ARCHIVUM MATHEMATICUM (BRNO)
Tomus 52 (2016), 277–289

A CANONICAL CONNECTION ON SUB-RIEMANNIAN CONTACT MANIFOLDS

Michael Eastwood and Katharina Neusser

Abstract. We construct a canonically defined affine connection in sub-Riemannian contact geometry. Our method mimics that of the Levi-Civita connection in Riemannian geometry. We compare it with the Tanaka-Webster connection in the three-dimensional case.

1. Introduction

Let M be a contact manifold with contact distribution H. Necessarily, the dimension of M is odd. The 3-dimensional case is special and we shall often concentrate our discussion on this case. For example, as observed in [7], the notion of a sub-Riemannian structure in this case coincides with Webster's notion of a pseudo-Hermitian structure [11]. From this point of view, there is the well-known Tanaka-Webster connection [10, 11], a canonically defined affine connection on pseudo-Hermitian manifolds in all dimensions but, in particular, on sub-Riemannian manifolds in dimension 3. We shall discuss this connection in detail §5 and compare it with what we constructed earlier in §4. There is yet another natural connection for sub-Riemannian contact structures due to Morimoto [8]. In contrast to the canonical connection we construct in §4, this one requires 'constant symbol,' which is however a vacuous condition in dimension 3.

We admit right away that our aim here is not to discuss 'the equivalence problem' for sub-Riemannian contact structures in the sense of Cartan nor study 'Jacobi curves' in sub-Riemannian geometry in the sense of Agrachev and Zelenko [2]. Instead, our more modest aim is to discuss and construct connections and partial connections (where, in the first instance, one differentiates only in the H-directions) as an invariant calculus in sub-Riemannian contact geometry, closely mimicking the construction of the Levi-Civita connection in the Riemannian setting.

2010 *Mathematics Subject Classification*: primary 53C17; secondary 53D10, 70G45.

Key words and phrases: contact manifold, sub-Riemannian geometry, partial connection, pseudo-Hermitian geometry.

The second author was supported by the grant GA CR P201/12/G028.

Received April 18, 2016, accepted September 20, 2016. Editor M. Čadek.

DOI: 10.5817/AM2016-5-277

2. Generalities on contact manifolds

Let M be a smooth manifold with tangent bundle $TM \to M$ and suppose $H \subset TM$ is a codimension 1 smooth subbundle. Equivalently, and this is our preferred point of view, we are given a smooth line subbundle $L \subset \wedge^1$, where $\wedge^1 = T^*M$ is the bundle of 1-forms on M. Thus, we have dual exact sequences

$$0 \to H \to TM \to L^* \to 0$$

and

$$0 \to L \to \wedge^1 \to \wedge^1_H \to 0\,. \tag{1}$$

2.1. **The Levi form.** The sequence (1) induces a short exact sequence

$$0 \to \wedge^1_H \otimes L \to \wedge^2 \to \wedge^2_H \to 0\,,$$

where $\wedge^2_H = \wedge^2(\wedge^1_H)$ and we may now consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \to & \wedge^1 & \to & \wedge^1_H & \to & 0 \\ & & & & {\scriptstyle d}\downarrow & & & & \\ 0 & \to & \wedge^1_H \otimes L & \to & \wedge^2 & \to & \wedge^2_H & \to & 0, \end{array}$$

where $d : \wedge^1 \to \wedge^2$ is the exterior derivative. From the Leibniz rule, it follows that the composition

$$L \to \wedge^1 \xrightarrow{d} \wedge^2 \to \wedge^2_H$$

is a homomorphism of vector bundles.

Definition 2.1. This composition $\mathcal{L} \in \Gamma(\wedge^2_H \otimes L^*)$ is called the *Levi form* of H. (It is the obstruction to the integrability of H.)

Definition 2.2. If $\mathcal{L}$ is non-degenerate, then (M, H) is said to be a *contact manifold*. (It follows that M is odd-dimensional.)

If $\theta \in \Gamma(L) \subset \Gamma(\wedge^1)$ is nowhere vanishing, non-degeneracy of the Levi form is equivalent to

$$\theta \wedge (d\theta)^n \equiv \theta \wedge \underbrace{d\theta \wedge \cdots \wedge d\theta}_{n} \text{ being nowhere vanishing}\,,$$

where $2n + 1$ is the dimension of M. Of course, locally there is no problem in choosing θ and such a choice is often useful. For global existence, the line bundle L should be orientable.

Definition 2.3. On a smooth manifold of dimension $2n + 1$, a *contact form* is a smooth 1-form θ such that $\theta \wedge (d\theta)^n$ is nowhere vanishing.

Remark 2.1. Some authors (e.g. [1]) define a contact manifold as a smooth manifold equipped with a contact form. In this article, however, a contact form is an extra (local) choice.

2.2. **Partial connections.** On a contact manifold it is natural to consider differentiation in the contact directions, i.e. along H. According to (1), this is equivalent to considering the composition

$$\wedge^0 \xrightarrow{d} \wedge^1 \to \wedge^1_H ,$$

which we denote by $d_H \colon \wedge^0 \to \wedge^1_H$. A consequence of the contact condition is that H is *bracket generating*, namely that repeated Lie brackets of fields from H eventually fill up all of TM (indeed, for a contact manifold, already $[H, H] = TM$ as regards sections of these bundles). A consequence of this is that the kernel of d_H consists of locally constant functions.

Remark 2.2. In [9] it is shown that $d_H \colon \wedge^0 \to \wedge^1_H$, is the first operator in an invariantly defined locally exact complex, known as the *Rumin* complex. It is an effective replacement for the de Rham complex (see also [4]).

Definition 2.4. Suppose M is a smooth contact manifold and V is a smooth vector bundle on M. A *partial connection* on V is a differential operator

$$\nabla_H \colon V \to \wedge^1_H \otimes V \quad \text{s.t. } \nabla_H(f\sigma) = f\nabla_H\sigma + d_Hf \otimes \sigma ,$$

for all $f \in \Gamma(\wedge^0)$ and $\sigma \in \Gamma(E)$.

Remark 2.3. A partial connection determines a differential operator

$$\nabla_H \colon \wedge^1 \otimes V \to \wedge^2_H \otimes V$$

characterised by

$$\nabla_H(\omega \otimes \sigma) = d_H\omega \otimes \sigma - \omega_H \wedge \nabla_H\sigma ,$$

where $d_H \colon \wedge^1 \to \wedge^2_H$ is the composition $\wedge^1 \xrightarrow{d} \wedge^2 \to \wedge^2_H$ and ω_H the image of ω under the projection $\wedge^1 \to \wedge^1_H$.

In fact, a partial connection on any bundle on any contact manifold may be promoted to a full connection as follows. The Levi form $\mathcal{L}\colon L \to \wedge^2_H$ is non-degenerate and so has an invariantly defined pointwise left inverse $\mathcal{L}^{-1} : \wedge^2_H \to L$.

Proposition 2.1. *A partial connection ∇_H on V uniquely determines a connection ∇ on V with the following two properties.*

- *the composition $V \xrightarrow{\nabla} \wedge^1 \otimes V \to \wedge^1_H \otimes V$ agrees with ∇_H,*
- *the composition $V \xrightarrow{\nabla} \wedge^1 \otimes V \xrightarrow{\nabla_H} \wedge^2_H \otimes V \xrightarrow{\mathcal{L}^{-1}\otimes \mathrm{Id}} L \otimes V$ vanishes.*

Proof. See [6, Proposition 3.5]. □

2.3. **Partial torsion.** Suppose $\nabla_H : \wedge^1 \to \wedge^1_H \otimes \wedge^1$ is a partial connection on the cotangent bundle. Then we have two linear differential operators between the same bundles, namely

(2)
- $d_H \colon \wedge^1 \to \wedge^2_H ,$
- the composition $\wedge^1 \xrightarrow{\nabla_H} \wedge^1_H \otimes \wedge^1 \to \wedge^1_H \otimes \wedge^1_H \to \wedge^2_H ,$

both of which satisfy a Leibniz rule, e.g.

$$d_H(f\omega) = f d_H\omega + d_Hf \wedge \omega_H .$$

Definition 2.5. The difference between the two differential operators in (2) is called the *partial torsion* τ_H of ∇_H. The Leibniz rule implies that it is a homomorphism of bundles, equivalently $\tau_H \in \Gamma(\wedge^2_H \otimes TM)$.

Remark 2.4. If ∇_H preserves L, then the projection of τ_H to $\wedge^2_H \otimes L^*$ is the Levi form $\mathcal{L}$ of H.

Lemma 2.1. *On any smooth contact manifold, there are partial connections on $\wedge^1$ with vanishing partial torsion.*

Proof. By partition of unity one can choose a partial connection ∇_H on $\wedge^1$ and then the general such partial connection is of the form $\nabla_H - \Gamma_H$ for an arbitrary homomorphism $\Gamma_H : \wedge^1 \to \wedge^1_H \otimes \wedge^1$. The partial torsion τ_H of ∇_H is then modified by the composition $\wedge^1 \xrightarrow{\Gamma_H} \wedge^1_H \otimes \wedge^1 \to \wedge^2_H$ and so we may always adopt such a modification to ensure that ∇_H is partially torsion-free, as required. $\square$

Remark 2.5. The remaining freedom in choosing a partially torsion-free connection on $\wedge^1$ is $\nabla_H \mapsto \nabla_H - \Gamma_H$, where

$$\Gamma_H \colon \wedge^1 \to \ker\colon \wedge^1_H \otimes \wedge^1 \to \wedge^2_H$$

is arbitrary.

Now let us suppose that $\theta \in \Gamma(L)$ is nowhere vanishing. Such a contact form may be used to effect a number of normalisations. Firstly, the line bundle L is trivialised. Secondly, a vector field T, called the *Reeb* field, may be uniquely characterised by

$$T \,\lrcorner\, \theta = 1 \qquad T \,\lrcorner\, d\theta = 0\,. \tag{3}$$

Consequently, the short exact sequence (1) splits and we may write

$$\wedge^1 = \begin{array}{c}\wedge^1_H\\ \oplus\\ \wedge^0\end{array} \quad \text{by means of} \quad \omega \mapsto \left[\begin{array}{c}\omega_H\\ T \,\lrcorner\, \omega\end{array}\right]. \tag{4}$$

Equivalently, we may identify $\wedge^1_H$ as a subbundle of $\wedge^1$ by means of

$$\wedge^1_H = \ker\colon \wedge^1 \xrightarrow{T \,\lrcorner\, -} \wedge^0$$

and $\wedge^2_H$ as a subbundle of $\wedge^2$ by means of

$$\wedge^2_H = \ker\colon \wedge^2 \xrightarrow{T \,\lrcorner\, -} \wedge^1\,.$$

In particular, the 2-form $d\theta$ may be viewed as a section of $\wedge^2_H$. It coincides with the image of θ under the Levi form $\mathcal{L}\colon L \to \wedge^2_H$. Thus, in the presence of a contact form θ, we obtain a non-degenerate 2-form $\Omega \equiv d\theta \in \Gamma(\wedge^2_H)$ on the contact distribution H. In any case, we may use the splitting (4) to insist that a partial connection on $\wedge^1$ have the form

$$\wedge^1 = \begin{array}{c}\wedge^1_H\\ \oplus\\ \wedge^0\end{array} \ni \left[\begin{array}{c}\sigma\\ \rho\end{array}\right] \xmapsto{\nabla_H} \left[\begin{array}{c}D_H\sigma + \Omega\rho\\ d_H\rho\end{array}\right] \in \begin{array}{c}\wedge^1_H \otimes \wedge^1_H\\ \oplus\\ \wedge^1_H\end{array}, \tag{5}$$

where $D_H\colon \wedge^1_H \to \wedge^1_H \otimes \wedge^1_H$ is a partial connection on $\wedge^1_H$. The form of ∇_H ensures that its partial torsion lies in $\wedge^2_H \otimes H$. Therefore, if we argue as in the proof of Lemma 2.1 to remove the remaining partial torsion, then we have shown the following:

Proposition 2.2. *If a contact form θ is used to split the* 1*-forms as*

$$\wedge^1 = \wedge^1_H \oplus \wedge^0,$$

then we may find partial connections on $\wedge^1$ of the form (5) *and free from partial torsion. The remaining freedom in choosing such connections is*

$$D_H \mapsto D_H - \Gamma_H \quad \textit{for} \quad \Gamma_H \colon \wedge^1_H \to \bigodot^2 \wedge^1_H$$

an arbitrary homomorphism.

Remark 2.6. The partial connection dual to (5) has the form

$$TM = \begin{array}{c} \wedge^0 \\ \oplus \\ H \end{array} \ni \begin{bmatrix} \lambda \\ X \end{bmatrix} \overset{\nabla_H}{\longmapsto} \begin{bmatrix} d_H\lambda + X \lrcorner \Omega \\ D_H X \end{bmatrix} \tag{6}$$

(where the otherwise expected minus sign in front of $X \lrcorner \Omega$ is taken care of by Ω being skew (and the convention that $X \lrcorner \Omega \in \Gamma(\wedge^1_H)$ inserts X into the *first* slot of Ω)). In particular, it follows that

$$\begin{array}{c} \wedge^0 \\ \oplus \\ H \end{array} \ni \begin{bmatrix} 1 \\ 0 \end{bmatrix} \overset{\nabla_H}{\longmapsto} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{array}{c} \wedge^1_H \\ \oplus \\ \wedge^0 \end{array} \ni \begin{bmatrix} 0 \\ 1 \end{bmatrix} \overset{\nabla_H}{\longmapsto} \begin{bmatrix} \Omega \\ 0 \end{bmatrix}.$$

Evidently, these two conditions are sufficient to guarantee that a partial connection on $\wedge^1$ have the form (5) and Proposition 2.2 may be invariantly reformulated as follows.

Theorem 2.1. *If θ is a contact form with associated Reeb field T, then we may find partial connections on the (co-)tangent bundle such that*
- $\nabla_H T = 0$,
- $\nabla_H \theta = (d\theta)_H$,
- ∇_H *is free from partial torsion,*

where $(d\theta)_H$ is the image of $d\theta$ under the composition

$$\wedge^2 \hookrightarrow \wedge^1 \otimes \wedge^1 \to \wedge^1_H \otimes \wedge^1.$$

The freedom in choosing such a partial connection lies in $\Gamma(\bigodot^2 \wedge^1_H \otimes H)$.

Remark 2.7. Theorem 2.1 bears a striking similarity to the usual story for connections on the (co-)tangent bundle in which torsion-free connections are free up to $\Gamma(\bigodot^2 \wedge^1 \otimes TM)$. This appealing feature is one of our reasons for advocating the construction in this article.

3. Sub-Riemannian contact geometry

A sub-Riemannian contact structure on a smooth manifold M is a contact distribution $H \subset TM$ equipped with a positive-definite symmetric form $g : \bigodot^2 H \to \mathbb{R}$. We do not suppose any particular compatibility between g and the Levi form. For any chosen contact form $\theta \in \Gamma(L) \subset \Gamma(\wedge^1)$, however, we can pointwise choose a co-frame for H in which $g \in \Gamma(\bigodot^2 \wedge_H^1)$ and $\Omega = d\theta \in \Gamma(\wedge_H^2)$ are simultaneously represented by the matrices

$$\left[\begin{array}{ccccc} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \mathrm{Id} & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccccc} 0 & \lambda_1 & \cdots & 0 & 0 \\ -\lambda_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_n \\ 0 & 0 & \cdots & -\lambda_n & 0 \end{array}\right], \tag{7}$$

respectively.

Proposition 3.1. *Locally, we can always choose a contact form θ so that*

$$\|\Omega\|^2 = 2n\,, \quad \textit{equivalently} \quad {\lambda_1}^2 + \cdots + {\lambda_n}^2 = n\,.$$

With this normalisation θ is then determined up to sign.

Proof. Replacing θ by $\hat{\theta} = \lambda\theta$, for λ a nowhere vanishing smooth function, gives $d\hat{\theta} = \lambda d\theta + d\lambda \wedge \theta$ but, since θ vanishes on H, as far as $\wedge_H^2$ is concerned we find that $\hat{\Omega} = \lambda\Omega$. The stated normalisation and freedom are clear. □

Remark 3.1. If M is three-dimensional, this normalisation asserts that $\lambda_1 = \pm 1$ and a choice of sign corresponds to a choice of orientation for H. In this case, we may define an endomorphism $J : H \to H$ by

$$g(JX, Y) = \Omega(X, Y), \quad \forall\, X, Y \in H$$

and our normalisation asserts that $J^2 = -\mathrm{Id}$. Thus, we have obtained a CR structure. Conversely, every *pseudo-Hermitian* structure in the sense of Webster [11] arises in this way. More precisely, we have shown the following (as already noted in [7]).

Proposition 3.2. *In three dimensions, an oriented sub-Riemannian contact structure is equivalent to a CR structure with a choice of contact form.*

Remark 3.2. Without the contact form, a three-dimensional CR structure coincides with an oriented 'sub-conformal' contact structure (as in [7]).

4. Construction of the partial connection

The existence and uniqueness of the Levi-Civita connection in Riemannian geometry is based on the algebraic fact that, for any finite-dimensional vector space V, the composition

$$\bigodot^2 V \otimes V \hookrightarrow V \otimes V \otimes V \xrightarrow{-\otimes-\odot-} V \otimes \bigodot^2 V \tag{8}$$

is an isomorphism. The same algebra underlies the following construction.

Theorem 4.1. *On any sub-Riemannian contact manifold, there is a unique partial connection $\nabla_H : \wedge^1 \to \wedge_H^1 \otimes \wedge^1$ with the following properties.*

- $\nabla_H T = 0$,
- $\nabla_H \theta = (d\theta)_H$,
- ∇_H *is free from partial torsion,*

– $\nabla_H g = 0$,

where θ is any local contact form normalised as in Proposition 3.1 and T is its associated Reeb field.

Proof. The only freedom in θ is to change its sign. Evidently, such a change respects the characterising properties of ∇_H so it suffices to work locally, choose θ, and employ Theorem 2.1 to find a global connection with the first three of our required properties and with remaining freedom

$$\nabla_H \mapsto \hat{\nabla}_H = \nabla_H - \Gamma_H \,,$$

for $\Gamma_H \in \Gamma(\bigodot^2 \wedge^1_H \otimes H)$. If we use the sub-Riemannian metric g to identify H with its dual $\wedge^1_H$, and write $\sigma : \bigodot^2 \wedge^1_H \otimes \wedge^1_H \xrightarrow{\simeq} \wedge^1_H \otimes \bigodot^2 \wedge^1_H$ for the isomorphism (8), then

$$\hat{\nabla}_H g = \nabla_H g - 2\sigma\Gamma_H$$

and so $\hat{\nabla}_H g = 0$ if and only if $\Gamma_H = \frac{1}{2}\sigma^{-1}\nabla_H g$, which shows both existence and uniqueness from our final requirement. ☐

5. Other constructions

As noted in [7] and echoed in Proposition 3.2, sub-Riemannian geometry in dimension 3 coincides with Webster's pseudo-Hermitian geometry [11]. For completeness, we briefly recount the story in higher dimensions as follows.

Definition 5.1. Suppose M is a smooth manifold of dimension $2n + 1$. An *almost CR structure* on M is a vector sub-bundle $H \subset TM$ of rank $2n$ with an endomorphism $J \colon H \to H$ such that $J^2 = -\mathrm{Id}$.

Definition 5.2. An almost CR structure is said to be *non-degenerate* if H is a contact distribution.

Definition 5.3. An almost CR structure is said to be *partially integrable* if and only if the L^*-valued form $\mathcal{L}(X, JY)$ on H is symmetric. Equivalently, for any contact form, the $\mathbb{C}$-valued form

$$\Omega(X, JY) - i\Omega(X, Y)$$

on H is Hermitian (and, in this case, non-degeneracy of the CR structure is equivalent to non-degeneracy of this Hermitian form).

It is observed in [5, p. 414] that partial integrability is implied by the more usual condition of integrability, which may be defined as follows.

Definition 5.4. An almost CR structure is said to be *integrable* if and only if $[H^{0,1}, H^{0,1}] \subseteq H^{0,1}$ where $H^{0,1} = \{X \in \mathbb{C}H \text{ s.t. } JX + iX = 0\}$. Evidently, this condition is vacuous in three dimensions (for then $H^{0,1}$ is a line bundle). A *CR structure* is an integrable almost CR structure.

Definition 5.5. A *pseudo-Hermitian* structure is a CR structure equipped with a choice of contact form. Such a structure is said to be *strictly pseudo-convex* if and only if the corresponding symmetric form $\Omega(X, JY)$ on H is positive-definite.

Proposition 5.1. *Always,*

$$\{\textit{strictly pseudo-convex pseudo-Hermitian structures}\}$$
$$\subseteq \{\textit{oriented sub-Riemannian contact structures}\}$$

with equality in 3 *dimensions.*

Proof. Using $\Omega(X, JY)$ as a sub-Riemannian metric and using J to orient H, it is clear in the co-frames (7) that CR geometry corresponds exactly to the case $\lambda_1 = \cdots = \lambda_n = 1$. □

5.1. **The Tanaka-Webster connection.** Since it is only in 3 dimensions that pseudo-Hermitian geometry coincides with sub-Riemannian geometry, we shall confine our discussion to this case. The construction [10, 11] of this canonical connection, written from the sub-Riemannian point of view, is as follows. Choose, a local co-frame θ, e_1, e_2 on M such that

$$d\theta = e_1 \wedge e_2 \quad \text{and} \quad Je_2 = e_1\,. \tag{9}$$

Notice that such a co-frame is determined up to

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \longmapsto \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{10}$$

for an arbitrary smooth function ϕ.

Lemma 5.1. *There is a smooth* 1*-form* ω *and smooth functions* A *and* B *uniquely characterised by*

$$\begin{array}{rcl} de_1 & = & \omega \wedge e_2 + A\theta \wedge e_1 + B\theta \wedge e_2 \\ de_2 & = & -\omega \wedge e_1 + B\theta \wedge e_1 - A\theta \wedge e_2 \end{array} \tag{11}$$

Proof. At each point, there are seemingly 6 equations here for 5 unknowns, namely the 3 coefficients of ω together with A and B. However, there is one relation namely

$$0 = d^2\theta = d(e_1 \wedge e_2) = de_1 \wedge e_2 - de_2 \wedge e_1\,,$$

which is exactly as required by the right hand side of (11). □

Notice that if we change our co-frame according to (10), then

$$d\hat{e}_1 = \hat{\omega} \wedge \hat{e}_2 + \hat{A}\theta \wedge \hat{e}_1 + \hat{B}\theta \wedge \hat{e}_2$$

$$d\hat{e}_2 = -\hat{\omega} \wedge \hat{e}_1 + \hat{B}\theta \wedge \hat{e}_1 - \hat{A}\theta \wedge \hat{e}_2\,,$$

where

$$\hat{\omega} = \omega - d\phi \quad \text{and} \quad \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix} = \begin{bmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}. \tag{12}$$

Theorem 5.1 (Tanaka-Webster)**.** *The connection on* Λ^1 *given by*

$$\nabla\theta = 0, \qquad \nabla e_1 = \omega \otimes e_2, \qquad \nabla e_2 = -\omega \otimes e_1 \tag{13}$$

in any chosen co-frame, does not depend on this choice.

Proof. There is no choice in θ. Otherwise, the required invariance follows by straightforward computation from $\hat{\omega} = \omega - d\phi$. □

We shall now use the co-frame θ, e_1, e_2 and its structure equations (11) to

- compute the torsion of the Tanaka-Webster connection,
- compute the curvature of the Tanaka-Webster connection,
- compute the partial connection of Theorem 4.1,
- promote it to a full connection via Proposition 2.1,
- and compare these two connections.

5.1.1. *Tanaka-Webster torsion.* The torsion of any connection on $\wedge^1$ is the difference between $d\colon \wedge^1 \to \wedge^2$ and the composition $\wedge^1 \xrightarrow{\nabla} \wedge^1 \otimes \wedge^1 \to \wedge^2$. According to (11), for the Tanaka-Webster connection, this is

$$\theta \mapsto e_1 \wedge e_2\,, \qquad e_1 \mapsto A\theta \wedge e_1 + B\theta \wedge e_2\,, \qquad e_2 \mapsto B\theta \wedge e_1 - A\theta \wedge e_2\,,$$

the first of which is just the Levi form and the rest may be written as

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \longmapsto \theta \wedge \begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{14}$$

Its invariance under change of co-frame (10) is equivalent to the second part of (12), which, for these purposes may be better rewritten as

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{B} & -\hat{A} \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} A & B \\ B & -A \end{bmatrix}.$$

In the standard expositions, the torsion is usually presented as a complex-valued quantity, equivalent to $A + iB$.

5.1.2. *Tanaka-Webster curvature.* The curvature of a general connection $E \to \wedge^1 \otimes E$ is the composition $E \xrightarrow{\nabla} \wedge^1 \otimes E \xrightarrow{\nabla} \wedge^2 \otimes E$ where

$$\nabla(\alpha \otimes \sigma) = d\alpha \otimes \sigma - \alpha \wedge \nabla\sigma \quad \text{characterises} \quad \nabla\colon \wedge^1 \otimes E \to \wedge^2 \otimes E\,.$$

Therefore, we may compute, according to (13), that

$$\begin{array}{ccccccc} \theta & \xrightarrow{\nabla} & 0 & & & & \\ e_1 & \xrightarrow{\nabla} & \omega \otimes e_2 & \xrightarrow{\nabla} & d\omega \otimes e_2 - \omega \wedge (-\omega \otimes e_1) & = & d\omega \otimes e_2 \\ e_2 & \xrightarrow{\nabla} & -\omega \otimes e_1 & \xrightarrow{\nabla} & -d\omega \otimes e_1 + \omega \wedge (\omega \otimes e_2) & = & -d\omega \otimes e_1\,. \end{array}$$

In other words, the curvature is determined by $d\omega$. Its invariance is clear from the first equation of (12). In fact, the curvature provides only one new scalar quantity, namely $d\omega \wedge \theta$, since $d\omega \wedge e_1$ and $d\omega \wedge e_2$ may be determined in terms of the torsion by differentiating the structure equations (11). It is traditionally captured by the real-valued function R determined by

$$d\omega \wedge \theta = R\,\theta \wedge e_1 \wedge e_2\,. \tag{15}$$

5.2. **The partial connection.** The same co-frame (9) may also be used to compute the partial connection. The characterising properties (3) of the Reeb field T show that it is, equivalently, determined by

$$T \lrcorner \theta = 1 \qquad T \lrcorner e_1 = 0 \qquad T \lrcorner e_2 = 0$$

whence the co-frame $\{\theta, e_1, e_2\}$ is compatible with the splitting (4). More specifically $\{e_1, e_2\}$ spans $\wedge^1_H \hookrightarrow \Lambda^1$ and θ trivialises $L \subset \wedge^1$. Therefore, if we consider the partial connection ∇_H on $\wedge^1$ defined by

$$\nabla_H \theta = e_1 \otimes e_2 - e_2 \otimes e_1 \quad \nabla_H e_1 = \omega_H \otimes e_2 \quad \nabla_H e_2 = -\omega_H \otimes e_1\,, \tag{16}$$

where ω is defined by (11) and ω_H is its image in $\wedge^1_H$, then we ensure that it has the form (5) and is free from partial torsion, as required by Theorem 4.1. Finally,

$$\begin{aligned}&\nabla_H(e_1 \otimes e_1 + e_2 \otimes e_2)\\ &\qquad = \omega_H \otimes e_2 \otimes e_1 + \omega_H \otimes e_1 \otimes e_2 - \omega_H \otimes e_1 \otimes e_2 - \omega_H \otimes e_2 \otimes e_1 = 0\end{aligned}$$

and all characterising properties of Theorem 4.1 are satisfied. Thus, apart from a minor modification whereby $\nabla_H \theta = d\theta$ replaces $\nabla\theta = 0$, the partial connection of Theorem 4.1 is induced by the Tanaka-Webster connection.

5.2.1. *Promotion of the partial connection.* We shall now take the partial connection defined by (16) and promote it to a full connection on $\wedge^1$ in line with Proposition 2.1. The general lift of (16) to a full connection is defined by

$$\begin{aligned}\nabla\theta &= \theta \otimes \alpha + e_1 \otimes e_2 - e_2 \otimes e_1\\ \nabla e_1 &= \theta \otimes \beta + \omega \otimes e_2\\ \nabla e_2 &= \theta \otimes \gamma - \omega \otimes e_1\end{aligned}$$

for 1-forms α, β, γ and if we now compute the composition

$$\wedge^1 \xrightarrow{\nabla} \wedge^1 \otimes \wedge^1 \xrightarrow{\nabla} \wedge^2 \otimes \wedge^1 \longrightarrow \wedge^2_H \otimes \wedge^1$$

for this lift, we find that

$$\theta \mapsto e_1 \wedge e_2 \otimes \alpha + (de_1 + e_2 \wedge \omega)_H \otimes e_2 - (de_2 - e_1 \wedge \omega)_H \otimes e_1 = e_1 \wedge e_2 \otimes \alpha$$

in accordance with (11), and then

$$\begin{aligned}e_1 &\mapsto e_1 \wedge e_2 \otimes \beta + (d\omega)_H \otimes e_2\\ e_2 &\mapsto e_1 \wedge e_2 \otimes \gamma - (d\omega)_H \otimes e_1\,.\end{aligned}$$

Therefore, we are obliged to take $\alpha = 0$ and

$$\beta = -Re_2 \quad \text{and} \quad \gamma = Re_1\,,$$

where $(d\omega)_H = R\, e_1 \wedge e_2$. In summary, our promoted connection is given by

$$\begin{aligned}\nabla\theta &= e_1 \otimes e_2 - e_2 \otimes e_1\\ \nabla e_1 &= (\omega - R\,\theta) \otimes e_2\\ \nabla e_2 &= (R\,\theta - \omega) \otimes e_1\end{aligned} \tag{17}$$

where R is the Tanaka-Webster curvature determined by (15).

5.3. **Comparison.** We may compare the promoted connection (17) with Tanaka--Webster. From (13) and (9) we find that their difference tensor, as a homomorphism $\wedge^1 \to \wedge^1 \otimes \wedge^1$, is given by

$$\wedge^1 = \begin{array}{c} \Lambda^1_H \\ \oplus \\ \wedge^0 \end{array} \ni \left[\begin{array}{c} \sigma \\ \rho \end{array} \right] \mapsto \left[\begin{array}{c} R\,\theta \otimes J\sigma + \Omega\rho \\ 0 \end{array} \right],$$

where R is the Webster-Tanaka curvature (15) and the 1-forms are split by the Reeb field corresponding to θ.

Remark 5.1. Recall that the two basic invariants of pseudo-Hermitian geometry are the torsion (14) and curvature (15). Finally, we remark that if we compute the full torsion of our promoted connection (17), then we find $\theta \mapsto 0$ and

$$\left[\begin{array}{c} e_1 \\ e_2 \end{array} \right] \longmapsto \theta \wedge \left[\begin{array}{cc} A & B+R \\ B-R & -A \end{array} \right] \left[\begin{array}{c} e_1 \\ e_2 \end{array} \right].$$

In this formula we see the basic invariants appearing together.

Remark 5.2. For any strictly pseudo-convex pseudo-Hermitian structure (in any dimension) there is, apart from the Tanaka–Webster connection, yet another canonical affine connection, namely the associated Weyl connection defined as in [5]. As partial connections on $\wedge^1$ they coincide, but as full connections they differ as computed in [5, Theorem 5.2.13]. In dimension 3 their difference tensor is simply a constant multiple of

$$\left[\begin{array}{c} \sigma \\ \rho \end{array} \right] \mapsto \left[\begin{array}{c} R\,\theta \otimes J\sigma \\ 0 \end{array} \right].$$

6. Outlook

It would be interesting to determine how our connection fits with other recent constructions in sub-Riemannian geometry. In particular, Agrachev, Barilari, and Rizzi [1] lament the lack of a canonical connection 'à la Levi-Civita,' instead deriving curvature invariants by other means, especially via Jacobi fields. These authors are working on contact manifolds but the approach via Jacobi fields is quite general, as shown by Barilari and Rizzi [3], following earlier work of Zelenko and Li [12].

A sub-Riemannian metric may be regarded as a Hamiltonian $T^*M \to \mathbb{R}$ defined as the composition

$$T^*M \to H^* \ni \omega_H \mapsto \tfrac{1}{2}\|\omega_H\|^2 .$$

The integral curves of the corresponding Hamiltonian vector field are called *extremals* and, for contact manifolds, their images down on M are precisely the *geodesics* for the sub-Riemannian distance.

In Riemannian geometry, perturbations of a given geodesic γ are controlled by *Jacobi fields* J^a satisfying the *Jacobi equation*

$$(X^a\nabla_a)^2J^c + R_{ab}{}^c{}_dJ^aX^bX^d = 0$$

along γ, where X^a is the unit velocity vector and ∇_a is the Levi-Civita connection with curvature tensor $R_{ab}{}^c{}_d$. The Bianchi symmetry says that

$$R_{ab}{}^c{}_d = \tfrac{2}{3}[R_{a(b}{}^c{}_{d)} - R_{b(a}{}^c{}_{d)}]$$

so the curvature tensor may, in some sense, be read off from the Jacobi equation, better so from the perturbation of extremals up on T^*M (more usually called the *geodesic spray*). In the sub-Riemannian case, the upshot is that 'curvature invariants' may be similarly read off from a 'Jacobi equation' on T^*M as in [1, 3, 12]. In [3] it is shown that these curvature invariants may also be derived from the (necessarily non-linear) Ehresmann connection introduced in [12].

We should emphasise that the connection we introduce does not preserve the contact distribution $H \subset TM$. Indeed, it is clear by design (6), that H is fed into $\wedge^0$ by the Levi form. In this regard, our connection more resembles the tractor connections from parabolic geometry [5].

Acknowledgement. We should like to thank the referee for drawing our attention to [1, 3, 12] and for suggesting various corrections.

References

[1] Agrachev, A.A., Barilari, D., Rizzi, L., *Sub-Riemannian curvature in contact geometry*, to appear in J. Geom. Anal.

[2] Agrachev, A.A., Zelenko, I., *Geometry of Jacobi curves I*, J. Dynam. Control Systems **8** (1) (2002), 93–140.

[3] Barilari, D., Rizzi, L., *On Jacobi fields and canonical connection in sub-Riemannian geometry*, arXiv:1506.01827.

[4] Bryant, R.L., Eastwood, M.G., Gover, A.R., Neusser, K., *Some differential complexes within and beyond parabolic geometry*, arXiv:1112.2142.

[5] Čap, A., Slovák, J., *Parabolic Geometries I: Background and General Theory*, Surveys and Monographs, vol. 154, Amer. Math. Soc., 2009.

[6] Eastwood, M.G., Gover, A.R., *Prolongation on contact manifolds*, Indiana Univ. Math. J. **60** (2011), 1425–1486.

[7] Falbel, E., Gorodski, C., Veloso, J.M., *Conformal sub-Riemannian geometry in dimension 3*, Mat. Contemp. **9** (1995), 61–73.

[8] Morimoto, T., *Cartan connection associated with a subriemannian structure*, Differential Geom. Appl. **26** (2008), 75–78.

[9] Rumin, M., *Un complexe de formes différentielles sur les variétés de contact*, Comptes Rendus Acad. Sci. Paris Math. **310** (1990), 401–404.

[10] Tanaka, N., *A differential geometric study on strongly pseudo-convex manifolds*, Lectures in Mathematics, Kyoto University, Kinokuniya, 1975.

[11] Webster, S.M., *Pseudo-Hermitian structures on a real hypersurface*, J. Differential Geom. **13** (1978), 25–41.

[12] Zelenko, I., Li, C., *Differential geometry of curves in Lagrange Grassmannians with given Young diagram*, Differential Geom. Appl. **27** (2009), 723–742.

School of Mathematical Sciences,
University of Adelaide,
SA 5005, Australia
E-mail: meastwoo@member.ams.org

Eduard Čech Institute & Mathematical Institute,
Charles University,
Prague, Czech Republic
E-mail: katharina.neusser@gmail.com