak{g}$. Then

$$\alpha(\det([e_i, e_j])) = \det(\alpha([e_i, e_j])) \neq 0\,,$$

which implies that $\det([e_i, e_j]) \neq 0$.

(2) $\Leftarrow$ (3) Let $p = \det([e_i, e_j]) \in S(\mathfrak{g})$. Since $p = p(e_1, \ldots, e_n) \neq 0$ and k is infinite, there exists $\lambda_i \in k$ such that $p(\lambda_1, \ldots, \lambda_n) \neq 0$ (see Theorem 3.76 of [26]). Let $\alpha\colon \mathfrak{g} \to k$ be the linear map defined by $\alpha(e_i) = \lambda_i$ for $i = 1, \ldots, n$. As before, extend $\alpha\colon \mathfrak{g} \to k$ to an algebra map $\alpha : S(\mathfrak{g}) \to k$. Then

$$\begin{aligned} \det(\alpha([e_i, e_j])) &= \alpha\big(\det([e_i, e_j])\big) \\ &= \alpha\big(p(e_1, \ldots, e_n)\big) \\ &= p\big(\alpha(e_1), \ldots, \alpha(e_n)\big) \\ &= p(\lambda_1, \ldots, \lambda_n) \\ &\neq 0\,. \end{aligned}$$

□

We now recall two examples. The first is Frobenius and the second is quasi-Frobenius but not Frobenius [6, 22].

Example 2.7. Let $\mathfrak{g}$ be the 4-dimensional Lie algebra with basis $\{x_1, \ldots, x_4\}$ and non-zero commutator relations:

$$[x_1, x_2] = \frac{1}{2}x_2 + x_3, \quad [x_1, x_3] = \frac{1}{2}x_3, \quad [x_1, x_4] = x_4, \quad [x_2, x_3] = x_4\,.$$

Then $\det([x_i, x_j]) = (x_4)^4 \neq 0$, where $[x_i, x_j]$ are regarded as elements of the symmetric algebra $S(\mathfrak{g})$. By Proposition 2.6, there exists a linear map $\alpha\colon \mathfrak{g} \to k$ for which $(\mathfrak{g}, \alpha)$ is a Frobenius Lie algebra.

Example 2.8. Let $\mathfrak{q}$ be the 4-dimensional Lie algebra with basis $\{x_1, \ldots, x_4\}$ and non-zero commutator relations:

$$[x_1, x_2] = x_3, \quad [x_1, x_3] = x_4 .$$

Since $\det([x_i, x_j]) = 0$, $\mathfrak{q}$ cannot be Frobenius by Proposition 2.6. However, it does admit the structure of a quasi-Frobenius Lie algebra. As an example of this, let β be the nondegenerate, skew-symmetric bilinear form given by

$$\beta = x_1^* \wedge x_4^* + x_2^* \wedge x_3^*$$

where $\{x_1^*, \ldots, x_4^*\}$ is the dual basis. A direct calculation shows that β satisfies the 2-cocycle condition. Hence, $(\mathfrak{q}, \beta)$ is quasi-Frobenius.

Definition 2.9. Let $(\mathfrak{g}_1, \beta_1)$ and $(\mathfrak{g}_2, \beta_2)$ be quasi-Frobenius Lie algebras. A *quasi-Frobenius Lie algebra homomorphism* from $(\mathfrak{g}_1, \beta_1)$ to $(\mathfrak{g}_2, \beta_2)$ is a Lie algebra homomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\varphi^*\beta_2 = \beta_1$, that is,

$$\beta_1(u, v) = \beta_2\big(\varphi(u), \varphi(v)\big), \quad \forall\, u, v \in \mathfrak{g}_1 . \tag{2.2}$$

If $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ satisfies (2.2) and is also a Lie algebra isomorphism, then φ is an isomorphism of quasi-Frobenius Lie algebras.

Proposition 2.10. *Let $\varphi\colon (\mathfrak{g}_1, \beta_1) \to (\mathfrak{g}_2, \beta_2)$ be a quasi-Frobenius Lie algebra map. If $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_2 < \infty$, then φ is an isomorphism of quasi-Frobenius Lie algebras.*

Proof. Since $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_2 < \infty$, it suffices to show that φ is injective. Let $u \in \mathfrak{g}_1$ be any nonzero element. Since β is nondegenerate, there exists $v \in \mathfrak{g}_1$ such that $\beta(u, v) \neq 0$. Hence,

$$\beta_2\big(\varphi(u), \varphi(v)\big) = \beta_1(u, v) \neq 0 ,$$

which implies that $\varphi(u) \neq 0$. This completes the proof. □

2.2. **Symplectic Lie Groups.** In this section, we recall the correspondence between *symplectic Lie groups* [4, 8] and quasi-Frobenius Lie algebras.

Definition 2.11. A *symplectic Lie group* is a pair (G, ω) where G is a Lie group and ω is a left-invariant symplectic form on G.

The next result shows that the Lie algebra of a symplectic Lie group is naturally a quasi-Frobenius Lie algebra.

Proposition 2.12. *Let (G, ω) be a symplectic Lie group. Then $(\mathfrak{g}, \omega_e)$ is a quasi-Frobenius Lie algebra.*

Proof. Let $\mathfrak{X}_l(G)$ denote the space of left-invariant vector fields on G and endow $\mathfrak{g} := T_eG$ with the Lie algebra structure of $\mathfrak{X}_l(G)$. Also, let $\tilde{x}$ denote the left-invariant vector field associated with $x \in \mathfrak{g}$. We now show that $(\mathfrak{g}, \omega_e)$ is a quasi-Frobenius Lie algebra. Since $\omega_g|_{T_gG}$ is nondegenerate for all $g \in G$ (in particular for $g = e$), it only remains to show that ω_e is a 2-cocycle of $\mathfrak{g}$ with values in $\mathbb{R}$ (where $\mathfrak{g}$ acts trivially on $\mathbb{R}$).

First, note that for any x, $y \in \mathfrak{g}$, $\omega(\tilde{x}, \tilde{y})$ is a constant function on G. Indeed, for $g \in G$

$$\begin{aligned}\big(\omega(\tilde{x}, \tilde{y})\big)(g) &:= \omega_g(\tilde{x}_g, \tilde{y}_g)\\ &= \omega_g((l_g)_* x, (l_g)_* y)\\ &= (l_g^* \omega)_e(x, y)\\ &= \omega_e(x, y)\end{aligned}$$

where the last equality follows from the fact that ω is left-invariant. This fact along with the fact the ω is closed implies that $\omega_e \in Z^2(\mathfrak{g}; \mathbb{R})$:

$$\begin{aligned}0 &= d\omega(\tilde{x}, \tilde{y}, \tilde{z})\\ &= \tilde{x}\big(\omega(\tilde{y}, \tilde{z})\big) - \tilde{y}\big(\omega(\tilde{x}, \tilde{z})\big) + \tilde{z}\big(\omega(\tilde{x}, \tilde{y})\big)\\ &\quad - \omega([\tilde{x}, \tilde{y}], \tilde{z}) + \omega([\tilde{x}, \tilde{z}], \tilde{y}) - \omega([\tilde{y}, \tilde{z}], \tilde{x})\\ &= -\omega([\tilde{x}, \tilde{y}], \tilde{z}) - \omega([\tilde{z}, \tilde{x}], \tilde{y}) - \omega([\tilde{y}, \tilde{z}], \tilde{x})\,.\end{aligned}$$

Evaluating the last equality at $e \in G$ and multiplying by -1 gives the 2-cocycle condition on ω_e:

$$\omega_e([x, y], z) + \omega_e([z, x], y) + \omega_e([y, z], x) = 0\,.$$

Hence, $(\mathfrak{g}, \omega_e)$ is a quasi-Frobenius Lie algebra. □

Proposition 2.13. *Let G be a Lie group whose Lie algebra $\mathfrak{g}$ carries the structure of a quasi-Frobenius Lie algebra with 2-cocycle β. Define $\widetilde{\beta} \in \Omega^2(G)$ by*

$$\widetilde{\beta}_g := (l_{g^{-1}})^* \beta \in \wedge^2 T_g^* G, \qquad \forall\ g \in G$$

where $l_g \colon G \to G$ is left translation by g. Then $(G, \widetilde{\beta})$ is a symplectic Lie group whose associated quasi-Frobenius Lie algebra is $(\mathfrak{g}, \widetilde{\beta}_e) = (\mathfrak{g}, \beta)$.

Proof. It follows immediately from the definition that $\widetilde{\beta}$ is left-invariant, that is, $(l_g)^* \widetilde{\beta} = \widetilde{\beta}$ for all $g \in G$. Moreover, since β is nondegenerate, $\widetilde{\beta}$ must be nondegenerate as well. To see that $d\widetilde{\beta} = 0$, it suffices to show that $d\widetilde{\beta}(\tilde{x}, \tilde{y}, \tilde{z}) = 0$ for all left-invariant vector fields $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$. Since $\widetilde{\beta}$ is left-invariant, it follows that $\widetilde{\beta}(\tilde{x}, \tilde{y}) = \widetilde{\beta}_e(x, y) = \beta(x, y)$ is a constant function on G for all left-invariant vector fields $\tilde{x}$ and $\tilde{y}$, where $\tilde{x}_e = x$ and $\tilde{y}_e = y$. In particular,

$$\widetilde{\beta}([\tilde{x}, \tilde{y}], \tilde{z}) = \beta([x, y], z)\,.$$

The proof of Proposition 2.12 shows that if $\widetilde{\beta}$ is left-invariant, we have

$$\begin{aligned}d\widetilde{\beta}(\tilde{x}, \tilde{y}, \tilde{z}) &= -\widetilde{\beta}([\tilde{x}, \tilde{y}], \tilde{z}) - \widetilde{\beta}([\tilde{z}, \tilde{x}], \tilde{y}) - \widetilde{\beta}([\tilde{y}, \tilde{z}], \tilde{x})\\ &= -\beta([x, y], z) - \beta([z, x], y) - \beta([y, z], x)\,.\end{aligned}$$

Since $\beta \in Z^2(\mathfrak{g}; \mathbb{R})$, the last equality must be zero. Hence, $(G, \widetilde{\beta})$ is a symplectic Lie group. □

Definition 2.14. Let (G, ω) and (H, σ) be symplectic Lie groups. A homomorphism of symplectic Lie groups is a Lie group homomorphism $\varphi \colon G \to H$ such that $\varphi^* \sigma = \omega$.

Lemma 2.15. *Let (G,ω) and (H,σ) be symplectic Lie groups and let $\varphi\colon G\to H$ be a Lie group homomorphism. Then $\varphi^*\sigma=\omega$ iff $(\varphi^*\sigma)_e=\omega_e$.*

Proof. ($\Rightarrow$) Suppose $(\varphi^*\sigma)=\omega$. By definition, $(\varphi^*\sigma)_g=\omega_g$ for all $g\in G$. In particular, the equality holds for $g=e$.

($\Leftarrow$) Now suppose $(\varphi^*\sigma)_e=\omega_e$. Let $g\in G$ and x, $y\in T_gG$. Then

$$\begin{aligned}(\varphi^*\sigma)_g(x,y)&=\sigma_{\varphi(g)}\big(\varphi_{*,g}(x),\varphi_{*,g}(y)\big)\\&=[(l_{\varphi(g^{-1})})^*\sigma_e]\big(\varphi_{*,g}(x),\varphi_{*,g}(y)\big)\\&=\sigma_e\big((l_{\varphi(g^{-1})}\circ\varphi)_{*,g}(x),(l_{\varphi(g^{-1})}\circ\varphi)_{*,g}(y)\big)\\&=\sigma_e\big((\varphi\circ l_{g^{-1}})_{*,g}(x),(\varphi\circ l_{g^{-1}})_{*,g}(y)\big)\\&=(\varphi^*\sigma)_e\big((l_{g^{-1}})_{*,g}(x),(l_{g^{-1}})_{*,g}(y)\big)\\&=\omega_e\big((l_{g^{-1}})_{*,g}(x),(l_{g^{-1}})_{*,g}(y)\big)\\&=[(l_{g^{-1}})^*\omega_e](x,y)\\&=\omega_g(x,y),\end{aligned}$$

where the second and last equalities follow from the left-invariance of σ and ω respectively and the fourth equality follows from the fact that φ is a group homomorphism. This completes the proof. $\square$

Proposition 2.16. *Let $\varphi\colon(G,\omega)\to(H,\sigma)$ be a homomorphism of symplectic Lie groups. Then*

$$\varphi_{*,e}\colon(\mathfrak{g},\omega_e)\to(\mathfrak{h},\sigma_e)$$

is a homomorphism of quasi-Frobenius Lie algebras.

Proof. This follows immediately from the properties of φ. $\square$

Proposition 2.17. *Let $\psi\colon(\mathfrak{g},\beta)\to(\mathfrak{h},\sigma)$ be a homomorphism of quasi-Frobenius Lie algebras. Let G be the simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and let H be any Lie group whose Lie algebra is $\mathfrak{h}$. Let $(G,\widetilde{\beta})$ and $(H,\widetilde{\sigma})$ be the symplectic Lie groups associated to $(\mathfrak{g},\beta)$ and $(\mathfrak{h},\sigma)$ respectively (see Proposition 2.13). Then there exists a unique symplectic Lie group homomorphism*

$$\widehat{\psi}\colon(G,\widetilde{\beta})\to(H,\widetilde{\sigma})$$

such that $\widehat{\psi}_{*,e}=\psi$.

Proof. Since G is simply connected, there exists a unique Lie group homomorphism $\widehat{\psi}\colon G\to H$ such that $\widehat{\psi}_{*,e}=\psi$. It only remains to show that $\widehat{\psi}^*\widetilde{\sigma}=\widetilde{\beta}$. By Lemma 2.15, it suffices to show that $(\widehat{\psi}^*\widetilde{\sigma})_e=\widetilde{\beta}_e=\beta$. To do this, let $x,y\in\mathfrak{g}$. Then

$$\begin{aligned}(\widehat{\psi}^*\widetilde{\sigma})_e(x,y)&=\widetilde{\sigma}_{\widehat{\psi}(e)}\big(\widehat{\psi}_{*,e}(x),\widehat{\psi}_{*,e}(y)\big)\\&=\widetilde{\sigma}_e\big(\psi(x),\psi(y)\big)\\&=\sigma\big(\psi(x),\psi(y)\big)\\&=(\psi^*\sigma)(x,y)\\&=\beta(x,y)\,.\end{aligned}$$

This completes the proof. □

Theorem 2.18. *Let* **SCSLG** *be the category of simply connected symplectic Lie groups and let* **qFLA** *be the category of finite dimensional quasi-Frobenius Lie algberas. Let F be the functor from* **SCSLG** *to* **qFLA** *which sends (G,ω) to $(\mathfrak{g},\omega_e)$ and $\varphi\colon (G,\omega)\to(H,\sigma)$ to $\varphi_{*,e}\colon(\mathfrak{g},\omega_e)\to(\mathfrak{h},\sigma_e)$. Then F is an equivalence of categories.*

Proof. Theorem 2.18 follows from the well known correspondence between simply connected Lie groups and finite dimensional Lie algebras combined with Proposition 2.12, Proposition 2.13, Proposition 2.16, and Proposition 2.17. □

As an example, we now recall the symplectic Lie group structure on the affine Lie group $A(n,\mathbb{R})$ (c.f., [2, 21, 22]).

Example 2.19. Recall that $A(n,\mathbb{R})$ is the Lie group consisting of $(n+1)\times(n+1)$ matrices of the form

$$A(n,\mathbb{R}) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A\in GL(n,\mathbb{R}),\ v\in\mathbb{R}^n \right\}.$$

The associated Lie algebra is then

$$\mathfrak{a}(n,\mathbb{R}) = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \mid A\in \mathfrak{gl}(n,\mathbb{R}),\ v\in\mathbb{R}^n \right\}.$$

From the definition of $A(n,\mathbb{R})$, we see that $A(n,\mathbb{R})$ is even dimensional with $\dim\ A(n,\mathbb{R}) = \dim\mathfrak{a}(n,\mathbb{R}) = n^2+n = n(n+1)$. Let E_{ij} denote the $(n+1)\times(n+1)$ matrix with 1 in the (i,j)-component and all other components zero. Then $\{E_{ij}\}_{1\le i\le n,\ 1\le j\le n+1}$ is a basis on $\mathfrak{a}(n,\mathbb{R})$. Let $\{E^*_{ij}\}_{1\le i\le n,\ 1\le j\le n+1}$ denote the corresponding dual basis. Define

$$\alpha = E^*_{12} + E^*_{23} + \cdots + E^*_{n,n+1}$$

and $\beta(X,Y) := -\delta\alpha(X,Y) = \alpha([X,Y])$ for all $X,Y\in\mathfrak{a}(n,\mathbb{R})$. Since

$$[E_{ij},E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}\,,$$

we see that

$$\beta(E_{ij},E_{kl}) = \delta_{jk}\delta_{l,i+1} - \delta_{li}\delta_{j,k+1}\,. \tag{2.3}$$

Careful consideration of (2.3) shows that $\beta := -\delta\alpha \in Z^2(\mathfrak{a}(n,\mathbb{R});\mathbb{R})$ is nondegenerate. Hence, $(\mathfrak{a}(n,\mathbb{R}),\alpha)$ is a Frobenius Lie algebra. (In particular, $(\mathfrak{a}(n,\mathbb{R}),\beta)$ is a quasi-Frobenius Lie algebra.) Let $\tilde\beta\in\Omega^2(A(n,\mathbb{R}))$ be the left-invariant 2-form on $A(n,\mathbb{R})$ associated to β. Then $(A(n,\mathbb{R}),\widetilde{\beta})$ is a symplectic Lie group. Furthermore, since $\beta := -\delta\alpha$, it follows that $\tilde\beta$ is exact. Specifically,

$$\widetilde{\beta} = -d\tilde\alpha$$

where $\tilde\alpha\in\Omega^1(A(n,\mathbb{R}))$ is the left-invariant 1-form on $A(n,\mathbb{R})$ associated to α.

2.3. **Lie bialgebras & the Drinfeld Double.**

Definition 2.20. A *Lie bialgebra* over a field k is a pair $(\mathfrak{g}, \gamma)$ where $\mathfrak{g}$ is a Lie algebra over k and $\gamma\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ is a skew-symmetric linear map such that

(1) $\gamma^*\colon \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$, where the dual map γ^* is restricted to $\mathfrak{g}^* \otimes \mathfrak{g}^* \subset (\mathfrak{g} \otimes \mathfrak{g})^*$;

(2) γ is a 1-cocycle on $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$, where $\mathfrak{g}$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ via the adjoint action.

γ is called the *cobracket* or *co-commutator.*

Condition 2 in Definition 2.20 is equivalent to the condition

$$\gamma([x,y]) = ad_x^{(2)}\,\gamma(y) - ad_y^{(2)}\gamma(x)\,, \quad \forall\, x, y \in \mathfrak{g}$$

where the linear map $ad_x^{(2)}\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the adjoint action of $x \in \mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$. Explicitly, $ad_x^{(2)}$ is defined via

$$ad_x^{(2)}(y \otimes z) = ad_x(y) \otimes z + y \otimes ad_x(z) = [x,y] \otimes z + y \otimes [x,z]$$

for y, $z \in \mathfrak{g}$.

Definition 2.21. Let $(\mathfrak{g}, \gamma_{\mathfrak{g}})$ and $(\mathfrak{h}, \gamma_{\mathfrak{h}})$ be Lie bialgebras. A *Lie bialgebra homomorphism* from $(\mathfrak{g}, \gamma_{\mathfrak{g}})$ to $(\mathfrak{h}, \gamma_{\mathfrak{h}})$ is a Lie algebra map $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ such that

$$(\varphi \otimes \varphi) \circ \gamma_{\mathfrak{g}} = \gamma_{\mathfrak{h}} \circ \varphi\,.$$

Example 2.22. Any Lie algebra $\mathfrak{g}$ can be turned into a Lie bialgebra by taking the cobracket $\gamma \equiv 0$. $(\mathfrak{g}, 0)$ is the *trivial* Lie bialgebra structure on $\mathfrak{g}$.

The next result shows that the notion of a Lie bialgebra is self-dual for the finite dimensional case.

Proposition 2.23. *Let $(\mathfrak{g}, \gamma_{\mathfrak{g}})$ be a finite dimensional Lie bialgebra and let $\gamma_{\mathfrak{g}^*}: \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ be the dual of the Lie bracket on $\mathfrak{g}$. Then $(\mathfrak{g}^*, \gamma_{\mathfrak{g}^*})$ is a Lie bialgebra, where the Lie bracket on $\mathfrak{g}^*$ is given by the dual of $\gamma_{\mathfrak{g}}$.*

For a Lie algebra $\mathfrak{g}$, the simplest way to obtain an element of $Z^1_{ad}(\mathfrak{g}; \mathfrak{g} \otimes \mathfrak{g})$ is to turn to the 0-cochains and take their coboundaries. This raises the following natural question: given $r \in \mathfrak{g} \otimes \mathfrak{g}$, when does $\delta r \in Z^1_{ad}(\mathfrak{g}; \mathfrak{g} \otimes \mathfrak{g})$ define a Lie bialgebra structure on $\mathfrak{g}$? To answer this question, let

$$r = \sum_i a_i \otimes b_i\,,$$

and define

$$[[r,r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]\,, \tag{2.4}$$

where

$$[r_{12}, r_{13}] := \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j \,, \tag{2.5}$$

$$[r_{12}, r_{23}] := \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j \,, \tag{2.6}$$

$$[r_{13}, r_{23}] := \sum_{i,j} = a_i \otimes a_j \otimes [b_i, b_j] \,. \tag{2.7}$$

Definition 2.24. A *coboundary Lie bialgebra* is a Lie bialgebra $(\mathfrak{g}, \gamma)$ such that $\gamma = \delta r$ for some $r \in \mathfrak{g} \otimes \mathfrak{g}$. The element r is called the *r-matrix.*

The next result provides a necessary and sufficient condition for an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ to define a Lie bialgebra structure on $\mathfrak{g}$.

Proposition 2.25. *Let $\mathfrak{g}$ be a Lie algebra. Then $(\mathfrak{g}, \delta r)$ is a Lie bialgebra iff*

(i) *$r+\sigma(r)$ is invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}\otimes\mathfrak{g}$, where $\sigma\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the unique linear map defined by $x \otimes y \mapsto y \otimes x$ for x, $y \in \mathfrak{g}$;*

(ii) *$[[r, r]]$ is invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.*

Proof. See pp. 51–54 of [7]. $\square$

The simplest way to ensure that condition (ii) of Proposition 2.25 is satisfied is to demand that

$$[[r, r]] = 0 \,. \tag{2.8}$$

Equation 2.8 is called the *classical Yang-Baxter equation* (CYBE). The CYBE motivates the following definition:

Definition 2.26. A coboundary Lie bialgebra $(\mathfrak{g}, \delta r)$ is *quasitriangular* if r is a solution of the CYBE. Furthermore, if r is skew-symmetric, that is, $r \in \mathfrak{g}\wedge\mathfrak{g} \subset \mathfrak{g}\otimes\mathfrak{g}$, then $(\mathfrak{g}, \delta r)$ is said to be *triangular.*

Example 2.27. Let $\mathfrak{g}$ be the two dimensional Lie algebra with basis x, y and commutator relation $[x, y] = x$. Define $r = y \wedge x$. Then $(\mathfrak{g}, \delta r)$ is a triangular Lie bialgebra, where $\gamma := \delta r$ is given explicitly by

$$\gamma(x) = 0 \,, \qquad \gamma(y) = x \wedge y \,.$$

Before turning to the Drinfeld double, we recall the following notion:

Definition 2.28. Let $\mathfrak{g}$ be a Lie algebra and let $\langle \cdot, \cdot \rangle$ be a bilinear form on $\mathfrak{g}$. $\mathfrak{g}$ is *ad-invariant* with respect to $\langle \cdot, \cdot \rangle$ if

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle \,, \quad \forall\ x, y, z \in \mathfrak{g} \,. \tag{2.9}$$

Now let $(\mathfrak{g}, \gamma_{\mathfrak{g}})$ be a finite dimensional Lie bialgebra and let $(\mathfrak{g}^*, \gamma_{\mathfrak{g}^*})$ be the associated dual Lie bialgebra. Consider the direct sum

$$\mathfrak{g} \oplus \mathfrak{g}^*$$

and equip it with the symmetric, nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle x + \xi, y + \eta \rangle = \xi(y) + \eta(x) \,,$$

where we write $x+\xi$ and $y+\eta$ for $(x,\xi),\ (y,\eta)\in\mathfrak{g}\oplus\mathfrak{g}^*$. The Drinfeld double of $(\mathfrak{g},\gamma_{\mathfrak{g}})$, denoted by $D(\mathfrak{g})$, is the unique quasitriangular Lie bialgebra which satisfies the following condtions:

(1) As a vector space,

$$D(\mathfrak{g})=\mathfrak{g}\oplus\mathfrak{g}^*\,.$$

(2) As a Lie algebra, $D(\mathfrak{g})$ is ad-invariant with respect to the inner product $\langle\cdot,\cdot\rangle$ and contains $\mathfrak{g}$ and $\mathfrak{g}^*$ as Lie subalgebras.

(3) The cobracket on $D(\mathfrak{g})$ is defined by $\gamma_D:=\gamma_{\mathfrak{g}}-\gamma_{\mathfrak{g}^*}$.

Let $[\cdot,\cdot]_D$, $[\cdot,\cdot]_{\mathfrak{g}}$, and $[\cdot,\cdot]_{\mathfrak{g}^*}$ denote the Lie brackets on $D(\mathfrak{g})$, $\mathfrak{g}$, and $\mathfrak{g}^*$ respectively. Condition (2) implies that

$$[x,y]_D=[x,y]_{\mathfrak{g}},\quad[\xi,\eta]_D=[\xi,\eta]_{\mathfrak{g}^*}\,,\quad[x,\xi]_D=ad^*_x\xi-ad^*_\xi x$$

for all $x,\,y\in\mathfrak{g}$ and $\xi,\eta\in\mathfrak{g}^*$, where ad^* denotes the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ and $\mathfrak{g}^*$ on $\mathfrak{g}$. Explicitly, $ad^*_x\colon\mathfrak{g}^*\to\mathfrak{g}^*$ and $ad^*_\xi\colon\mathfrak{g}\to\mathfrak{g}$ are defined by $ad^*_x:=-\,ad^t_x$ and $ad^*_\xi:=-\,ad^t_\xi$ where ad^t_x and ad^t_ξ are the ordinary duals of $ad_x\colon\mathfrak{g}\to\mathfrak{g}$ and $ad_\xi\colon\mathfrak{g}^*\to\mathfrak{g}^*$. In dealing with the Drinfeld double, we will drop the "D", "$\mathfrak{g}$", and "$\mathfrak{g}^*$" that appear as subscripts in the Lie brackets of $D(\mathfrak{g})$, $\mathfrak{g}$, and $\mathfrak{g}^*$ respectively. Condition (2) implies that the triple $(D(\mathfrak{g}),\mathfrak{g},\mathfrak{g}^*)$ is a *Manin triple* with respect to the inner product $\langle\cdot,\cdot\rangle$. In fact, there is a one to one correspondence between finite dimensional Lie bialgebras and Manin triples (see [7]).

Lastly, condition (3) implies that $D(\mathfrak{g})$ is quasitriangular with r-matrix

$$r=\sum_i e_i\otimes e_i^*$$

where $e_1,\dots,e_n$ is any basis on $\mathfrak{g}$ and $e_1^*,\dots,e_n^*$ is the corresponding dual basis.

Example 2.29. Let $(\mathfrak{g},\gamma)$ be the 2-dimensional Lie bialgebra with basis x,y satisfying $[x,y]=x$ and cobracket $\gamma(x)=0$ and $\gamma(y)=x\wedge y$. Let x^*,y^* denote the corresponding dual basis. The commutator relations on $D(\mathfrak{g})$ are

$$\begin{gathered}[x,y]=x\,,\quad[x^*,y^*]=y^*\,,\quad[x,x^*]=-y^*,\quad[x,y^*]=0\\ [y,x^*]=x^*+y\,,\qquad\qquad[y,y^*]=-x\,.\end{gathered}$$

The r-matrix is $r=x\otimes x^*+y\otimes y^*$.

3. $\mathfrak{g}$-quasi-Frobenius Lie Algebras

We begin with the formal definition:

Definition 3.1. A $\mathfrak{g}$-*quasi-Frobenius Lie algebra* is a triple $(\mathfrak{q},\beta,\rho)$ such that $(\mathfrak{q},\beta)$ is a quasi-Frobenius Lie algebra and $\rho\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{q})$, $x\mapsto\rho_x$ is a left $\mathfrak{g}$-module structure on $\mathfrak{q}$ such that

(i) ρ_x is a derivation on $\mathfrak{q}$ for all $x\in\mathfrak{g}$,

(ii) $\beta(\rho_x(u),v)+\beta(u,\rho_x(v))=0$ for all $x\in\mathfrak{g}$, $u,v\in\mathfrak{q}$ ($\mathfrak{g}$-invariance).

In this section, we prove a result for the general construction of $\mathfrak{g}$-quasi-Frobenius Lie algebras. Before doing so, we make the following observation:

Proposition 3.2. *Let* $(\mathfrak{q},\beta)$ *be a quasi-Frobenius Lie algebra and let* $\mathrm{Aut}(\mathfrak{q},\beta)$ *be the automorphism group of* $(\mathfrak{q},\beta)$. *Then* $\mathrm{Aut}(\mathfrak{q},\beta)$ *is an embedded Lie subgroup of* $GL(\mathfrak{q})$.

Proof. As a set, $\mathrm{Aut}(\mathfrak{q},\beta)=\mathrm{Aut}(\mathfrak{q})\cap \mathrm{Sp}(\mathfrak{q},\beta)$ where $\mathrm{Aut}(\mathfrak{q})$ is the group of automorphisms of the Lie algebra $\mathfrak{q}$ and $\mathrm{Sp}(\mathfrak{q},\beta)$ is the group of linear symplectomorphisms of $(\mathfrak{q},\beta)$, where the latter is regarded as a symplectic vector space. Since $\mathrm{Aut}(\mathfrak{q})$ and $\mathrm{Sp}(\mathfrak{q},\beta)$ are both closed subgroups of $GL(\mathfrak{q})$, each being the zero set of a collection of polynomials, $\mathrm{Aut}(\mathfrak{q},\beta)$ is also a closed subgroup of $GL(\mathfrak{q})$. By the closed subgroup theorem [27], $\mathrm{Aut}(\mathfrak{q},\beta)$ is an embedded Lie subgroup of $GL(\mathfrak{q})$. □

Proposition 3.3. *Let* $(\mathfrak{q},\beta)$ *be a quasi-Frobenius Lie algebra and let*

$$\rho\colon G\to \mathrm{Aut}(\mathfrak{q},\beta)\subset GL(\mathfrak{q})\,,\quad g\mapsto \rho_g$$

be a Lie group homomorphism. Define

$$\rho':=\rho_{*,e}\colon \mathfrak{g}\to\mathfrak{gl}(\mathfrak{q}),\quad x\mapsto\rho'_x\,.$$

Then $(\mathfrak{q},\beta,\rho')$ *is a* $\mathfrak{g}$*-quasi-Frobenius Lie algebra. In particular, if* G *is any Lie subgroup of* $\mathrm{Aut}(\mathfrak{q},\beta)$, *then* $(\mathfrak{q},\beta)$ *admits the structure of a* $\mathfrak{g}$*-quasi-Frobenius Lie algebra.*

Proof. Since ρ is a Lie group homomorphism, it immediately follows that $\rho'\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{q})$ is a representation of $\mathfrak{g}$ on $\mathfrak{q}$. We now show that

$$\rho_x([u,v])=[\rho_x(u),v]+[u,\rho_x(v)] \tag{3.1}$$

and

$$\beta(\rho_x(u),v)+\beta\big(u,\rho_x(v)\big)=0 \tag{3.2}$$

for all $x\in\mathfrak{g}$ and $u,v\in\mathfrak{q}$. To do this, fix a basis $e_1,e_2,\dots,e_n$ on $\mathfrak{q}$. Since $\rho_{\exp(tx)}(u),\ \rho_{\exp(tx)}(v)\in\mathfrak{q}$, we have

$$\rho_{\exp(tx)}(u)=\sum_i a_i(t)e_i,\qquad \rho_{\exp(tx)}(v)=\sum_i b_i(t)e_i \tag{3.3}$$

for some smooth functions $a_i(t)$, $b_i(t)$, $i=1,\dots,n$. Hence,

$$\rho'_x(u)=\sum_i \dot a_i(0)e_i\,,\qquad \rho'_x(v)=\sum_i \dot b_i(0)e_i\,. \tag{3.4}$$

Since $\rho_g\in\mathrm{Aut}(\mathfrak{q},\beta)$ for all $g\in G$, we have

$$\rho_{\exp(tx)}([u,v])=[\rho_{\exp(tx)}(u),\rho_{\exp(tx)}(v)]\,. \tag{3.5}$$

Substituting (3.3) into the right side of (3.5) and applying $\frac{d}{dt}\,|_{t=0}$ to both sides of (3.5) gives

$$\begin{aligned}\rho'_x([u,v]) &= \frac{d}{dt}\Big|_{t=0}[\rho_{\exp(tx)}(u),\rho_{\exp(tx)}(v)]\\ &= \frac{d}{dt}\Big|_{t=0}\sum_{i,j} a_i(t)b_j(t)[e_i,e_j]\\ &= \sum_{i,j}(\dot{a}_i(0)b_j(0)[e_i,e_j]+a_i(0)\dot{b}_j(0)[e_i,e_j])\\ &= [\rho'_x(u),v]+[u,\rho'_x(v)]\,, \end{aligned} \tag{3.6}$$

which proves (3.1).

For equation (3.2), note that

$$\beta(\rho_{\exp(tx)}(u),\rho_{\exp(tx)}(v))=\beta(u,v) \tag{3.7}$$

since $\rho_g\in\mathrm{Aut}(\mathfrak{q},\beta)$ for all $g\in G$. Substituting (3.3) into the left side of (3.7) and applying $\frac{d}{dt}\big|_{t=0}$ to both sides of (3.7) gives

$$\beta(\rho'_x(u),v)+\beta\big(u,\rho'_x(v)\big)=0\,.$$

This completes the proof. □

A trivial example of a $\mathfrak{g}$-quasi-Frobenius Lie algebra is obtained by equipping any quasi-Frobenius Lie algebra with the trivial $\mathfrak{g}$-action. We now consider a more interesting example which is an application of Proposition 3.3.

Example 3.4. Let $\mathfrak{q}$ be the 4-dimensional Lie algebra $\{e_1,e_2,e_3,e_4\}$ whose non-zero commutator relations are given by [6]:

$$[e_1,e_2]=e_2\,,\quad [e_1,e_3]=e_3\,,\quad [e_1,e_4]=2e_4\,,\quad [e_2,e_3]=e_4\,.$$

Let $\alpha\colon\mathfrak{q}\to\mathbb{R}$ be the linear map defined by $\alpha(e_i)=0$ for $i=1,2,3$ and $\alpha(e_4)=1$. Define $\beta(u,v):=\alpha([u,v])$ for all $u,v\in\mathfrak{q}$. Then the matrix representation of β with respect to the basis $\{e_1,e_2,e_3,e_4\}$ is

$$(\beta_{ij})=\begin{pmatrix}0&0&0&2\\0&0&1&0\\0&-1&0&0\\-2&0&0&0\end{pmatrix}.$$

Hence, β is nondegenerate which shows that $(\mathfrak{q},\alpha)$ is a Frobenius Lie algbera. Let G be the set of linear isomorphisms on $\mathfrak{q}$ whose matrix representations with respect to $\{e_1,e_2,e_3,e_4\}$ is given by

$$\left\{\begin{pmatrix}1&0&0&0\\0&b&c&0\\0&0&1/b&0\\a&0&0&1\end{pmatrix}\;\Bigg|\; a,c\in\mathbb{R},\ b>0\right\}. \tag{3.8}$$

A direct calculation shows that G is a 3-dimensional non-abelian, connected Lie subgroup of $Aut(\mathfrak{q},\beta)$. Let $\rho: G\to\mathrm{Aut}(\mathfrak{q},\beta)\subset GL(\mathfrak{q})$ be the inclusion map (which is clearly a Lie group homomorphism). Proposition 3.3 implies that $(\mathfrak{q}, \beta, \rho')$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra, where $\rho' := \rho_{*,e}$. As a Lie algebra, $\mathfrak{g}$ has basis

(3.9)
$$x_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad x_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x_3 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where we have identified G with its matrix represenations in (3.8). The non-zero commutator relations are

$$[x_2, x_3] = 2x_3.$$

Let $a = a_1x_1 + a_2x_2 + a_3x_3 \in \mathfrak{g}$. Since $\rho\colon G \to \mathrm{Aut}(\mathfrak{q}, \beta) \subset GL(\mathfrak{q})$ is just the inclusion map, it follows that the matrix representation of $\rho'_a\colon \mathfrak{q} \to \mathfrak{q}$ with respect to the basis $\{e_1, e_2, e_3, e_4\}$ is simply

$$\rho'_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 \\ 0 & 0 & -a_2 & 0 \\ a_1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.10}$$

Since $(\mathfrak{q}, \beta, \rho')$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra by Proposition 3.3, ρ'_a acts on $\mathfrak{q}$ via derivations and satisfies

$$\beta(\rho'_a(u), v) + \beta\big(u, \rho'_a(v)\big) = 0$$

for all $u, v \in \mathfrak{q}$.

For later use, we conclude this section with the following natural definition:

Definition 3.5. Let $(\mathfrak{q}, \beta, \phi)$ and $(\mathfrak{r}, \sigma, \mu)$ be $\mathfrak{g}$-quasi-Frobenius Lie algebras. A homomorphism from $(\mathfrak{q}, \beta, \phi)$ to $(\mathfrak{r}, \sigma, \mu)$ is a homomorphism

$$\psi\colon (\mathfrak{q}, \beta) \to (\mathfrak{r}, \sigma)$$

of quasi-Frobenius Lie algebras which is also $\mathfrak{g}$-equivariant, that is,

$$\psi \circ \phi_x = \mu_x \circ \psi$$

for all $x \in \mathfrak{g}$.

3.1. **Categorical Formulation.** In this section, we apply the idea of categorification to quasi-Frobenius Lie algebras. The upshot of this is the notion of a *quasi-Frobenius Lie object*, which can be viewed as the analogue of a Frobenius object in the current setting. The starting point for this particular step is the categorification of Lie algebra due to Goyvaerts and Vercruysse [12]:

Definition 3.6. A *Lie object* in an additive symmetric monoidal category $(\mathcal{C}, \otimes, I, \Phi, l, r, c)$ is a pair (L, b) where L is an object of $\mathcal{C}$ and $b\colon L \otimes L \to L$ is a morphism such that

(i) $b + b \circ c = 0_{L\otimes L, L}$,
(ii) $b \circ (\mathrm{id}_L \otimes b) \circ (\mathrm{id}_{L\otimes(L\otimes L)} + c_{L\otimes L, L} \circ \Phi^{-1}_{L,L,L} + \Phi_{L,L,L} \circ c_{L, L\otimes L}) = 0_{L\otimes(L\otimes L), L}$.

Remark 3.7. With regard to the notation in Definition 3.6, $\otimes$ is the monoidal product; I is the unit object; Φ is the associator; l and r are the left and right unit maps respectively; and c is the braiding.

Example 3.8. Let $\mathbf{Vect}_k$ be the symmetric monoidal additive category of finite dimensional vector spaces over k. It follows readily from Definition 3.6 that a Lie object (L, b) in $\mathbf{Vect}_k$ is precisely a finite dimensional Lie algebra L over k with Lie bracket $[x, y] := b(x, y)$.

Definition 3.9. A *quasi-Frobenius Lie object* in an additive symmetric monoidal category $(\mathcal{C}, \otimes, I, \Phi, l, r, c)$ is a triple $(L, b, \overline{\beta})$ such that

(1) (L, b) is a Lie object.

(2) L has a left dual object L^* (where $\varepsilon\colon L^* \otimes L \to I$ and $\eta\colon I \to L \otimes L^*$ denote the evaluation and coevaluation morphisms respectively).

(3) $\overline{\beta}\colon L \xrightarrow{\sim} L^*$ is an isomorphism such that the induced morphism

$$\beta := \varepsilon \circ (\overline{\beta} \otimes \mathrm{id}_L)\colon L \otimes L \to I\,,$$

satisfies

$$\beta + \beta \circ c_{L,L} = 0_{L\otimes L, I}$$

and

$$\beta \circ (b \otimes id_L) \circ [\mathrm{id}_{(L\otimes L)\otimes L} + \Phi^{-1}_{L,L,L} \circ c_{L\otimes L,L} + c_{L,L\otimes L} \circ \Phi_{L,L,L}] = 0_{(L\otimes L)\otimes L,I}\,.$$

If there exists a morphism $\alpha\colon L \to I$ such that $\beta = \alpha \circ b$, then $(L, b, \overline{\beta})$ is called a *Frobenius Lie object.*

Example 3.10. Let $(L, b, \overline{\beta})$ be a quasi-Frobenius Lie object in $\mathbf{Vect}_k$. Then its easy to see that L is a quasi-Frobenius Lie algebra over k with Lie bracket $[x, y] := b(x, y)$ and $\beta\colon L \otimes L \to k$ (as defined in (3) of Definition 3.9) is the nondegenerate 2-cocycle in the Lie algebra cohomology of L. Likewise, a Frobenius Lie object in $\mathbf{Vect}_k$ is just a Frobenius Lie algebra.

Proposition 3.11. *The category* $\mathbf{Rep}(\mathfrak{g})$ *of finite dimensional left* $\mathfrak{g}$*-modules over* k *is an additive symmetric monoidal category where every object has a left dual and*

(i) *the monoidal product is the usual tensor product of left* $\mathfrak{g}$*-modules and* $\mathfrak{g}$*-linear maps;*

(ii) *the unit object is* k *with the trivial* $\mathfrak{g}$*-action;*

(iii) *the associator* Φ *is the trivial one;*

(iv) *for any object* (V, ρ) *in* $\mathbf{Rep}(\mathfrak{g})$*, the left and right morphisms* $l_V : k\otimes V \xrightarrow{\sim} V$ *and* $r_V : V \otimes k \xrightarrow{\sim} V$ *are the trivial ones;*

(v) *for objects* (V, ρ)*,* (W, ϕ) *in* $\mathbf{Rep}(\mathfrak{g})$*, the braiding* $c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ *is simply the linear map that sends* $v \otimes w \in V \otimes W$ *to* $w \otimes v \in W \otimes V$*;*

(vi) *the left dual of an object* (V, ρ) *in* $\mathbf{Rep}(\mathfrak{g})$ *is the dual representation* (V^*, ρ^*) *(i.e.,* $\rho^*_x := -\rho^t_x\colon V^* \to V^*$ *for* $x \in \mathfrak{g}$*, where* ρ^t_x *is the dual or transpose of* $\rho_x : V \to V$*);*

(vii) *the evaluation morphism is* $\varepsilon\colon V^* \otimes V \to k$, $\varepsilon(\xi, v) := \xi(v)$ *and the coevaluation morphism is* $\eta\colon k \to V \otimes V^*$, $1 \mapsto \sum_i e_i \otimes \delta^i$ *where* e_i *is any basis of* V *and* δ^i *is the corresponding dual basis.*

Proof. It is an easy exercise to verify that $(\mathbf{Rep}(\mathfrak{g}), \otimes, k, \Phi, l, r, c)$ satisfies all the axioms of an additive symmetric monoidal category. $\square$

The next result establishes the categorical formulation of $\mathfrak{g}$-quasi-Frobenius Lie algberas.

Proposition 3.12. *A quasi-Frobenius Lie object in* $\mathbf{Rep}(\mathfrak{g})$ *is a* $\mathfrak{g}$*-quasi-Frobenius Lie algebra.*

Proof. By definition, a quasi-Frobenius Lie object in $\mathbf{Rep}(\mathfrak{g})$ consists of a representation $(\mathfrak{q}, \rho)$ of $\mathfrak{g}$ together with $\mathfrak{g}$-linear maps

$$b : \mathfrak{q} \otimes \mathfrak{q} \to \mathfrak{q}\,, \qquad \overline{\beta} : \mathfrak{q} \xrightarrow{\sim} \mathfrak{q}^*\,,$$

which satisfy conditions (1) and (3) of Definition 3.9.

We begin by verifying that $(\mathfrak{q}, \beta)$ is a quasi-Frobenius Lie algebra. To start, note that condition (1) of Definition 3.9 implies that $\mathfrak{q}$ is a Lie algebra with Lie bracket $[u, v] := b(u, v)$. From Definition 3.9, the morphism $\beta\colon \mathfrak{q} \otimes \mathfrak{q} \to k$ is given explicitly as

$$\beta(u, v) = \varepsilon(\overline{\beta}(u), v) = \overline{\beta}(u)(v)\,.$$

Condition (3) of Definition 3.9 implies that β is a 2-cocycle of $\mathfrak{q}$ with values in k (where $\mathfrak{q}$ acts trivially on k). Furthermore, since $\overline{\beta}\colon \mathfrak{q} \xrightarrow{\sim} \mathfrak{q}^*$ is an isomorphism, it follows that β is nondegenerate. Hence, $(\mathfrak{q}, \beta)$ is a quasi-Frobenius Lie algebra.

Since $\overline{\beta}$ is $\mathfrak{g}$-linear (being a morphism of $\mathbf{Rep}(\mathfrak{g})$), we have

$$\overline{\beta}(\rho_x(u))(v) = \rho_x^*(\overline{\beta}(u))(v) = -\overline{\beta}(u)(\rho_x(v))\,, \quad \forall\, u, v \in \mathfrak{q} \tag{3.11}$$

where we recall that $\rho_x^* := -\rho_x^t$. Expressing the left and right most sides of (3.11) in terms of β gives

$$\beta(\rho_x(u), v) = -\beta(u, \rho_x(v))\,,$$

which proves the $\mathfrak{g}$-invariance of β, that is, $\beta(\rho_x(u), v) + \beta(u, \rho_x(v)) = 0$.

Since b is also $\mathfrak{g}$-linear, we also have

$$\begin{aligned}
\rho_x([u, v]) &= \rho_x(b(u \otimes v)) \\
&= b(\overline{\rho}_x(u \otimes v)) \\
&= b(\rho_x(u) \otimes v) + b(u \otimes \rho_x(v)) \\
&= [\rho_x(u), v] + [u, \rho_x(v)]\,,
\end{aligned}$$

where $\overline{\rho}_x$ in the second equality denotes the induced left $\mathfrak{g}$-module structure on $\mathfrak{q} \otimes \mathfrak{q}$. Hence, $(\mathfrak{q}, \beta, \rho)$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra. $\square$

4. The Geometry of $\mathfrak{g}$-quasi-Frobenius Lie algebras

4.1. G-Symplectic Lie groups.

Definition 4.1. Let G be a Lie group. A *G-symplectic Lie group* is a triple (Q, ω, φ) where (Q, ω) is a symplectic Lie group and

$$\varphi \colon G \times Q \to Q\,, \quad (g, q) \mapsto \varphi_g(q) := \varphi(g, q)$$

is a smooth left action on Q such that $\varphi_g \colon (Q, \omega) \to (Q, \omega)$ is an isomorphism of symplectic Lie groups.

Notation 4.2. When dealing with multiple Lie groups, we will denote the identity element of each group simply as e as opposed to e_G for G, e_Q for Q, and so on when there is no risk of confusion.

Proposition 4.3. *Let (Q, ω, φ) be a G-symplectic Lie group with action*

$$\varphi \colon G \times Q \to Q\,, \quad (g, q) \mapsto \varphi_g(q) := \varphi(g, q)\,.$$

Define

$$\begin{aligned} \varphi' &\colon G \to GL(\mathfrak{q})\,, \quad g \mapsto \varphi'_g := (\varphi_g)_{*,e} \colon \mathfrak{q} \to \mathfrak{q} \\ \varphi'' &\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{q})\,, \quad\ \ x \mapsto \varphi''_x := (\varphi')_{*,e}(x) \colon \mathfrak{q} \to \mathfrak{q}\,. \end{aligned}$$

Then

(i) *φ' is a representation of G on $\mathfrak{q}$ such that $\varphi'_g \in \mathrm{Aut}(\mathfrak{q}, \omega_e)$ for all $g \in G$.*

(ii) *$(\mathfrak{q}, \omega_e, \varphi'')$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra.*

Proof. Since φ is a left action of G on Q and $\varphi_g(e) = e$ for all $g \in G$, we have

$$\begin{aligned} \varphi'_g \circ \varphi'_h &= (\varphi_g)_{*,e} \circ (\varphi_h)_{*,e} \\ &= (\varphi_g \circ \varphi_h)_{*,e} \\ &= (\varphi_{gh})_{*,e} = \varphi'_{gh}\,. \end{aligned}$$

Hence, φ' is a representation of G on $\mathfrak{q}$. Furthermore, since $\varphi_g \colon Q \to Q$ is both a Lie group isomorphism and a symplectomorphism, it follows that $\varphi'_g \colon \mathfrak{q} \to \mathfrak{q}$ is a Lie algebra isomorphism and

$$\omega_e(u, v) = \big((\varphi_g)^*\omega\big)_e(u, v) = \omega_e\big((\varphi_g)_{*,e}(u), (\varphi_g)_{*,e}(v)\big) = \omega_e\big(\varphi'_g(u), \varphi'_g(v)\big)\,,$$

which shows that $\varphi'_g \in \mathrm{Aut}(\mathfrak{q}, \omega_e)$ for all $g \in G$. This proves (i).

Statement (ii) follows from an application of Proposition 3.3 to the quasi-Frobenius Lie algebra $(\mathfrak{q}, \omega_e)$ with Lie group homomorphism $\varphi' \colon G \to \mathrm{Aut}(\mathfrak{q}, \omega_e) \subset GL(\mathfrak{q})$. This completes the proof. ☐

Remark 4.4. We will refer to $(\mathfrak{q}, \omega_e, \varphi'')$ in Proposition 4.3 as the $\mathfrak{g}$-quasi-Frobenius Lie algebra associated to the G-symplectic Lie group (Q, ω, φ).

The next result provides a means of constructing G-symplectic Lie groups.

Proposition 4.5. *Let (Q, ω) be a simply connected symplectic Lie group, let G be a Lie group, and let $\rho\colon G \to \operatorname{Aut}(\mathfrak{q}, \omega_e)$, $g \mapsto \rho_g$ be a Lie group homomorphism. Then there exists a unique smooth left-G action*

$$\widehat{\rho}\colon G \times Q \to Q\,, \quad (g, q) \mapsto \widehat{\rho}_g(q)\,,$$

such that $(Q, \omega, \widehat{\rho})$ is a G-symplectic Lie group and $(\widehat{\rho}_g)_{*,e} = \rho_g$. In particular, if G is any Lie subgroup of $\operatorname{Aut}(\mathfrak{q}, \omega_e)$ and $G \neq \{e\}$, then (Q, ω) admits the structure of a G-symplectic Lie group with a nontrivial G-action.

Proof. Let $\rho\colon G \to \operatorname{Aut}(\mathfrak{q}, \omega_e)$, $g \mapsto \rho_g$ be a Lie group homomorphism. Since Q is simply connected and $\rho_g \in \operatorname{Aut}(\mathfrak{q}, \omega_e)$ for all $g \in G$, it follows from Proposition 2.17 that there exists a unique homomorphism of symplectic Lie groups

$$\widehat{\rho}_g\colon (Q, \omega) \to (Q, \omega)$$

such that $(\widehat{\rho}_g)_{*,e} = \rho_g$ for all $g \in G$. Furthermore, for g, $h \in G$, we have

$$\begin{aligned}(\widehat{\rho}_g \circ \widehat{\rho}_h)_{*,e} &= (\widehat{\rho}_g)_{*,e} \circ (\widehat{\rho}_h)_{*,e} \\ &= \rho_g \circ \rho_h = \rho_{gh} = (\widehat{\rho}_{gh})_{*,e}\,.\end{aligned} \tag{4.1}$$

Since $\widehat{\rho}_g \circ \widehat{\rho}_h$ and $\widehat{\rho}_{gh}$ are Lie group homomorphisms and Q is connected, equation (4.1) implies that

$$\widehat{\rho}_g \circ \widehat{\rho}_h = \widehat{\rho}_{gh}\,. \tag{4.2}$$

Hence,

$$\widehat{\rho}\colon G \times Q \to Q\,, \qquad (g, q) \mapsto \widehat{\rho}_g(q)$$

is a left (not necessarily smooth) G-action. We now show that $\widehat{\rho}$ is smooth. To do this, set $\widehat{\rho}(g, q) = \widehat{\rho}_g(q)$ for $g \in G$, $q \in Q$ and let U be an open neighborhood of $0 \in \mathfrak{q}$ such that

$$\exp|_U\colon U \xrightarrow{\sim} \exp(U)$$

is a diffeomorphism. The naturality of the exponential map implies that

$$\widehat{\rho}(g, q) = \exp \circ \rho_g \circ (\exp|_U)^{-1}(q)\,, \quad \forall\ (g, q) \in G \times \exp(U)\,. \tag{4.3}$$

Since the right side of (4.3) is smooth on $G \times \exp(U)$, it follows that $\widehat{\rho}\,|_{G\times\exp(U)}$ is also smooth. Now fix an arbitrary element q_0 of Q and define

$$f\colon G \to Q\,, \qquad g \mapsto \widehat{\rho}(g, q_0)\,.$$

We now show that f is smooth. Since Q is connected, $\exp(U)$ generates Q. Hence, there exists $q_{0,1}, \ldots, q_{0,k} \in \exp(U)$ such that

$$q_0 = q_{0,1} q_{0,2} \cdots q_{0,k}\,.$$

Since $\widehat{\rho}_g\colon Q \to Q$ is a Lie group homomorphism for all $g \in G$, we have

$$f(g) := \widehat{\rho}(g, q_0) = \widehat{\rho}(g, q_{0,1})\widehat{\rho}(g, q_{0,2}) \ldots \widehat{\rho}(g, q_{0,k}) \in Q\,. \tag{4.4}$$

Since $(g, q_{0,i}) \in G \times \exp(U)$ for $i = 1, \ldots, k$, it follows that the right side of (4.4) depends smoothly on g. Hence, f is smooth. Now, for all $(g, q) \in G \times (q_0 \exp(U))$, we have

$$\begin{aligned} \widehat{\rho}(g, q) &= \widehat{\rho}(g, q_0 q_0^{-1} q) \\ &= \widehat{\rho}(g, q_0)\widehat{\rho}(g, q_0^{-1} q) \\ &= f(g)[(\widehat{\rho}|_{G\times\exp(U)}) \circ (\mathrm{id}_G \times l_{q_0^{-1}})(g, q)]\,, \end{aligned} \tag{4.5}$$

where $l_{q_0^{-1}} \colon Q \to Q$ is left translation by q_0^{-1}. Since f and $\widehat{\rho}|_{G\times\exp(U)}$ are both smooth, it follows that the right side of (4.5) is smooth on $G \times (q_0 \exp(U))$. Hence, $\widehat{\rho}\,|_{G\times(q_0\exp(U))}$ is smooth. Since $q_0 \in Q$ is arbitrary, it follows that $\widehat{\rho}$ is smooth on $G \times Q$. This completes the proof. $\square$

We now illustrate Proposition 4.5 with a simple example:

Example 4.6. Let Q be the 2-dimensional non-abelian Lie group

$$Q = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0,\ b \in \mathbb{R} \right\}. \tag{4.6}$$

Note that Q is simply connected, being diffeomorphic to $\mathbb{R}_+ \times \mathbb{R}$. The associated Lie algebra is

$$\mathfrak{q} = \left\{ \begin{pmatrix} \overline{a} & \overline{b} \\ 0 & 0 \end{pmatrix} \mid \overline{a},\ \overline{b} \in \mathbb{R} \right\}. \tag{4.7}$$

A convenient basis for $\mathfrak{q}$ is then

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.8}$$

where we note that

$$[e_1, e_2] = e_2\,. \tag{4.9}$$

Let $\alpha \colon \mathfrak{q} \to \mathbb{R}$ be the linear map defined by $\alpha(e_1) = 0$ and $\alpha(e_2) = 1$. Then $(\mathfrak{q}, \alpha)$ is a Frobenius Lie algebra. Let $\widetilde{\beta}$ be the left-invariant symplectic form on Q defined by $\widetilde{\beta}_e = \beta$, where $\beta(u, v) := \alpha([u, v])$ for $u,\ v \in \mathfrak{q}$.

For $\lambda \in \mathbb{R}$, let $\rho_\lambda \colon \mathfrak{q} \to \mathfrak{q}$ be the linear isomorphism defined by

$$\rho_\lambda(e_1) := e_1 + \lambda e_2\,, \quad \rho_\lambda(e_2) := e_2\,.$$

Then it is a straightforward exercise to show that $\rho_\lambda \in \mathrm{Aut}(\mathfrak{q}, \omega_e)$ and

$$\rho \colon \mathbb{R} \xrightarrow{\sim} \mathrm{Aut}(\mathfrak{q}, \omega_e)\,, \quad \lambda \mapsto \rho_\lambda$$

is a Lie group isomorphism. Proposition 4.5 implies that (Q, ω) admits the structure of an $\mathbb{R}$-symplectic Lie group with unique action $\widehat{\rho} \colon \mathbb{R} \times Q \to Q$ satisfying $(\widehat{\rho}_\lambda)_{*,e} = \rho_\lambda$.

We now compute the action $\widehat{\rho}$ explicitly. Let $u \in \mathfrak{q}$. Then

$$u = \overline{a}e_1 + \overline{b}e_2 = \begin{pmatrix} \overline{a} & \overline{b} \\ 0 & 0 \end{pmatrix} \tag{4.10}$$

for some a, $b \in \mathbb{R}$. Using the naturality of the exponential map, we have

$$\widehat{\rho}_\lambda \circ \exp(u) = \exp \circ \rho_\lambda(u)\,. \tag{4.11}$$

A direct calculation shows that

$$\exp(u) = \begin{pmatrix} e^{\overline{a}} & \mu(\overline{a})\overline{b} \\ 0 & 1 \end{pmatrix}, \tag{4.12}$$

where $\mu\colon \mathbb{R} \to \mathbb{R}_+$ is the nonzero smooth function given by $\mu(t) = \frac{1}{t}(e^t - 1)$ for $t \neq 0$ and $\mu(0) = 1$. Note that every element of Q is in the image of the exponential map. Indeed, given

$$q = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

for $a > 0$, $b \in \mathbb{R}$, one simply sets $\overline{a} = \ln a$ and $\overline{b} = b/\mu(\ln a)$ in (4.12) to obtain $\exp(u) = q$. The left side of (4.11) is

$$\exp \circ \rho_\lambda(u) = \exp \begin{pmatrix} \overline{a} & \lambda\overline{a} + \overline{b} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{\overline{a}} & \mu(\overline{a})(\lambda\overline{a} + \overline{b}) \\ 0 & 1 \end{pmatrix}. \tag{4.13}$$

Hence,

$$\widehat{\rho}_\lambda \begin{pmatrix} e^{\overline{a}} & \mu(\overline{a})\overline{b} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\overline{a}} & \mu(\overline{a})(\lambda\overline{a} + \overline{b}) \\ 0 & 1 \end{pmatrix}. \tag{4.14}$$

Setting $\overline{a} = \ln a$ and $\overline{b} = b/\mu(\ln a)$ for $a > 0$ and $b \in \mathbb{R}$, we obtain

$$\widehat{\rho}_\lambda \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & \lambda(a-1) + b \\ 0 & 1 \end{pmatrix}. \tag{4.15}$$

Since (Q, ω, φ) is an $\mathbb{R}$-symplectic Lie group by Proposition 4.5, $\widehat{\rho}_\lambda$ is both a Lie group isomorphism and a symplectomorphism of (Q, ω) which satisfies $(\widehat{\rho}_\lambda)_{*,e} = \rho_\lambda$.

In anticipation of the next section, we introduce the following definition:

Definition 4.7. Let (Q, ω, φ) and (R, τ, χ) be G-symplectic Lie groups. A homomorphism of G-symplectic Lie groups from (Q, ω, φ) to (R, τ, χ) is a homomorphism

$$\Psi\colon (Q, \omega) \to (R, \tau)$$

of symplectic Lie groups which is also G-equivariant, that is, $\Psi(\varphi_g(q)) = \chi_g(\Psi(q))$ for all $g \in G$ and $q \in Q$.

4.2. **The Equivalence.** In this section, we show that the category of finite dimensional $\mathfrak{g}$-quasi-Frobenius Lie algebras is equivalent to the category of simply connected G-symplectic Lie groups, where G is also simply connected. We begin with the following result.

Proposition 4.8. *Let $\Psi\colon (Q, \omega, \varphi) \to (R, \tau, \chi)$ be a homomorphism of G-symplectic Lie groups. Then*

$$\Psi_{*,e}\colon (\mathfrak{q}, \omega_e, \varphi'') \to (\mathfrak{r}, \tau_e, \chi'')$$

is a homomorphism of $\mathfrak{g}$-quasi-Frobenius Lie algebras, where φ'' and χ'' are defined as in Proposition 4.3.

Proof. By definition, $\Psi\colon (Q,\omega) \to (R,\tau)$ is a homomorphism of symplectic Lie groups. This implies that

$$\Psi_{*,e}\colon (\mathfrak{q},\omega_e) \to (\mathfrak{r},\tau_e)$$

is a homomorphism of quasi-Frobenius Lie algebras. It only remains to show that $\Psi_{*,e}$ is $\mathfrak{g}$-equivariant. Since Ψ is G-equivariant, we have

$$\Psi \circ \varphi_g = \chi_g \circ \Psi\,, \quad \forall\ g \in G\,.$$

This in turn implies that

$$\Psi_{*,e} \circ \varphi'_g = \chi'_g \circ \Psi_{*,e}\,, \quad \forall\ g \in G\,, \tag{4.16}$$

where $\varphi'_g := (\varphi_g)_{*,e}\colon \mathfrak{q} \to \mathfrak{q}$ and $\chi'_g := (\chi_g)_{*,e}\colon \mathfrak{r} \to \mathfrak{r}$. Let $x \in \mathfrak{g}$ and set $g = \exp(tx)$ in (4.17). Applying $\frac{d}{dt}\,|_{t=0}$ to both sides then gives

$$\Psi_{*,e} \circ \varphi''_x = \chi''_x \circ \Psi_{*,e}\,. \tag{4.17}$$

This in turn completes the proof. □

Lemma 4.9. *Let $(\mathfrak{q},\beta,\phi)$ be a $\mathfrak{g}$-quasi-Frobenius Lie algebra and let G be the simply connected Lie group whose Lie algebra is $\mathfrak{g}$. Then there exists a unique Lie group homomorphism $f\colon G \to GL(\mathfrak{q})$, $g \mapsto f_g$ such that $f_{*,e} = \phi$ and $f_g \in \mathrm{Aut}(\mathfrak{q},\beta)$ for all $g \in G$.*

Proof. Since G is simply connected and $\phi\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{q})$ is a Lie algebra map, there exists a unique Lie group homomorphism $f\colon G \to GL(\mathfrak{q})$ such that $f_{*,e} = \phi$. We now show that $f_g \in \mathrm{Aut}(\mathfrak{q},\beta)$ for all $g \in G$. Fix $x \in \mathfrak{g}$. To simplify notation, let

$$f_t := f_{\exp(tx)}\colon \mathfrak{q} \to \mathfrak{q}\,. \tag{4.18}$$

Define $A\colon \mathbb{R} \times \mathfrak{q} \times \mathfrak{q} \to \mathbb{R}$ by

$$A(t,u,v) := \beta(f_t(u), f_t(v)) - \beta(u,v)\,. \tag{4.19}$$

Since $f_{*,e} = \phi$ and $(\mathfrak{q},\beta,\phi)$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra, we have

$$\frac{d}{dt}\,|_{t=0}\ A(t,u,v) = \beta(\phi_x(u),v) + \beta(u,\phi_x(v)) = 0\,, \quad \forall\ u,v \in \mathfrak{q}\,. \tag{4.20}$$

Furthermore, since f is a group homomorphism and

$$\exp((t+s)x) = \exp(tx)\exp(sx)\,,$$

we have

$$A(t+s,u,v) = A(t,f_s(u),f_s(v)) + A(s,u,v)\,, \quad \forall\ u,v \in \mathfrak{q}\,. \tag{4.21}$$

Equations (4.20) and (4.21) imply

$$\frac{d}{dt}\,|_{t=s}\ A(t,u,v) = \frac{d}{dt}\,|_{t=0}\ A(t+s,u,v) = 0 + 0 = 0\,. \tag{4.22}$$

Hence, for fixed u, $v \in \mathfrak{q}$, $A(t,u,v)$ is a constant. Since $A(0,u,v) = 0$, it follows that $A(t,u,v) = 0$ for all $t \in \mathbb{R}$. Hence,

$$\beta\big(f_t(u), f_t(v)\big) = \beta(u,v)\,, \quad \forall\ t \in \mathbb{R}\,. \tag{4.23}$$

In particular,

$$\beta\big(f_{\exp(x)}(u), f_{\exp(x)}(v)\big) = \beta(u,v)\,. \tag{4.24}$$

Now define $B\colon \mathbb{R}\times\mathfrak{q}\times\mathfrak{q}\times\mathfrak{q}\to\mathfrak{q}$ by

$$B(t,u,v,w)=\beta\big([f_t(u),f_t(v)]-f_t([u,v]),f_t(w)\big)\,. \tag{4.25}$$

Equation (4.23) implies that

$$B(t,u,v,w)=\beta\big([f_t(u),f_t(v)],f_t(w)\big)-\beta([u,v],w)\,. \tag{4.26}$$

Using (4.26) and the fact that $(\mathfrak{q},\beta,\phi)$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra, we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}B(t,u,v,w)&=\beta([\phi_x(u),v],w)+\beta([u,\phi_x(v)],w)+\beta\big([u,v],\phi_x(w)\big)\\&=\beta(\phi_x([u,v]),w)+\beta\big([u,v],\phi_x(w)\big)\\&=0\,,\quad\forall\, u,v,w\,.\end{aligned} \tag{4.27}$$

From (4.26), we also have

$$\begin{aligned}B(t+s,u,v,w)&=\beta\big([f_t(f_s(u)),f_t(f_s(v))],f_t(f_s(w))\big)-\beta([u,v],w)\\&=B\big(t,f_s(u),f_s(v),f_s(w)\big)+\beta\big([f_s(u),f_s(v)],f_s(w)\big)-\beta([u,v],w)\end{aligned} \tag{4.28}$$

Equations (4.27) and (4.28) now imply

$$\frac{d}{dt}\,\Big|_{t=s}\,B(t,u,v,w)=\frac{d}{dt}\,\Big|_{t=0}\,B(t+s,u,v,w)=0+0+0=0\,. \tag{4.29}$$

From (4.29), it follows that for fixed u, v, w, $B(t,u,v,w)$ is a constant for all $t\in\mathbb{R}$. Hence, $B(t,u,v,w)=B(0,u,v,w)=0$ for all $t\in\mathbb{R}$ and $u,v,w\in\mathfrak{q}$. In particular,

$$B(1,u,v,w)=\beta\big([f_1(u),f_1(v)]-f_1([u,v]),f_1(w)\big)=0\,,\quad\forall\, u,v,w\in\mathfrak{q}\,. \tag{4.30}$$

Since β is non-degenerate and $f_1:=f_{\exp(x)}\in GL(\mathfrak{q})$, it follows that

$$f_{\exp(x)}([u,v])=[f_{\exp(x)}(u),f_{\exp(x)}(v)]\,. \tag{4.31}$$

Since G is connected, $x\in\mathfrak{g}$ is arbitrary, and f is a group homomorphism, equations (4.24) and (4.31) imply that

$$\beta\big(f_g(u),f_g(v)\big)=\beta(u,v)\,,\quad f_g([u,v])=[f_g(u),f_g(v)] \tag{4.32}$$

for all $g\in G$. Hence, $f_g\in\operatorname{Aut}(\mathfrak{q},\beta)$ for all $g\in G$. This completes the proof. ☐

Proposition 4.10. *Let $(\mathfrak{q},\beta,\phi)$ be a $\mathfrak{g}$-quasi-Frobenius Lie algebra. Let G and Q be the simply connected Lie groups associated to $\mathfrak{g}$ and $\mathfrak{q}$ respectively and let $\widetilde{\beta}\in\Omega^2(Q)$ be the left-invariant 2-form associated to β. Then there exists a unique left action $\overline{\phi}\colon G\times Q\to Q$ such that $(Q,\widetilde{\beta},\overline{\phi})$ is a G-symplectic Lie group whose associated $\mathfrak{g}$-quasi-Frobenius Lie algebra is*

$$(\mathfrak{q},\widetilde{\beta}_e,\overline{\phi}'')=(\mathfrak{q},\beta,\phi)\,,$$

where $\overline{\phi}''$ is defined as in Proposition 4.3.

Proof. By Proposition 2.13, $(Q,\widetilde{\beta})$ is a symplectic Lie group. Since G is simply connected, Lemma 4.9 shows that there exists a unique Lie group homomorphism

$$f\colon G\to GL(\mathfrak{q})\,,\quad g\mapsto f_g$$

such that $f_{*,e} = \phi\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{q})$ and $f_g \in \mathrm{Aut}(\mathfrak{q}, \beta)$ for all $g \in G$. Since Q is simply connected, Proposition 4.5 shows that there exists a unique smooth left G-action

$$\overline{\phi}\colon G \times Q \to Q\,, \qquad (g, q) \mapsto \overline{\phi}_g(q)$$

such that $(Q, \widetilde{\beta}, \overline{\phi})$ is a G-symplectic Lie group and $(\overline{\phi}_g)_{*,e} = f_g$. Setting $\overline{\phi}'_g := (\overline{\phi}_g)_{*,e}$ as in Proposition 4.3, we have

$$\overline{\phi}'' := \overline{\phi}'_{*,e} = f_{*,e} = \phi\,.$$

This completes the proof. $\square$

Proposition 4.11. *Let $\psi\colon (\mathfrak{q}, \beta, \phi) \to (\mathfrak{r}, \sigma, \mu)$ be a homomorphism of $\mathfrak{g}$-quasi-Frobenius Lie algebras. Let G be the simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and let $(Q, \widetilde{\beta}, \overline{\phi})$ and $(R, \widetilde{\sigma}, \overline{\mu})$ be the simply connected G-symplectic Lie groups associated to $(\mathfrak{q}, \beta, \phi)$ and $(\mathfrak{r}, \sigma, \mu)$ respectively by Proposition 4.10. Then there exists a unique homomorphism of G-symplectic Lie groups*

$$\widehat{\psi}\colon (Q, \widetilde{\beta}, \overline{\phi}) \to (R, \widetilde{\sigma}, \overline{\mu})$$

such that $\widehat{\psi}_{*,e} = \psi$.

Proof. By Proposition 2.17, there exists a unique homomorphism of symplectic Lie groups $\widehat{\psi}\colon (Q, \widetilde{\beta}) \to (R, \widetilde{\sigma})$ such that $\widehat{\psi}_{*,e} = \psi$. We now verify that $\widehat{\psi}$ is G-equivariant.

Let $\overline{\phi}'\colon G \to \mathrm{Aut}(\mathfrak{q}, \beta)$, $g \mapsto \overline{\phi}'_g$ and $\overline{\mu}'\colon G \to \mathrm{Aut}(\mathfrak{r}, \sigma)$, $g \mapsto \overline{\mu}'_g$ be defined as in Proposition 4.3. Fix $x \in \mathfrak{g}$. To simplify notation, let

$$\overline{\phi}'_t := \overline{\phi}'_{\exp(tx)}\,, \quad \overline{\mu}'_t := \overline{\mu}'_{\exp(tx)}\,.$$

Define $B\colon \mathbb{R} \times \mathfrak{q} \times \mathfrak{r} \to \mathbb{R}$ by

$$\begin{aligned} B(t, u, v) &:= \sigma\big(\psi \circ \overline{\phi}'_t(u) - \overline{\mu}'_t \circ \psi(u), \overline{\mu}'_t(v)\big) \\ &= \sigma\big(\psi \circ \overline{\phi}'_t(u), \overline{\mu}'_t(v)\big) - \sigma\big(\overline{\mu}'_t \circ \psi(u), \overline{\mu}'_t(v)\big) \\ &= \sigma\big(\psi \circ \overline{\phi}'_t(u), \overline{\mu}'_t(v)\big) - \sigma(\psi(u), v)\,, \end{aligned} \tag{4.33}$$

where the third equality follows from the fact that $\overline{\mu}'_t \in \mathrm{Aut}(\mathfrak{r}, \sigma)$. Hence,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} B(t, u, v) &= \sigma(\psi \circ \phi_x(u), v) + \sigma\big(\psi(u), \mu_x(v)\big) \\ &= \sigma(\mu_x \circ \psi(u), v) + \sigma\big(\psi(u), \mu_x(v)\big) \\ &= 0\,, \quad \forall\, u \in \mathfrak{q},\ v \in \mathfrak{r} \end{aligned} \tag{4.34}$$

where the second equality follows from the fact that ψ is $\mathfrak{g}$-equivariant (i.e., $\psi \circ \phi_x = \mu_x \circ \psi$) and the third equality follows from the fact that $(\mathfrak{r}, \sigma, \mu)$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra with 2-cocycle σ and $\mathfrak{g}$-action μ. Next note that

$$B(t+s, u, v) = B\big(t, \overline{\phi}'_s(u), \overline{\mu}'_s(v)\big) + \sigma\big(\psi(\overline{\phi}'_s(u)), \overline{\mu}'_s(v)\big) - \sigma(\psi(u), v)\,. \tag{4.35}$$

Hence,

$$\frac{d}{dt}\Big|_{t=s} B(t, u, v) = \frac{d}{dt}\Big|_{t=0} B(t+s, u, v) = 0 + 0 - 0 = 0\,, \tag{4.36}$$

where the first zero follows from (4.34). Hence,

$$B(t,u,v)=B(0,u,v)=0\,,\quad \forall t\in\mathbb{R}\,,\ u\in\mathfrak{q}\ ,v\in\mathfrak{r}\,. \tag{4.37}$$

In particular, $B(1,u,v)=0$ for all u, $v\in\mathfrak{r}$. Since σ is nondegenerate and $\overline{\mu}'_t\colon\mathfrak{r}\to\mathfrak{r}$ is also a linear isomorphism for all t, it follows that

$$\psi\circ\overline{\phi}'_t=\overline{\mu}'_t\circ\psi\,,\quad \forall t\in\mathbb{R}\,. \tag{4.38}$$

In particular, we have

$$\psi\circ\overline{\phi}'_{\exp(x)}=\overline{\mu}'_{\exp(x)}\circ\psi\,. \tag{4.39}$$

Since $x\in\mathfrak{g}$ was arbitrary, (4.39) must hold for all $x\in\mathfrak{g}$. Since G is connected, every element $g\in G$ is of the form $g=\exp(x_1)\ldots\exp(x_k)$ for some $x_i\in\mathfrak{g}$, $i=1,\ldots,k$. It follows from this and the fact that $\overline{\phi}'$ and $\overline{\mu}'$ are group homomorphisms that

$$\psi\circ\overline{\phi}'_g=\overline{\mu}'_g\circ\psi\,,\quad \forall\ g\in G\,. \tag{4.40}$$

Equation (4.40) combined with the fact that (1) Q is connected, (2) $\widehat{\psi}\circ\overline{\phi}_g$ and $\overline{\mu}_g\circ\widehat{\psi}$ are both Lie group homomorphisms $\forall\ g\in G$, and (3)

$$(\widehat{\psi}\circ\overline{\phi}_g)_{*,e}=\psi\circ\overline{\phi}'_g=\overline{\mu}'_g\circ\psi=(\overline{\mu}_g\circ\widehat{\psi})_{*,e},\quad \forall\ g\in G \tag{4.41}$$

imply that $\widehat{\psi}\circ\overline{\phi}_g=\overline{\mu}_g\circ\widehat{\psi}$ for all $g\in G$. In other words, $\widehat{\psi}$ is G-equivariant and this completes the proof. □

We conclude the paper with the following generalization of Theorem 2.18.

Theorem 4.12. *Let G be a simply connected Lie group and let G-**SCSLG** be the category of simply connected G-symplectic Lie groups and let $\mathfrak{g}$-**qFLA** be the category of finite dimensional $\mathfrak{g}$-quasi-Frobenius Lie algebras. Let $\widehat{F}$ be the functor from G-**SCSLG** to $\mathfrak{g}$-**qFLA** which sends the object (Q,ω,φ) to $(\mathfrak{q},\omega_e,\varphi'')$, where φ'' is defined as in Proposition 4.3 and the morphism $\Psi\colon(Q,\omega,\varphi)\to(R,\tau,\chi)$ to*

$$\Psi_{*,e}\colon(\mathfrak{q},\omega_e,\varphi'')\mapsto(\mathfrak{r},\tau_e,\chi'')\,.$$

Then $\widehat{F}$ is an equivalence of categories.

Proof. Theorem 4.12 follows from Theorem 2.18, Proposition 4.3, Proposition 4.8, Proposition 4.10, and Proposition 4.11. □

5. $D(\mathfrak{g})$-quasi-Frobenius Lie algberas

Let $(\mathfrak{g},\gamma)$ be a finite dimensional Lie bialgebra. We begin with the following observation:

Proposition 5.1. *Let V be a vector space over k and let $\rho\colon D(\mathfrak{g})\to\mathfrak{gl}(V)$ be a linear map (not necessarily a representation). Define*

$$\varphi:=\rho\,|_{\mathfrak{g}}\colon\mathfrak{g}\to\mathfrak{gl}(V),\qquad \psi:=\rho\,|_{\mathfrak{g}^*}\colon\mathfrak{g}^*\to\mathfrak{gl}(V).$$

The following statements are equivalent.

(i) *ρ is a representation of $D(\mathfrak{g})$ on V.*

(ii) *φ and ψ are representations of $\mathfrak{g}$ and $\mathfrak{g}^*$ on V which satisfy*

$$\psi_{ad^*_x\,\xi} - \varphi_{ad^*_\xi\,x} = \varphi_x \circ \psi_\xi - \psi_\xi \circ \varphi_x\,, \quad \forall\, x \in \mathfrak{g}\,,\ \xi \in \mathfrak{g}^*\,. \tag{5.1}$$

Proof. (i)$\Rightarrow$(ii) Since ρ is a representation of $D(\mathfrak{g})$ on V, it follows immediately that φ and ψ must be representations of $\mathfrak{g}$ and $\mathfrak{g}^*$ on V respectively. For (5.1), we note that

$$[x,\xi] = ad^*_x\,\xi - ad^*_\xi x \quad \forall\, x \in \mathfrak{g},\ \xi \in \mathfrak{g}^*\,.$$

Since ρ is a representation and $\varphi := \rho\,|_{\mathfrak{g}}$ and $\psi := \rho\,|_{\mathfrak{g}^*}$, we have

$$\psi_{ad^*_x\,\xi} - \varphi_{ad^*_\xi\,x} = \rho_{[x,\xi]} = \rho_x\rho_\xi - \rho_\xi\rho_x = \varphi_x\psi_\xi - \psi_\xi\varphi_x\,,$$

which proves (5.1).

(i)$\Leftarrow$(ii) Let $a = x + \xi \in D(\mathfrak{g})$. Then

$$\begin{aligned}
\rho_{[x+\xi,y+\eta]} &= \rho_{[x,y]} + \rho_{[x,\eta]} + \rho_{[\xi,y]} + \rho_{[\xi,\eta]} \\
&= \varphi_{[x,y]} + \psi_{ad^*_x\eta} - \varphi_{ad^*_\eta x} + \varphi_{ad^*_\xi y} - \psi_{ad^*_y\xi} + \psi_{[\xi,\eta]} \\
&= \varphi_x \circ \varphi_y - \varphi_y \circ \varphi_x + \varphi_x \circ \psi_\eta - \psi_\eta \circ \varphi_x \\
&\quad + \psi_\xi \circ \varphi_y - \varphi_y \circ \psi_\xi + \psi_\xi \circ \psi_\eta - \psi_\eta \circ \psi_\xi \\
&= (\varphi_x + \psi_\xi) \circ (\varphi_y + \psi_\eta) - (\varphi_y + \psi_\eta) \circ (\varphi_x + \psi_\xi) \\
&= \rho_{x+\xi} \circ \rho_{y+\eta} - \rho_{y+\eta} \circ \rho_{x+\xi}\,.
\end{aligned}$$

This proves that $\rho\colon D(\mathfrak{g}) \to \mathfrak{gl}(V)$ is a representation of $D(\mathfrak{g})$ on V. □

Proposition 5.2. *Let $(\mathfrak{q},\beta)$ be a quasi-Frobenius Lie algebra and let $\rho\colon D(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{q})$ be a linear map (not necessarily a representation). Define $\varphi := \rho|_{\mathfrak{g}}$ and $\psi := \rho|_{\mathfrak{g}^*}$. Then $(\mathfrak{q},\beta,\rho)$ is a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra iff the following conditions are satisfied:*

(a) $\psi_{ad^*_x\xi} - \varphi_{ad^*_\xi x} = \varphi_x \circ \psi_\xi - \psi_\xi \circ \varphi_x, \quad \forall\, x \in \mathfrak{g},\ \xi \in \mathfrak{g}^*$

(b) *$(\mathfrak{q},\beta,\varphi)$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra.*

(c) *$(\mathfrak{q},\beta,\psi)$ is a $\mathfrak{g}^*$-quasi-Frobenius Lie algebra.*

Proof. By Proposition 5.1, ρ is left $D(\mathfrak{g})$-module structure on $\mathfrak{q}$ iff φ and ψ are left $\mathfrak{g}$ and $\mathfrak{g}^*$-module structures on $\mathfrak{q}$ respectively which satisfy condition (a). Since $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ as a vector space, it follows that $\rho\colon D(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{q})$ satisfies conditions (i) and (ii) of Definition 3.1 iff $\phi\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{q})$ and $\psi\colon \mathfrak{g}^* \to \mathfrak{gl}(\mathfrak{q})$ both satisfy conditions (i) and (ii) of Definition 3.1. This completes the proof. □

Proposition 5.3. *Let $\mathfrak{g}$ be a finite dimensional quasitriangular Lie bialgebra with r-matrix $r = \sum_i a_i \otimes b_i$. Let $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(V)$, $x \mapsto \varphi(x)$ be a representation of $\mathfrak{g}$ on V. Define $\psi\colon \mathfrak{g}^* \to \mathfrak{gl}(V)$, $\xi \mapsto \psi(\xi)$ by*

$$\psi(\xi) := \sum_i \xi(a_i)\varphi(b_i)\,, \quad \forall\, \xi \in \mathfrak{g}^*\,. \tag{5.2}$$

Then ψ is a representation of $\mathfrak{g}^*$ on V.

Proof. We need to show that

$$\psi([\xi,\eta]) = \psi(\xi)\psi(\eta) - \psi(\eta)\psi(\xi)\,. \tag{5.3}$$

We now expand the left side of (5.3):

$$\begin{aligned}\psi([\xi,\eta]) &= \sum_j [\xi,\eta](a_j)\varphi(b_j)\\ &= \sum_j (\xi\otimes\eta)((\delta r)(a_j))\varphi(b_j)\\ &= \sum_{i,j}\xi([a_j,a_i])\eta(b_i)\varphi(b_j) + \sum_{i,j}\xi(a_i)\eta([a_j,b_i])\varphi(b_j)\,. \end{aligned}\tag{5.4}$$

The right side of (5.3) expands as

$$\begin{aligned}\psi(\xi)\psi(\eta)-\psi(\eta)\psi(\xi) &= \sum_{i,j}\xi(a_i)\eta(a_j)\varphi(b_i)\varphi(b_j) - \sum_{i,j}\eta(a_j)\xi(a_i)\varphi(b_j)\varphi(b_i)\\ &= \sum_{i,j}\xi(a_i)\eta(a_j)\varphi([b_i,b_j]). \end{aligned}\tag{5.5}$$

The CYBE can be rewritten as

$$\sum_{i,j} a_i\otimes a_j\otimes [b_i,b_j] = \sum_{i,j}[a_j,a_i]\otimes b_i\otimes b_j + \sum_{i,j} a_i\otimes [a_j,b_i]\otimes b_j\,. \tag{5.6}$$

Applying $\xi\otimes\eta\otimes\varphi$ to both sides of (5.6) gives

$$\sum_{i,j}\xi(a_i)\eta(a_j)\varphi([b_i,b_j]) = \sum_{i,j}\xi([a_j,a_i])\eta(b_i)\varphi(b_j) + \sum_{i,j}\xi(a_i)\eta([a_j,b_i])\varphi(b_j)\,. \tag{5.7}$$

Equations (5.4), (5.5), and (5.7) imply

$$\psi(\xi)\psi(\eta)-\psi(\eta)\psi(\xi) = \psi([\xi,\eta])\,.$$

This completes the proof. □

Corollary 5.4. *Let $\mathfrak{g}$ be a finite dimensional quasitriangular Lie bialgebra with r-matrix $r=\sum_i a_i\otimes b_j$ and let $(\mathfrak{q},\beta,\varphi)$ be a $\mathfrak{g}$-quasi-Frobenius Lie algebra. Define $\psi\colon \mathfrak{g}^*\to\mathfrak{gl}(\mathfrak{q})$, $\xi\mapsto\psi(\xi)$ by*

$$\psi(\xi) := \sum_i \xi(a_i)\varphi(b_i)\,,$$

where $\varphi(b_i) := \varphi_{b_i}\colon \mathfrak{q}\to\mathfrak{q}$. Then $(\mathfrak{q},\beta,\psi)$ is a $\mathfrak{g}^*$-quasi-Frobenius Lie algebra.

Proof. Immediate. □

Proposition 5.5. *Let $\mathfrak{g}$ be a finite dimensional quasitriangular Lie bialgebra with r-matrix $r=\sum_i a_i\otimes b_i$. Let $\varphi\colon\mathfrak{g}\to\mathfrak{gl}(V)$, $x\mapsto\varphi(x)$ be a representation of $\mathfrak{g}$ on V. Define $\psi\colon\mathfrak{g}^*\to\mathfrak{gl}(V)$, $\xi\mapsto\psi(\xi)$ according to Proposition 5.3. Define $\rho\colon D(\mathfrak{g})\to\mathfrak{gl}(V)$, $a\mapsto\rho(a)$ by*

$$\rho(x+\xi) := \varphi(x)+\psi(\xi)\,,\quad \forall\, x\in\mathfrak{g},\ \xi\in\mathfrak{g}^*\,. \tag{5.8}$$

Then ρ is a representation of $D(\mathfrak{g})$ on V.

Proof. By Proposition 5.1, it suffices to show that

$$\psi(ad_x^* \xi) - \varphi(ad_\xi^* x) = \varphi(x)\psi(\xi) - \psi(\xi)\varphi(x)\,. \tag{5.9}$$

We begin by expanding the left side of (5.9). First,

$$\begin{aligned} \psi(ad_x^* \xi) &= \sum_i (ad_x^* \xi)(a_i)\varphi(b_i) \\ &= \sum_i \xi([a_i, x])\varphi(b_i)\,. \end{aligned} \tag{5.10}$$

By Proposition 2.25, $\sum_i a_i \otimes b_i + \sum_i b_i \otimes a_i$ is invariant under the adjoint action of $\mathfrak{g}$. Hence,

$$\sum_i [a_i, x] \otimes b_i = \sum_i a_i \otimes [x, b_i] + \sum_i [x, b_i] \otimes a_i + \sum_i b_i \otimes [x, a_i]\,. \tag{5.11}$$

Equations (5.10) and (5.11) now imply

$$\psi(ad_x^* \xi) = \sum_i \xi(a_i)\varphi([x, b_i]) + \sum_i \xi([x, b_i])\varphi(a_i) + \sum_i \xi(b_i)\varphi([x, a_i])\,. \tag{5.12}$$

Next, we note that

$$ad_\xi^* x = \sum_i \xi(b_i)[x, a_i] + \sum_i \xi([x, b_i])a_i\,. \tag{5.13}$$

From (5.12) and (5.13), we have

$$\psi(ad_x^* \xi) - \varphi(ad_\xi^* x) = \sum_i \xi(a_i)\varphi([x, b_i])\,. \tag{5.14}$$

For the right side of (5.9), we have

$$\begin{aligned} \varphi(x)\psi(\xi) - \psi(\xi)\varphi(x) &= \sum_i \xi(a_i)\varphi(x)\varphi(b_i) - \sum_i \xi(a_i)\varphi(b_i)\varphi(x) \\ &= \sum_i \xi(a_i)\varphi([x, b_i]) \\ &= \psi(ad_x^* \xi) - \varphi(ad_\xi^* x)\,, \end{aligned} \tag{5.15}$$

where the last equality follows from (5.14). This completes the proof. $\square$

Theorem 5.6. *Let $\mathfrak{g}$ be a finite dimensional quasitriangular Lie bialgebra. Let $(\mathfrak{q}, \beta, \varphi)$ be any $\mathfrak{g}$-quasi-Frobenius Lie algebra. Then there exists a representation $\rho : D(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{q})$ such that $\rho\mid_{\mathfrak{g}} = \varphi$ and $(\mathfrak{q}, \beta, \rho)$ is a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra.*

Proof. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be the r-matrix associated to $\mathfrak{g}$ and let $\psi\colon \mathfrak{g}^* \to \mathfrak{gl}(\mathfrak{q})$ be the representation of $\mathfrak{g}^*$ on $\mathfrak{q}$ determined by φ and r according to Proposition 5.3. By Corollary 5.4, $(\mathfrak{q}, \beta, \psi)$ is a $\mathfrak{g}^*$-quasi-Frobenius Lie algebra. Define $\rho\colon D(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{q})$ by

$$\rho(x + \xi) := \varphi(x) + \psi(\xi)\,, \quad \forall\, x \in \mathfrak{g},\ \xi \in \mathfrak{g}^*\,.$$

By Proposition 5.5, ρ is a representation of $D(\mathfrak{g})$ on $\mathfrak{q}$. Since $(\mathfrak{q}, \beta, \varphi)$ and $(\mathfrak{q}, \beta, \psi)$ are $\mathfrak{g}$ and $\mathfrak{g}^*$-quasi-Frobenius Lie algebras and $\rho\mid_{\mathfrak{g}} = \varphi$ and $\rho\mid_{\mathfrak{g}^*} = \psi$ (by definition), it follows that $(\mathfrak{q}, \beta, \rho)$ is a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra. $\square$

Corollary 5.7. *Let $\mathfrak{g}$ be any finite dimensional Lie algebra and let $(\mathfrak{q}, \beta, \varphi)$ be any $\mathfrak{g}$-quasi-Frobenius Lie algbera. Let $D(\mathfrak{g})$ be the Drinfeld double of the Lie bialgebra $(\mathfrak{g}, \gamma)$ where $\gamma \equiv 0$. Define $\rho\colon D(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{q})$ by $\rho(x+\xi) = \varphi(x)$ for all $x \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$. Then $(\mathfrak{q}, \beta, \rho)$ is a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra.*

Proof. $(\mathfrak{g}, \gamma)$ is naturally a quasitriangular Lie bialgebra with r-matrix $r \equiv 0 \in \mathfrak{g}\otimes\mathfrak{g}$. Corollary 5.7 now follows as a special case of the proof of Theorem 5.6. $\square$

We conclude the paper with an example.

Example 5.8. Let $(\mathfrak{q}, \beta)$ be the 4-dimensional quasi-Frobenius Lie algebra from Example 3.4. For convenience, we recall its structure: $\mathfrak{q}$ has basis $\{e_1, e_2, e_3, e_4\}$ with non-zero commutator relations given by

$$[e_1, e_2] = e_2\,, \quad [e_1, e_3] = e_3\,, \quad [e_1, e_4] = 2e_4\,, \quad [e_2, e_3] = e_4\,,$$

and the matrix representation of β with respect to $\{e_1, e_2, e_3, e_4\}$ is

$$(\beta_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}.$$

Let $(\mathfrak{g}, \delta r)$ be the 2-dimensional triangular Lie bialgebra from Examples 2.27 and 2.29. Once again, we recall the structure for convenience. $\mathfrak{g}$ has basis $\{x, y\}$ with commutator relation $[x, y] = x$ and r-matrix $r = y \wedge x$. Let $\{x^*, y^*\}$ denote the corresponding dual basis. The commutator relations on $D(\mathfrak{g})$ are

$$\begin{gathered}[x, y] = x\,, \quad [x^*, y^*] = y^*\,, \quad [x, x^*] = -y^*\,, \quad [x, y^*] = 0 \\ [y, x^*] = x^* + y\,, \quad [y, y^*] = -x\,.\end{gathered}$$

Let $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{q})$ be the linear map defined by

$$\varphi_x(e_1) = 0\,, \quad \varphi_x(e_2) = 0\,, \quad \varphi_x(e_3) = e_2\,, \quad \varphi_x(e_4) = 0$$

$$\varphi_y(e_1) = 0\,, \quad \varphi_y(e_2) = -\frac{1}{2}e_2\,, \quad \varphi_y(e_3) = \frac{1}{2}e_3\,, \quad \varphi_y(e_4) = 0\,.$$

Consideration of Example 3.4 (or a direct calculation) shows that $(\mathfrak{q}, \beta, \varphi)$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra. By Theorem 5.6, there exists a representation $\rho\colon D(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{q})$ such that $\rho\,|_{\mathfrak{g}} = \varphi$ and $(\mathfrak{q}, \beta, \rho)$ is a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra. We now compute ρ explicitly. From the proof of Theorem 5.6, this amounts to computing the representation $\psi\colon \mathfrak{g}^* \to \mathfrak{gl}(\mathfrak{q})$ which is determined by φ and $r = y \wedge x$ according to Proposition 5.3:

$$\psi_{x^*} = -\varphi_y\,, \quad \psi_{y^*} = \varphi_x\,.$$

ρ is then uniquely defined by $\rho\,|_{\mathfrak{g}} = \varphi$ and $\rho|_{\mathfrak{g}^*} = \psi$.

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References

[1] Abrams, L., *Two dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications **5** (1996), 569–587.

[2] Agaoka, Y., *Uniqueness of left invariant symplectic structures on the affine Lie group*, Proc. Amer. Math. Soc. **129** (9) (2001), 2753–2762, (electronic).

[3] Atiyah, M.F., *Topological quantum field theory*, Publ. Math. Inst. Hautes Études Sci. **68** (1988), 175–186.

[4] Baues, O., Cortés, V., *Symplectic Lie Groups I–III*, `arXiv:1307.1629`.

[5] Boyom, N., *Models for solvable symplectic Lie groups*, Indiana Univ. Math. J. **42** (4) (1993), 1149–1168.

[6] Burde, D., *Characteristically nilpotent Lie algebras and symplectic structures*, Forum Math. **18** (5) (2006), 769–787.

[7] Chari, V., Pressley, A., *A Guide to Quantum Groups*, Cambridge University Press, 1994.

[8] Chu, B., *Symplectic homogenous spaces*, Trans. Amer. Math. Soc. **197** (1974), 145–159.

[9] Drinfeld, V.G., *Hamiltonian structures on Lie groups, Lie bialgebras, and the geometric meaning of the classical Yang-Baxter equations*, Dokl. Akad. Nauk SSSR **268** (2) (1983), 285–287, (Russian).

[10] Etingof, P., Schiffman, O., *Lectures on Quantum Groups*, Lect. Math. Phys., Int. Press, 1998.

[11] Golubitsky, M., Guillemin, V., *Stable mappings and their singularities*, Grad. Texts in Math., vol. 14, Springer, Berlin, 1973.

[12] Goyvaerts, I., Vercruysse, J., *A Note on the Categorification of Lie Algebras*, Lie Theory and Its Applications in Physics, Springer Proceedings in Math. & Stat., 2013, pp. 541–550.

[13] Guillemin, V., Pollack, A., *Differential Topology*, Prentice-Hall, 1974.

[14] Helgason, S., *Differential Geometry, Lie groups, and Symmetric Spaces*, Pure Appl. Math., 1978.

[15] Kaufmann, R., *Orbifolding Frobenius algebras*, Internat. J. Math. **14** (6) (2003), 573–617.

[16] Kaufmann, R., Pham, D., *The Drinfel'd double and twisting in stringy orbifold theory*, Internat. J. Math. **20** (5) (2009), 623–657.

[17] Kosmann-Schwarzbach, Y., *Poisson-Drinfel'd groups*, Publ. Inst. Rech. Math. Av. **5** (12) (1987).

[18] Kosmann-Schwarzbach, Y., *Lie Bialgebras, Poisson Lie groups and dressing transformations*, Integrability of nonlinear systems (Pondicherry, 1996), vol. 495, Lecture Notes in Phys., 1997, pp. 104–170.

[19] Lee, J., *Introduction to Smooth Manifolds*, Springer-Verlag, New York Inc., 2003.

[20] Lu, J., Weinstein, A., *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. Differential Geom. **31** (2) (1990), 501–526.

[21] Mikami, K., *Symplectic and Poisson structures on some loop groups*, Contemp. Math. **179** (1994), 173–192.

[22] Ooms, A., *On Lie algebras having a primitive universal enveloping algebra*, J. Algebra **32** (1974), 488–500.

[23] Ooms, A., *On Frobenius Lie algebras*, Comm. Algebra **8** (1) (1980), 13–52.

[24] Semenov-Tian-Shansky, M.A., *What is a classical r-matrix?*, Funct. Anal. Appl. **17** (1983), 259–272.

[25] Turaev, V., *Homotopy field theory in dimension 2 and group-algebras*, `arXiv.org:math/9910010`, (1999).

[26] Vinberg, E.B., *A course in algebra*, Graduate Studies in Math., vol. 56, AMS, 2003.

[27] Warner, F., *Foundations of Differentiable Manifolds and Lie Groups*, Springer, 1983.

[28] Witten, E., *Topological quantum field theory*, Comm. Math. Phys. **117** (1988), 353–386.

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