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KILLING SPINOR-VALUED FORMS AND THE CONE CONSTRUCTION

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Abstract. On a pseudo-Riemannian manifold $\mathcal{M}$ we introduce a system of partial differential Killing type equations for spinor-valued differential forms, and study their basic properties. We discuss the relationship between solutions of Killing equations on $\mathcal{M}$ and parallel fields on the metric cone over $\mathcal{M}$ for spinor-valued forms.

1. Introduction

The subject of the present article are the systems of over-determined partial differential equations for spinor-valued differential forms, classified as a type of Killing equations. The solution spaces of these systems of PDE's are termed Killing spinor-valued differential forms. A central question in geometry asks for pseudo-Riemannian manifolds admitting non-trivial solutions of Killing type equations, namely how the properties of Killing spinor-valued forms relate to the underlying geometric structure for which they can occur.

Killing spinor-valued forms are closely related to Killing spinors and Killing forms with Killing vectors as a special example. Killing spinors are both twistor spinors and eigenspinors for the Dirac operator, and real Killing spinors realize the limit case in the eigenvalue estimates for the Dirac operator on compact Riemannian spin manifolds of positive scalar curvature. There is a classification of complete simply connected Riemannian manifolds equipped with real Killing spinors, leading to the construction of manifolds with the exceptional holonomy groups G_2 and $\mathsf{Spin}(7)$, see [8], [1]. Killing vector fields on a pseudo-Riemannian manifold are the infinitesimal generators of isometries, hence they influence its geometrical properties. In particular, on compact manifolds of negative Ricci curvature there are no non-trivial Killing vector fields, while on manifolds of non-positive Ricci curvature are all Killing vector fields parallel. A generalization of Killing vector fields are Killing forms, characterized by the fact that their covariant derivative is totally skew-symmetric tensor field.

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There is rather convenient tool allowing to describe invariant systems of partial differential equations. It is based on Stein-Weiss gradient operators constructed by decomposing the covariant derivative into individual invariant components, see [13]. Among prominent examples of Stein-Weiss gradients are the twistor-like operators, which are overdetermined operators corresponding to the highest weight gradient component. It can be shown that the solution spaces of Killing type equations are always in the kernel of corresponding twistor operators. On the other hand, equations given by vanishing of twistor operators are equivalent to weaker systems called conformal Killing equations.

The main result of the present article establishes a correspondence between special solutions of Killing equations for spinor-valued forms on the base pseudo-Riemannian manifold $\mathcal{M}$ and certain parallel spinor-valued forms on the metric cone $\overline{\mathcal{M}}$ over $\mathcal{M}$. Since the existence of parallel spinor-valued forms can be reduced to a holonomy problem, one can at least partially classify the manifolds admitting these special solutions by enumerating all possible holonomy groups. In particular, for compact irreducible Riemannian manifolds one can exploit the Berger-Simmons classification, cf. [3] and [10]. This was already done by Bär in [1] for real Killing spinors and by Semmelmann in [9] for ordinary Killing forms.

Let us briefly describe the content of our article. After general introduction in Section 1, we employ in Section 2 the representation theory of $\mathsf{Spin}(n_+, n_-)$ and produce invariant decompositions which offer a deeper insight into subsequent treatment of spinor-valued forms. We show that the space of spinor-valued forms is highly reducible and as a special case we discuss the primitive spinor-valued forms. We also introduce the so called generalized twistor modules as distinguished components in the decomposition of tensor products with the dual of the fundamental vector representation. Then we give rather straightforward definition of Killing equations on spinor-valued p-form fields in Section 3 and prove a basic property characterizing its relationship to other types of Killing equations. In Section 4, we introduce the metric cone over the base pseudo-Riemannian manifold $\mathcal{M}$, and discuss the lifts of both spinor-valued form fields and Killing equations on $\mathcal{M}$ to its metric cone $\overline{\mathcal{M}}$. The conclusion is that for any degree p, there is an injection from special Killing spinor-valued p-forms on $\mathcal{M}$ to parallel spinor-valued $(p+1)$-forms on $\overline{\mathcal{M}}$ for the Levi-Civita connection on the metric cone. The variance ϵ in the signature (n_+, n_-) for $\mathcal{M}$ and $(\overline{n}_+, \overline{n}_-)$ for $\overline{\mathcal{M}}$ is built into the definition of Killing number of special Killing spinor-valued forms on $\mathcal{M}$, and at the same time appears in the formulas for the connection on the metric cone $\overline{\mathcal{M}}$.

2. Spinor-valued forms

Spinors and form representations. The spinor-valued forms originate in the tensor product of forms (i.e., the alternating tensors) and spinors. We recall the Clifford algebra $\mathsf{Cl}(n_+, n_-)$, constructed from the n-dimensional pseudo-Euclidean space $\mathsf{V} = \mathbb{R}^{n_+, n_-} = (\mathbb{R}^n, g)$ equipped with the standard symmetric bilinear form g of signature (n_+, n_-). The complex spinor space S arises as an irreducible complex $\mathsf{Cl}(n_+, n_-)$-module. We denote by $\mathsf{A}^p = \bigwedge^p \mathsf{V}^*$ the space of ordinary forms of degree

$p \in \{0, \dots, n\}$, namely $\mathsf{V}^* = \mathsf{A}^1$ denotes the dual of V. The space of spinor-valued forms of degree p is defined as the tensor product $\mathsf{SA}^p = \mathsf{A}^p \otimes \mathsf{S}$.

Since V is naturally embedded into $\mathsf{Cl}(n_+, n_-)$ as its subspace of generators, the module structure of S is realized by the Clifford multiplication '$\cdot$' of spinors by vectors. Equivalently, the multiplication can be viewed as a $\mathsf{Cl}(n_+, n_-)$-valued 1-form denoted $\gamma\cdot$. A convenient way to write the defining relations of the Clifford algebra $\mathsf{Cl}(n_+, n_-)$ is

$$\mathrm{sym}(\gamma\cdot \otimes \gamma\cdot) = -2g\,, \tag{1}$$

where sym denotes the symmetrization over form indices. To complete our notation, we recall the usual exterior product '$\wedge$' of two forms, the interior product '$\lrcorner$' of a vector and a form and finally the orthogonal dual '*' mapping vectors to 1-forms via the isomorphism induced by g .

It is straightforward to verify a few basic relations useful in the computations with spinor-valued forms:

$$\begin{aligned} X \cdot (\gamma\cdot \wedge \Phi) + \gamma\cdot \wedge (X \cdot \Phi) &= -2X^* \wedge \Phi\,,\\ X \cdot (\gamma^*\!\!\cdot \lrcorner\, \Phi) + \gamma^*\!\!\cdot \lrcorner\, (X \cdot \Phi) &= -2X \lrcorner\, \Phi\,,\\ X \lrcorner\, (\gamma\cdot \wedge \Phi) + \gamma\cdot \wedge (X \lrcorner\, \Phi) &= X \cdot \Phi\,,\\ X^* \wedge (\gamma^*\!\!\cdot \lrcorner\, \Phi) + \gamma^*\!\!\cdot \lrcorner\, (X^* \wedge \Phi) &= X \cdot \Phi\,, \end{aligned} \tag{2}$$

for all $\Phi \in \mathsf{SA}^p$ and $X \in \mathsf{V}$. Note that $\gamma\cdot$ in the formulas acts simultaneously on the spinor part by the Clifford multiplication and on the form part by the exterior or interior product, respectively.

Spin**-invariant decompositions.** Let us briefly recall the case of ordinary forms as discussed in, cf. [9]. The space $\mathsf{V}^* \otimes \mathsf{A}^p$ decomposes with respect to the orthogonal group $\mathsf{O}(n_+, n_-)$ as

$$\mathsf{V}^* \otimes \mathsf{A}^p \cong \mathsf{A}^{p-1} \oplus \mathsf{A}^{p+1} \oplus \mathsf{A}^{p,1}\,. \tag{3}$$

For any $\alpha \in \mathsf{A}^p$ and $X \in \mathsf{V}$, the projections on the first two components are given simply by the interior and exterior products,

$$\mathrm{p}_1(X^* \otimes \alpha) = X \lrcorner\, \alpha\,, \qquad \mathrm{p}_2(X^* \otimes \alpha) = X^* \wedge \alpha\,. \tag{4}$$

Consequently, the remaining component called the *twistor module* for A^p is the common kernel of $\mathrm{p}_1, \mathrm{p}_2$

$$\mathsf{A}^{p,1} = \mathrm{Ker}(\mathrm{p}_1) \cap \mathrm{Ker}(\mathrm{p}_2)\,. \tag{5}$$

The situation is more complicated for the spinor-valued forms SA^p. Firstly, SA^p is reducible with respect to the spin group $\mathsf{Spin}(n_+, n_-)$. Its decomposition can be obtained using the technique of the Howe dual pairs, for details see [11]. The algebraic operators

$$X = \gamma\cdot\wedge\,, \qquad Y = -\gamma^*\!\!\cdot\lrcorner\,, \qquad H = [X, Y]\,, \tag{6}$$

commute with the action of $\mathsf{Spin}(n_+, n_-)$ and span a Lie algebra isomorphic to the Lie algebra $\mathfrak{sl}(2)$. In particular, we have

$$H(\Phi) = (n-2p)\,\Phi \quad \text{for all} \quad \Phi \in \mathsf{SA}^p\,, \tag{7}$$

and further analysis shows that SA^p decomposes as[1]

$$\mathsf{SA}^p \cong \mathsf{SA}^0_0 \oplus \cdots \oplus \mathsf{SA}^l_0, \quad l = \min\{p, n-p\}. \tag{8}$$

The component defined as the kernel of Y,

$$\mathsf{SA}^q_0 = \{\Phi \in \mathsf{SA}^q \mid \gamma^*_\cdot \lrcorner\, \Phi = 0\} \tag{9}$$

for $q \in \{0, \dots, [n/2]\}$ is called the space of primitive spinor-valued forms.

In order to decompose the space $\mathsf{V}^* \otimes \mathsf{SA}^p$, we first consider projections analogous to (4) and in addition one given by the Clifford multiplication,

$$\mathrm{p}_1(X^* \otimes \Phi) = X \lrcorner\, \Phi\,, \quad \mathrm{p}_2(X^* \otimes \Phi) = X^* \wedge \Phi\,, \quad \mathrm{p}_3(X^* \otimes \Phi) = X \cdot \Phi\,, \tag{10}$$

for $\Phi \in \mathsf{SA}^p$ and $X \in \mathsf{V}$. In the case $p = 0$ the decomposition degenerates:

$$\mathsf{V}^* \otimes \mathsf{S} = \mathsf{SA}^1 \cong \mathsf{S} \oplus \mathsf{SA}^1_0\,. \tag{11}$$

The (classical) *twistor module* for S is just the kernel of Clifford multiplication,

$$\mathsf{SA}^1_0 = \mathrm{Ker}(\mathrm{p}_3)\,, \tag{12}$$

the first two projections being trivial. The same applies also to the case $p = n$, $\mathsf{SA}^n \cong \mathsf{SA}^0 = \mathsf{S}$.

If $p \in \{1, \dots, n-1\}$, the *twistor module* for SA^p is once again defined as the common kernel of all projections,

$$\mathsf{SA}^{p,1} = \mathrm{Ker}(\mathrm{p}_1) \cap \mathrm{Ker}(\mathrm{p}_2) \cap \mathrm{Ker}(\mathrm{p}_3)\,. \tag{13}$$

However, it turns out that the three projections are not independent, in fact, the decomposition looks like

$$\mathsf{V}^* \otimes \mathsf{SA}^p \cong (\mathsf{SA}^{p-1} \oplus \mathsf{SA}^{p+1} \oplus \mathsf{SA}^p) \,/\, \mathsf{S} \oplus \mathsf{SA}^{p,1}\,. \tag{14}$$

In other words, the first three components share just two copies of the spinor space $\mathsf{SA}^0_0 = \mathsf{S}$. Hence we need to modify the projections to make them independent and such a modification is rather complicated. Moreover, due to the reducibility of SA^p there are multiplicities in the full decomposition of $\mathsf{V}^* \otimes \mathsf{SA}^p$ to irreducible summands and the choice of said modification is not unique. However, the multiplicities disappear in the restriction to the subspace of primitive spinor-valued forms and all projections are essentially unique. For more details and explicit formulas, see [17].

[1]More precisely, the decomposition is irreducible only for n odd. For n even, each of the summands further decomposes into two irreducible components analogously to the decomposition of the spinor space into half-spinors $\mathsf{S} = \mathsf{S}^+ \oplus \mathsf{S}^-$.

3. Killing equations

We assume $(\mathcal{M}, g)$ is an oriented and spin pseudo-Riemannian manifold of dimension n and signature (n_+, n_-), and ∇ is the Levi-Civita covariant derivative. As usual we denote the tangent bundle by $\mathcal{T}(\mathcal{M})$ and the Lie algebra of smooth vector fields by $\mathcal{X}(\mathcal{M})$. We shall consider tensor fields on $\mathcal{M}$ given by smooth sections of vector bundles associated to a class of $\mathsf{Spin}(n_+, n_-)$-representations discussed in the previous Section 2.

Killing forms. Killing vector fields can be characterized as the vector fields, whose flow preserves the metric g. In terms of the Levi-Civita covariant derivative, a Killing vector field K fulfills

$$g(\nabla_X K, Y) + g(\nabla_Y K, X) = 0 \quad \text{for all} \quad X, Y \in \mathcal{T}(\mathcal{M})\,. \tag{15}$$

The skew-symmetry of the covariant derivative of K generalizes to the definition of Killing form as a differential form α fulfilling (cf., [16], [9])

$$\nabla_X \alpha = \tfrac{1}{p+1}\, X \lrcorner \, \mathrm{d}\alpha \tag{16}$$

for all $X \in \mathcal{T}(\mathcal{M})$, where p is the degree of α and $\mathrm{d}\alpha$ denotes the usual exterior differential of α. By the polarization identity, (16) is equivalent to

$$X \lrcorner \, \nabla_X \alpha = 0, \tag{17}$$

and this implies that the Killing forms provide quadratic first integrals of the geodesic equation, cf. [15]: for α a Killing form and X the geodesic vector field on $\mathcal{M}$,

$$\nabla_X (X \lrcorner \, \alpha) = 0\,. \tag{18}$$

By (3), the covariant derivative can be decomposed on three invariant first-order operators: codifferential $\mathrm{d}^* \colon \Omega^p(\mathcal{M}) \to \Omega^{p-1}(\mathcal{M})$ given by $\mathrm{p}_1 \circ \nabla$, exterior differential $\mathrm{d} \colon \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M})$ given by $\mathrm{p}_2 \circ \nabla$, and the twistor operator $\mathrm{T} \colon \Omega^p(\mathcal{M}) \to \Omega^{p,1}(\mathcal{M})$ given by projecting ∇ on the twistor module. Here $\Omega^p(\mathcal{M})$ denotes the space of differential forms of degree p and $\Omega^{p,1}(\mathcal{M})$ the space of tensor fields corresponding to the representation $\mathsf{A}^{p,1}$. As for Killing forms, (16) is equivalent to

$$\mathrm{T}\,\alpha = 0\,, \quad \text{and} \quad \mathrm{d}^*\alpha = 0\,. \tag{19}$$

In particular, Killing forms are in the kernel of the twistor operator, i.e., they are conformal Killing forms. For more detailed discussion, see [9].

Killing spinors. We denote by $\mathcal{P}_{\mathsf{Spin}}(\mathcal{M})$ a chosen spin structure on $\mathcal{M}$ and $\mathcal{S}(\mathcal{M})$ the associated spinor bundle. The Levi-Civita connection uniquely lifts to a spin connection and by abuse of notation we denote the induced covariant derivative on spinors and tensor-spinor fields ∇ as well. $\Sigma(\mathcal{M})$ denotes the space of spinor fields and $\Sigma\Omega^1_0(\mathcal{M})$ the space of primitive spinor-valued differential 1-forms corresponding to the representation SA^1_0.

A Killing spinor is a spinor field Ψ such that

$$\nabla_X \Psi = aX \cdot \Psi, \quad X \in \mathcal{T}(\mathcal{M})\,, \tag{20}$$

where $a \in \mathbb{C}$ is called the Killing number of Ψ. Killing spinors are also intimately related to the underlying geometry of $\mathcal{M}$, c.f. [7], [2].

The algebraic decomposition (11) yields two invariant first-order differential operators: Dirac operator $\mathrm{D}\colon \Sigma(\mathcal{M}) \to \Sigma(\mathcal{M})$ given by $\mathrm{p}_3 \circ \nabla$, and twistor operator $\mathrm{T}\colon \Sigma(\mathcal{M}) \to \Sigma\Omega^1_0(\mathcal{M})$ given by projecting ∇ on the twistor module. The Killing equation (20) is equivalent to

$$\mathrm{T}\,\Psi = 0, \quad \text{and} \quad \mathrm{D}\,\Psi = -na\Psi\,. \tag{21}$$

In particular, Killing spinors are in the kernel of the twistor operator, i.e., they are conformal Killing (or, twistor) spinors.

Killing spinor-valued forms. The $\mathsf{Cl}(n_+, n_-)$-valued 1-form $\gamma\cdot$, see (1), is invariant for the action of $\mathsf{Spin}(n_+, n_-)$ and hence globally defined on $\mathcal{M}$. Since the Levi-Civita connection is metric, so is the spin connection and subsequently we also have $\nabla(\gamma\cdot) = 0$.

Definition 1. A Killing spinor-valued form is a spinor-valued differential form Φ of degree $p \in \{1, \dots, n-1\}$ such that

$$\nabla_X \Phi = a\Big(X \cdot \Phi - \tfrac{1}{p+1}\, X \lrcorner (\gamma\cdot \wedge \Phi)\Big) + \tfrac{1}{p+1}\, X \lrcorner \mathrm{d}\Phi, \quad X \in \mathcal{T}(\mathcal{M})\,, \tag{22}$$

where $\mathrm{d}\Phi$ is the covariant exterior differential of Φ and $a \in \mathbb{C}$ is called the Killing number of Φ.

The equation (22) first appeared in theoretical physics in the context of Kaluza--Klein supergravity, cf. [6], [5]. In geometry, the equation was introduced first in a simplified form corresponding to $a = 0$ in [12] and in its general form in [17]. As in the case of differential forms, we can reformulate (22) using the polarization identity:

$$X \lrcorner \nabla_X \Phi = aX \lrcorner (X \cdot \Phi)\,, \tag{23}$$

hence Killing spinor-valued forms also yield invariants along the geodesics of $\mathcal{M}$, but only in the case $a = 0$. A consequence of (23) is

Proposition 2. *Let Φ be a spinor-valued Killing form with Killing number $a = 0$ and let X be the geodesic vector field on $\mathcal{M}$. Then*

$$\nabla_X (X \lrcorner \Phi) = 0\,, \tag{24}$$

i.e., $X \lrcorner \Phi$ is covariantly constant along the geodesics.

The Killing spinor-valued forms can be directly constructed out of Killing forms and Killing spinors.

Proposition 3. *Let α be a Killing form of degree $p \in \{1, \dots, n-1\}$ and Ψ a Killing spinor with Killing number a. Then $\Phi = \alpha \otimes \Psi$ is a Killing spinor-valued form with Killing number a.*

Proof. Let $\{X_1, \ldots, X_n\}$ be an orthonormal frame. We first compute the exterior covariant derivative of Φ using (20),

$$\begin{aligned} \mathrm{d}\Phi &= \sum_{i=1}^{n} X_i^* \wedge \nabla_{X_i}\Phi = \sum_{i=1}^{n} X_i^* \wedge (\nabla_{X_i}\alpha \otimes \Psi + \alpha \otimes \nabla_{X_i}\Psi) \\ &= \mathrm{d}\alpha \otimes \Psi + a\gamma\cdot \wedge \Phi\,. \end{aligned}$$

Now by (16) and one more time (20), we get

$$\begin{aligned} \nabla_X\Phi &= \nabla_X\alpha \otimes \Psi + \alpha \otimes \nabla_X\Psi = \tfrac{1}{p+1}\, X \lrcorner (\mathrm{d}\alpha \otimes \Psi) + aX \cdot \Phi \\ &= a\Big(X \cdot \Phi - \tfrac{1}{p+1}\, X \lrcorner (\gamma\cdot \wedge \Phi)\Big) + \tfrac{1}{p+1}\, X \lrcorner \mathrm{d}\Phi\,. \end{aligned}$$ □

We conclude this section with a description of the spinor-valued Killing forms in terms of $\mathsf{Spin}(n_+, n_-)$-invariant first-order operators:

codifferential	$\mathrm{d}^*\colon \Sigma\Omega^p(\mathcal{M}) \to \Sigma\Omega^{p-1}(\mathcal{M})$	given by $\mathrm{p}_1 \circ \nabla$,
covariant exterior differential	$\mathrm{d}\colon \Sigma\Omega^p(\mathcal{M}) \to \Sigma\Omega^{p+1}(\mathcal{M})$	given by $\mathrm{p}_2 \circ \nabla$,
twisted Dirac operator	$\mathrm{D}\colon \Sigma\Omega^p(\mathcal{M}) \to \Sigma\Omega^{p}(\mathcal{M})$	given by $\mathrm{p}_3 \circ \nabla$,
Twistor operator	$\mathrm{T}\colon \Sigma\Omega^p(\mathcal{M}) \to \Sigma\Omega^{p,1}(\mathcal{M})$	given by projecting ∇ on the twistor module .

Here $\Sigma\Omega^p(\mathcal{M})$ denotes the space of spinor-valued differential forms of degree p and $\Sigma\Omega^{p,1}(\mathcal{M})$ the space of tensor-spinor fields corresponding to the representation $\mathsf{SA}^{p,1}$. The equation (22) is then equivalent to the system of three differential equations

$$\begin{aligned} &\mathrm{T}\,\Phi = 0\,, \quad \mathrm{d}^*\Phi = a\gamma^*_\cdot \lrcorner \Phi\,, \\ \text{and} \qquad &\mathrm{D}\,\Phi = \tfrac{1}{p+1}\Big(-ap(n+2)\Phi - \gamma\cdot \wedge \mathrm{d}^*\Phi + \gamma^*_\cdot \lrcorner \mathrm{d}\Phi\Big)\,. \end{aligned} \tag{25}$$

In particular, we have

Proposition 4. *Killing spinor-valued forms are in the kernel of the twistor operator, i.e., they are a special case of conformal Killing spinor-valued forms.*

For detailed computations and further discussion, see [17].

4. The cone construction

Metric cone. The ε-metric cone over pseudo-Riemannian manifold $(\mathcal{M}, g)$ is the warped product $(\overline{\mathcal{M}} = \mathcal{M} \times \mathbb{R}_+, \overline{g} = r^2 g + \varepsilon\, \mathrm{d}r^2)$, where r is the coordinate function on $\mathbb{R}_+$ and $\varepsilon = \pm 1$. Note that the signature of $\overline{g}$ is $(\overline{n}_+, \overline{n}_-)$ with

$$\overline{n}_+ = n_+ + (1+\varepsilon)/2 \quad \text{and} \quad \overline{n}_- = n_- + (1-\varepsilon)/2\,. \tag{26}$$

The canonical projections $\mathrm{p}_1\colon \overline{\mathcal{M}} \to \mathcal{M}$ and $\mathrm{p}_2\colon \overline{\mathcal{M}} \to \mathbb{R}_+$ naturally split the tangent bundle of $\overline{\mathcal{M}}$ as a direct sum of pull-back bundles

$$\mathcal{T}(\overline{\mathcal{M}}) = \mathrm{p}_1^*\mathcal{T}(\mathcal{M}) \oplus \mathrm{p}_2^*\mathcal{T}(\mathbb{R}_+). \tag{27}$$

We associate to a vector field $X \in \mathcal{X}(\mathcal{M})$ or to a p-form $\alpha \in \Omega^p(\mathcal{M})$ a vector field $\overline{X} \in \mathcal{X}(\overline{\mathcal{M}})$ or a p-form $\overline{\alpha} \in \Omega^p(\overline{\mathcal{M}})$, respectively, by

$$\overline{X} = \tfrac{1}{r}\, \mathrm{p}_1^*(X), \quad \overline{\alpha} = r^p\, \mathrm{p}_1^*(\alpha)\,. \tag{28}$$

We also denote by ∂_r and $\mathrm{d}r$ the pull-backs to $\overline{\mathcal{M}}$ of the canonical unit vector field and coordinate 1-form on $\mathbb{R}_+$, respectively.

In order to express the covariant derivative $\overline{\nabla}$ induced by the Levi-Civita connection on $\overline{\mathcal{M}}$ in terms of ∇ on $\mathcal{M}$, we first compute the commutators

$$[\overline{X}, \overline{Y}] = \tfrac{1}{r}\overline{[X,Y]}\,, \quad [\overline{X}, \partial_r] = \tfrac{1}{r}\,\overline{X}\,, \quad \text{for all} \quad X, Y \in \mathcal{X}(\mathcal{M})\,. \tag{29}$$

Subsequently, we have

$$\begin{aligned} \overline{\nabla}_{\overline{X}}\overline{Y} &= \tfrac{1}{r}(\overline{\nabla_X Y} - \varepsilon g(X,Y)\,\partial_r)\,, \qquad & \overline{\nabla}_{\partial_r}\overline{X} &= 0, \\ \overline{\nabla}_{\overline{X}}\partial_r &= \tfrac{1}{r}\,\overline{X}\,, & \overline{\nabla}_{\partial_r}\partial_r &= 0\,, \end{aligned} \tag{30}$$

and dually for $\alpha \in \Omega^p(\mathcal{M})$

$$\begin{aligned} \overline{\nabla}_{\overline{X}}\overline{\alpha} &= \tfrac{1}{r}(\overline{\nabla_X\alpha} - \mathrm{d}r \wedge \overline{(X \lrcorner\, \alpha)})\,, \qquad & \overline{\nabla}_{\partial_r}\overline{\alpha} &= 0\,, \\ \overline{\nabla}_{\overline{X}}(\mathrm{d}r) &= \tfrac{1}{r}\,\varepsilon\overline{X}^*\,, & \overline{\nabla}_{\partial_r}(\mathrm{d}r) &= 0\,. \end{aligned} \tag{31}$$

Remark. It follows from the comparison

$$\widetilde{X} = \mathrm{p}_1^*(X) = r\overline{X}\,, \qquad \widetilde{\alpha} = \mathrm{p}_1^*(\alpha) = \tfrac{1}{r^p}\,\overline{\alpha}\,,$$

that our formulas (30), (31) are equivalent to the frequently used formulas

$$\begin{aligned} \overline{\nabla}_{\widetilde{X}}\widetilde{Y} &= \widetilde{\nabla_X Y} - r\varepsilon g(X,Y)\,\partial_r\,, \qquad & \overline{\nabla}_{\partial_r}\widetilde{X} &= \tfrac{1}{r}\widetilde{X}\,, \\ \overline{\nabla}_{\widetilde{X}}\widetilde{\alpha} &= \widetilde{\nabla_X\alpha} - \tfrac{1}{r}\mathrm{d}r \wedge \widetilde{(X \lrcorner\, \alpha)}\,, & \overline{\nabla}_{\partial_r}\widetilde{\alpha} &= -\tfrac{p}{r}\,\widetilde{\alpha}\,, \end{aligned}$$

cf. [9]. The advantage of our conventions is that the lifts of vector fields and differential forms on the cone are always parallel in the radial direction. Moreover, the inner product of vector fields is preserved.

Let $f = (X_1, \dots, X_n)$ be a local orthonormal frame on $\mathcal{M}$ and ω_i^{jk} the corresponding local connection form on $\mathcal{M}$,[2]

$$\nabla_{X_i} Y = \sum_{j,k=1}^{n} \omega_i^{jk}\, g(Y, X_j)\, X_k\,, \quad \text{for all} \quad Y \in \mathcal{X}(\mathcal{M})\,. \tag{32}$$

Then $\overline{f} = (\overline{X}_1, \dots, \overline{X}_n, \partial_r)$ is a local orthonormal frame on $\overline{\mathcal{M}}$ and from (30) we get the corresponding local connection form $\overline{\omega}_i^{jk}$ on $\overline{\mathcal{M}}$,

$$\begin{aligned} \overline{\omega}_i^{jk} &= \tfrac{1}{r}\,\omega_i^{jk}\,, \qquad & \overline{\omega}_i^{j(n+1)} &= -\overline{\omega}_i^{(n+1)j} = -\tfrac{1}{r}\,\varepsilon\,\delta_i^j\,, \\ \overline{\omega}_{n+1}^{jk} &= 0\,, & \overline{\omega}_{n+1}^{j(n+1)} &= -\overline{\omega}_{n+1}^{(n+1)j} = 0\,, \end{aligned} \tag{33}$$

using the fact $\overline{g}(\partial_r, \partial_r) = \varepsilon$.

[2]Note that we have raised the index j, which corresponds to an isomorphism between the Lie algebra $\mathfrak{so}(n_+, n_-)$ and the space of skew-symmetric bivectors. This is convenient for subsequent computations of the spin connection (cf. Lemma 5) without explicit sign changes depending on the signature (n_+, n_-).

Spinors on the cone. The Clifford algebra $\mathsf{Cl}(n_+, n_-)$ is a subalgebra of $\mathsf{Cl}(\overline{n}_+, \overline{n}_-)$ and similarly the spin group $\mathsf{Spin}(n_+, n_-)$ is a subgroup of $\mathsf{Spin}(\overline{n}_+, \overline{n}_-)$. The corresponding complex spinor spaces S and $\overline{\mathsf{S}}$ can be related as $\mathsf{Cl}(n_+, n_-)$-modules by the isomorphisms

(a) if n is even then $\overline{\mathsf{S}} \cong \mathsf{S}$,

(b) and if n is odd then $\overline{\mathsf{S}} \cong \mathsf{S} \oplus \widehat{\mathsf{S}}$, where $\widehat{\mathsf{S}}$ is a second irreducible complex $\mathsf{Cl}(n_+, n_-)$-module not isomorphic to S.

In both cases there is a unique, up to a $\mathsf{Cl}(n_+, n_-)$-equivariant isomorphism, embedding $\mathsf{S} \subset \overline{\mathsf{S}}$. We also introduce two other modified embeddings $\varphi_\pm \colon \mathsf{S} \to \overline{\mathsf{S}}$, given by

$$\varphi_\pm(\Psi) = (1 \mp \sqrt{\varepsilon}\, e_{n+1}) \cdot \Psi, \tag{34}$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of V and $\{e_1, \dots, e_n, e_{n+1}\}$ is an orthonormal basis of $\overline{\mathsf{V}} = \mathbb{R}^{\overline{n}_+, \overline{n}_-}$. All the subsequent formulas are valid for both choices of the square root sign, so we choose $\sqrt{\varepsilon} = 1$ for $\varepsilon = 1$ and $\sqrt{\varepsilon} = \mathsf{i}$ for $\varepsilon = -1$, respectively. A straightforward computation based on $(e_{n+1}\cdot)^2 = -\varepsilon$ shows

$$\begin{aligned} \varphi_\pm(e_i \cdot \Psi) &= \pm\sqrt{\varepsilon}\, e_i \cdot e_{n+1} \cdot \varphi_\pm(\Psi)\,, \\ \varphi_\pm(e_i \cdot e_j \cdot \Psi) &= e_i \cdot e_j \cdot \varphi_\pm(\Psi)\,, \\ \varphi_+ \circ \varphi_-(\Psi) &= \varphi_- \circ \varphi_+(\Psi) = 2\Psi\,. \end{aligned} \tag{35}$$

In particular, the embeddings $\varphi_\pm$ are $\mathsf{Spin}(n_+, n_-)$-equivariant and injective. In a slightly different notation, this construction can be found in [1], [2, pp. 17–19].

The cone $\overline{\mathcal{M}}$ is clearly homotopy equivalent to $\mathcal{M}$, hence any spin structure on $\mathcal{M}$ determines a unique spin structure on $\overline{\mathcal{M}}$. In more detail, we construct the spin structure $\mathcal{P}_{\mathsf{Spin}}(\overline{\mathcal{M}})$ by taking the pull-back of the spin structure $\mathcal{P}_{\mathsf{Spin}}(\mathcal{M})$ to $\overline{\mathcal{M}}$ and extending the structure group,

$$\mathcal{P}_{\mathsf{Spin}}(\overline{\mathcal{M}}) = \mathrm{p}_1^* \mathcal{P}_{\mathsf{Spin}}(\mathcal{M}) \times_{\mathsf{Spin}(n_+, n_-)} \mathsf{Spin}(\overline{n}_+, \overline{n}_-)\,. \tag{36}$$

This extension is compatible with the above construction of the orthonormal frame $\overline{f}$ from f, namely, if f_s is a lift of f then $\overline{f}_s = \mathrm{p}_1^*(f_s)$ is a lift of $\overline{f}$.

Hence we can reduce the structure group of natural bundles on the cone to $\mathsf{Spin}(n_+, n_-)$, in particular, the spinor bundle is given by

$$\mathcal{S}(\overline{\mathcal{M}}) = \mathrm{p}_1^* \mathcal{P}_{\mathsf{Spin}}(\mathcal{M}) \times_{\mathsf{Spin}(n_+, n_-)} \overline{\mathsf{S}}\,, \tag{37}$$

and the pull-back $\mathrm{p}_1^* \mathcal{S}(\mathcal{M})$ is canonically a subbundle of $\mathcal{S}(\overline{\mathcal{M}})$. Now we use the equivariant embeddings $\varphi_\pm$ and associate to a spinor field $\Psi \in \Sigma(\mathcal{M})$ spinor fields $\overline{\Psi}_\pm \in \Sigma(\overline{\mathcal{M}})$ by

$$\overline{\Psi}_\pm = (1 \mp \sqrt{\varepsilon}\, \partial_r) \cdot \mathrm{p}_1^* \Psi\,. \tag{38}$$

The two choices of the sign yield inequivalent though analogous results and we shall consider both of them.

Lemma 5. *Let Ψ be a spinor field on $\mathcal{M}$ and $\overline{\Psi}_\pm$ the associated spinor fields on the cone $\overline{\mathcal{M}}$. The covariant derivative of $\overline{\Psi}_\pm$ is given by the equations*

$$\overline{\nabla}_{\overline{X}}(\overline{\Psi}_\pm) = \tfrac{1}{r}\overline{\Big(\nabla_X(\Psi) \mp \tfrac{1}{2}\sqrt{\varepsilon}\, X \cdot \Psi\Big)}_\pm\,, \quad \overline{\nabla}_{\partial_r}(\overline{\Psi}_\pm) = 0\,, \tag{39}$$

for all $X \in \mathcal{T}(\mathcal{M})$.

Proof. Let f be a local orthonormal frame field on $\mathcal{M}$ and f_s its lift to a spin frame field. The covariant derivative of Ψ is in general given by

$$\nabla_{X_i}\Psi = \nabla_{X_i}[f_s, s] = \Big[f_s,\ X_i(s) + \tfrac{1}{4}\sum_{j,k}\omega_i^{jk}\, e_j \cdot e_k \cdot s\Big]\,, \tag{40}$$

where s is the S-valued function which corresponds to Ψ with respect to f_s. Next let $\overline{f}$ and $\overline{f}_s$ be the associated frame fields on $\overline{\mathcal{M}}$. We substitute into (40) the formulas (33) for the connection form on $\overline{\mathcal{M}}$ and compute using (35):

$$\begin{aligned}
\overline{\nabla}_{\overline{X}_i}\overline{\Psi}_\pm &= \overline{\nabla}_{\overline{X}_i}\Big[\overline{f}_s, \varphi_\pm(\mathrm{p}_1^* s)\Big] \\
&= \Big[\overline{f}_s,\ \overline{X}_i(\varphi_\pm(\mathrm{p}_1^* s)) + \tfrac{1}{4}\sum_{j,k}\overline{\omega}_i^{jk}\, e_j \cdot e_k \cdot \varphi_\pm(\mathrm{p}_1^* s) \\
&\qquad + \tfrac{1}{2}\sum_j \overline{\omega}_i^{j(n+1)}\, e_j \cdot e_{n+1} \cdot \varphi_\pm(\mathrm{p}_1^* s)\Big] \\
&= \tfrac{1}{r}\Big[\overline{f}_s,\ \mathrm{p}_1^* X_i(\varphi_\pm(\mathrm{p}_1^* s)) + \tfrac{1}{4}\sum_{j,k}\omega_i^{jk}\, e_j \cdot e_k \cdot \varphi_\pm(\mathrm{p}_1^* s) \\
&\qquad - \tfrac{1}{2}\,\varepsilon \sum_j \delta_i^j\, e_j \cdot e_{n+1} \cdot \varphi_\pm(\mathrm{p}_1^* s)\Big] \\
&= \tfrac{1}{r}\Big[\overline{f}_s,\ \varphi_\pm\Big(\mathrm{p}_1^*\Big(X_i(s) + \tfrac{1}{4}\sum_{j,k}\omega_i^{jk}\, e_j \cdot e_k \cdot s \mp \tfrac{1}{2}\sqrt{\varepsilon}\, e_i \cdot s\Big)\Big)\Big] \\
&= \tfrac{1}{r}\overline{\Big(\nabla_{X_i}(\Psi) \mp \tfrac{1}{2}\sqrt{\varepsilon}\, X_i \cdot \Psi\Big)}_\pm\,,
\end{aligned}$$

where the indices i, j, k always run through $1, \dots, n$. The proof of the second equality is trivial. □

Finally, to a spinor-valued p-form $\Phi \in \Sigma\Omega^p(\mathcal{M})$ we associate spinor-valued p-forms $\overline{\Phi}_\pm \in \Sigma\Omega^p(\overline{\mathcal{M}})$ by

$$\overline{\Phi}_\pm = r^p(1 \mp \sqrt{\varepsilon}\,\partial_r) \cdot \mathrm{p}_1^*\Phi\,. \tag{41}$$

Combining (31) and (39) we get

$$\overline{\nabla}_{\overline{X}}\overline{\Phi}_\pm = \tfrac{1}{r}\Big(\overline{(\nabla_X\Phi \mp \tfrac{1}{2}\sqrt{\varepsilon}\, X \cdot \Phi)}_\pm - \mathrm{d}r \wedge \overline{(X \lrcorner \Phi)}_\pm\Big), \qquad \overline{\nabla}_{\partial_r}\overline{\Phi} = 0\,, \tag{42}$$

for all $\Phi \in \Sigma\Omega^p(\mathcal{M})$ and $X \in \mathcal{T}(\mathcal{M})$.

Killing equations and the cone. The rest of this paper is devoted to the main results which establish a correspondence between special solutions of the Killing equations on $\mathcal{M}$ and suitable parallel sections on the cone $\overline{\mathcal{M}}$.

Comparing (39) with (20) we immediately get the correspondence for Killing spinors, c.f. [1] for the Riemannian case and [4] for the general pseudo-Riemannian case.

Corollary 6. *Let Ψ be a spinor field on $\mathcal{M}$. The associated spinor field $\overline{\Psi}_\pm$ on $\overline{\mathcal{M}}$ is parallel if and only if Ψ is a Killing spinor with the Killing number $a = \pm\frac{1}{2}\sqrt{\varepsilon}$.*

The next correspondence holds only for special Killing forms introduced by Tachibana and Yu in [14] by an additional second order condition. Here we present a slightly generalized version of Semmelmann's result from [9] by considering also the case $\varepsilon = -1$.

Definition 7. A special Killing p-form is a Killing p-form α fulfilling

$$\nabla_X(\mathrm{d}\alpha) = bX^* \wedge \alpha, \quad \text{for all} \quad X \in \mathcal{T}(\mathcal{M}), \tag{43}$$

where $b \in \mathbb{R}$ is arbitrary constant.

Proposition 8. *Let α be a p-form on $\mathcal{M}$. The $(p+1)$-form β on $\overline{\mathcal{M}}$ defined by*

$$\beta = \mathrm{d}r \wedge \overline{\alpha} + \tfrac{1}{p+1}\,\overline{\mathrm{d}\alpha} \tag{44}$$

is parallel for $\overline{\nabla}$ if and only α is a special Killing form with constant $b = -\varepsilon(p+1)$.

Proof. We compute the covariant derivative of β using (31):

$$\overline{\nabla}_{\overline{X}}\beta = \tfrac{1}{r}\Big(\mathrm{d}r \wedge \Big(\overline{\nabla_X\alpha - \tfrac{1}{p+1}\, X \lrcorner\, \mathrm{d}\alpha}\Big) + \overline{\tfrac{1}{p+1}\nabla_X(\mathrm{d}\alpha) + \varepsilon X^* \wedge \alpha}\Big),$$
$$\overline{\nabla}_{\partial_r}\beta = 0.$$

The claim now follows from (16) and (43). $\square$

Analogously to the case of forms itself, the correspondence for Killing spinor-valued forms holds only for special Killing spinor-valued forms. In this case the second order condition which fits the cone construction has rather complicated form.

Definition 9. A special Killing spinor-valued p-form is a Killing spinor-valued p-form Φ fulfilling

$$\begin{aligned}\nabla_X(\mathrm{d}\Phi) &= bX^* \wedge \Phi + a\Big(X \cdot \mathrm{d}\Phi + \tfrac{1}{p+1}\,\gamma\cdot \wedge (X \lrcorner\, \mathrm{d}\Phi)\Big)\\ &\quad + a^2\Big(2X^* \wedge \Phi + \tfrac{2p+1}{p+1}\,\gamma\cdot \wedge (X \cdot \Phi) + \tfrac{1}{p+1}\,\gamma\cdot \wedge (\gamma\cdot \wedge (X \lrcorner\, \Phi))\Big),\end{aligned} \tag{45}$$

for all $X \in \mathcal{T}(\mathcal{M})$, where $a \in \mathbb{C}$ is the Killing number of Φ and $b \in \mathbb{R}$ is another arbitrary constant.

The exact form of all the terms containing $\gamma\cdot$ in both defining equations (22) and (45) is prescribed purely by algebraic constraints deduced from the decomposition (14). For an illustration of the algebraic constraints in the case of primitive spinor-valued forms, see Lemma 11.

Proposition 10. *Let Φ be a spinor-valued p-form on $\mathcal{M}$. The spinor-valued $(p+1)$-form $\Xi_\pm$ on the cone $\overline{\mathcal{M}}$ defined by*

$$\Xi_\pm = \mathrm{d}r \wedge \overline{\Phi}_\pm \mp \tfrac{1}{2(p+1)}\sqrt{\varepsilon}\,\overline{\gamma\cdot \wedge \Phi}_\pm + \tfrac{1}{p+1}\,\overline{\mathrm{d}\Phi}_\pm \tag{46}$$

is parallel if and only if Φ *is special Killing with Killing number* $a = \pm\frac{1}{2}\sqrt{\varepsilon}$ *and constant* $b = -\varepsilon(p+1)$.

Proof. We compute the covariant derivative of $\Xi_\pm$ using (31) and (42):

$$\begin{aligned}
\overline{\nabla}_{\overline{X}}\Xi_\pm &= \tfrac{1}{r}\Big(\Big(\overline{(\varepsilon X^* \wedge \Phi)}_\pm + \mathrm{d}r \wedge \overline{\Big(\nabla_X \Phi \mp \tfrac{1}{2}\sqrt{\varepsilon}\, X \cdot \Phi\Big)}_\pm\Big)\\
&\quad \mp \tfrac{1}{2(p+1)}\sqrt{\varepsilon}\Big(\overline{(\nabla_X(\gamma\cdot \wedge \Phi) \mp \tfrac{1}{2}\sqrt{\varepsilon}\, X \cdot (\gamma\cdot \wedge \Phi))}_\pm - \mathrm{d}r \wedge \overline{(X \lrcorner (\gamma\cdot \wedge \Phi))}_\pm\Big)\\
&\quad + \tfrac{1}{p+1}\Big(\overline{(\nabla_X(\mathrm{d}\Phi) \mp \tfrac{1}{2}\sqrt{\varepsilon}\, X \cdot \mathrm{d}\Phi)}_\pm - \mathrm{d}r \wedge \overline{(X \lrcorner \mathrm{d}\Phi)}_\pm\Big)\Big),\\
\overline{\nabla}_{\partial_r}\Xi_\pm &= 0\,.
\end{aligned}$$

We now separate the radial and tangential components in the first equation of the last display and get that $\Xi_\pm$ is parallel if and only if for all $X \in \mathcal{T}(\mathcal{M})$

$$\begin{aligned}
\nabla_X \Phi &= \pm\tfrac{1}{2}\sqrt{\varepsilon}\Big(X \cdot \Phi - \tfrac{1}{p+1} X \lrcorner (\gamma\cdot \wedge \Phi)\Big) + \tfrac{1}{p+1}\, X \lrcorner \mathrm{d}\Phi\,,\\
\nabla_X(\mathrm{d}\Phi) &= -\varepsilon(p+1)\, X^* \wedge \Phi \pm \tfrac{1}{2}\sqrt{\varepsilon}\, X \cdot \mathrm{d}\Phi \pm \tfrac{1}{2}\sqrt{\varepsilon}\,\gamma\cdot \wedge \nabla_X \Phi\\
&\quad - \tfrac{1}{4}\,\varepsilon\, X \cdot (\gamma\cdot \wedge \Phi)\,.
\end{aligned}$$

Next we substitute the first equation into the second one and further rearrange using the relations (2):

$$\begin{aligned}
\nabla_X(\mathrm{d}\Phi) &= -\varepsilon(p+1)\, X^* \wedge \Phi \pm \tfrac{1}{2}\sqrt{\varepsilon}\Big(X \cdot \mathrm{d}\Phi + \tfrac{1}{p+1}\gamma\cdot \wedge (X \lrcorner \mathrm{d}\Phi)\Big)\\
&\quad + \tfrac{1}{4}\,\varepsilon\,\Big(\gamma\cdot \wedge \Big(X \cdot \Phi - \tfrac{1}{p+1} X \lrcorner (\gamma\cdot \wedge \Phi)\Big) - X \cdot (\gamma\cdot \wedge \Phi)\Big)\\
&= -\varepsilon(p+1)\, X^* \wedge \Phi \pm \tfrac{1}{2}\sqrt{\varepsilon}\Big(X \cdot \mathrm{d}\Phi + \tfrac{1}{p+1}\gamma\cdot \wedge (X \lrcorner \mathrm{d}\Phi)\Big)\\
&\quad + \tfrac{1}{4}\,\varepsilon\Big(2X^* \wedge \Phi + \tfrac{2p+1}{p+1}\,\gamma\cdot \wedge (X \cdot \Phi) + \tfrac{1}{p+1}\gamma\cdot \wedge (\gamma\cdot \wedge (X \lrcorner \Phi))\Big)\,.
\end{aligned}$$

The claim now follows from (22) and (45). $\square$

Let us recall the notion of primitive Killing spinor-valued p-form, see (9). The above correspondence applies also to this case, and we show that it maps primitive spinor-valued forms back to primitive spinor-valued forms.

Lemma 11. *Let* Φ *be a primitive Killing spinor-valued* p*-form on* $\mathcal{M}$ *with Killing number* a*. Then it holds*

$$\gamma^*_\cdot \lrcorner \mathrm{d}\Phi = -a(n+2)\Phi\,. \tag{47}$$

In particular, we note that $\mathrm{d}\Phi$ *does not need to be primitive.*

Proof. First recall that $\nabla(\gamma\cdot) = 0$. Hence the hypothesis implies $\nabla_X \Phi$ is primitive for all $X \in \mathcal{T}(\mathcal{M})$ and we compute using (2), (7) and (22):

$$
\begin{aligned}
0 &= \nabla_X(\gamma^{*}\!\cdot \lrcorner\, \Phi) = \gamma^{*}\!\cdot \lrcorner\, \nabla_X \Phi \\
&= a\gamma^{*}\!\cdot \lrcorner \Big(X\cdot\Phi - \tfrac{1}{p+1}\, X \lrcorner (\gamma\cdot \wedge \Phi) \Big) + \tfrac{1}{p+1}\, \gamma^{*}\!\cdot \lrcorner\, (X \lrcorner \mathrm{d}\Phi) \\
&= a\Big(-2X \lrcorner \Phi - X\cdot(\gamma^{*}\!\cdot \lrcorner\, \Phi) + \tfrac{1}{p+1}\, X \lrcorner (\gamma^{*}\!\cdot \lrcorner (\gamma\cdot \wedge \Phi)) \Big) \\
&\quad - \tfrac{1}{p+1}\, X \lrcorner (\gamma^{*}\!\cdot \lrcorner\, \mathrm{d}\Phi) \\
&= a\Big(-2X \lrcorner \Phi - \tfrac{n-2p}{p+1}\, X \lrcorner \Phi + \tfrac{1}{p+1}\, X \lrcorner (\gamma\cdot \wedge (\gamma^{*}\!\cdot \lrcorner\, \Phi)) \Big) \\
&\quad - \tfrac{1}{p+1}\, X \lrcorner (\gamma^{*}\!\cdot \lrcorner\, \mathrm{d}\Phi) \\
&= -\tfrac{1}{p+1}\, X \lrcorner \left(a(n+2)\Phi + \gamma^{*}\!\cdot \lrcorner\, \mathrm{d}\Phi \right),
\end{aligned}
$$

The claim now follows. $\square$

Lemma 12. *Let Φ be a spinor-valued p-form on $\mathcal{M}$ and $\overline{\Phi}_\pm$ the associated spinor-valued p-forms on the cone $\overline{\mathcal{M}}$. Then*

$$
\overline{\gamma}^{*}\!\cdot \lrcorner\, \overline{\Phi}_\pm = \pm\sqrt{\varepsilon}\, \partial_r \cdot \overline{(\gamma^{*}\!\cdot \lrcorner\, \Phi)}_\pm\,. \tag{48}
$$

Proof. We can relate the orthogonal duals of the Clifford multiplication 1-forms on $\mathcal{M}$ and $\overline{\mathcal{M}}$, respectively, by

$$
\overline{\gamma}^{*}\!\cdot \lrcorner\, \mathrm{p}_1^{*}\Phi = \mathrm{p}_1^{*}(\gamma^{*}\!\cdot \lrcorner\, \Phi) + \varepsilon\partial_r \cdot (\partial_r \lrcorner\, \mathrm{p}_1^{*}\Phi) = \mathrm{p}_1^{*}(\gamma^{*}\!\cdot \lrcorner\, \Phi)\,.
$$

Now we substitute (41) and compute using (2) and the fact $(\partial_r\cdot)^2 = -\varepsilon$,

$$
\begin{aligned}
\overline{\gamma}^{*}\!\cdot \lrcorner\, \overline{\Phi}_\pm &= r^p\, \overline{\gamma}^{*}\!\cdot \lrcorner\, (1 \mp \sqrt{\varepsilon}\, \partial_r)\cdot \mathrm{p}_1^{*}\Phi \\
&= r^p((1 \pm \sqrt{\varepsilon}\, \partial_r)\cdot(\overline{\gamma}^{*}\!\cdot \lrcorner\, \mathrm{p}_1^{*}\Phi) \pm 2\partial_r \lrcorner\, \mathrm{p}_1^{*}\Phi) \\
&= r^p(1 \pm \sqrt{\varepsilon}\, \partial_r)\cdot(\overline{\gamma}^{*}\!\cdot \lrcorner\, \mathrm{p}_1^{*}\Phi) \\
&= \pm\sqrt{\varepsilon}\, r^p\, \partial_r \cdot (1 \mp \sqrt{\varepsilon}\, \partial_r)\cdot(\overline{\gamma}^{*}\!\cdot \lrcorner\, \mathrm{p}_1^{*}\Phi) \\
&= \pm\sqrt{\varepsilon}\, r^p\, \partial_r \cdot (1 \mp \sqrt{\varepsilon}\, \partial_r)\cdot \mathrm{p}_1^{*}(\gamma^{*}\!\cdot \lrcorner\, \Phi) \\
&= \pm\sqrt{\varepsilon}\, \partial_r \cdot \overline{(\gamma^{*}\!\cdot \lrcorner\, \Phi)}_\pm\,,
\end{aligned}
$$

proving the claim. $\square$

Proposition 13. *Let Φ be a primitive Killing spinor-valued p-form on $\mathcal{M}$ with Killing number $a = \pm\frac{1}{2}\sqrt{\varepsilon}$. Then the spinor-valued differential $(p+1)$-form $\Xi_\pm$ on the cone $\overline{\mathcal{M}}$ constructed in (46) is primitive as well.*

Proof. By (2), (7), (47) and (48), we calculate:

$$\begin{aligned}
\overline{\gamma}^{*}\lrcorner\, \Xi_{\pm} &= \overline{\gamma}^{*}\lrcorner \Big(\mathrm{d}r \wedge \overline{\Phi}_{\pm} \mp \tfrac{1}{2(p+1)}\sqrt{\varepsilon}\,\overline{\gamma\cdot \wedge \Phi}_{\pm} + \tfrac{1}{p+1}\,\overline{\mathrm{d}\Phi}_{\pm}\Big)\\
&= \varepsilon\partial_r \cdot \overline{\Phi}_{\pm} \mp \sqrt{\varepsilon}\,\mathrm{d}r \wedge \partial_r \cdot \overline{(\gamma^{*}\lrcorner\, \Phi)}_{\pm}\\
&\quad - \tfrac{1}{2(p+1)}\,\varepsilon\partial_r \cdot \overline{(\gamma^{*}\lrcorner\,(\gamma\cdot \wedge \Phi))}_{\pm} \pm \tfrac{1}{p+1}\sqrt{\varepsilon}\,\partial_r \cdot \overline{(\gamma^{*}\lrcorner\, \mathrm{d}\Phi)}_{\pm}\\
&= \varepsilon\partial_r \cdot \overline{\Phi}_{\pm} + \tfrac{n-2p}{2(p+1)}\,\varepsilon\partial_r \cdot \overline{\Phi}_{\pm}\\
&\quad - \tfrac{1}{2(p+1)}\,\varepsilon\partial_r \cdot \overline{(\gamma\cdot \wedge (\gamma^{*}\lrcorner\, \Phi)}_{\pm} - \tfrac{n+2}{2(p+1)}\,\varepsilon\partial_r \cdot \overline{\Phi}_{\pm} = 0\,.
\end{aligned}$$

The proof is complete. □

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