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ASYMPTOTIC REPRESENTATIONS OF SOLUTIONS OF THE NONAUTONOMOUS ORDINARY DIFFERENTIAL n-TH ORDER EQUATIONS

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Abstract. Asymptotic representations of some classes of solutions of nonautonomous ordinary differential n-th order equations which somewhat are close to linear equations are established.

1. Introduction

We consider the differential equation

$$y^{(n)} = \alpha_0 p(t) y |\ln |y||^{\sigma} \,, \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p\colon [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \leq +\infty$[1].

A solution y of the equation (1.1), which is defined and which is not equal to zero on an interval $[t_y, \omega[\subset [a, \omega[$, is called a $P_\omega(\lambda_0)$-solution if it satisfies the conditions:

$$\lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{either } 0, \\ \text{or } \pm\infty \end{cases} \quad (k = \overline{0, n-1}) \,, \quad \lim_{t\uparrow\omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t) y^{(n-2)}(t)} = \lambda_0 \,. \tag{1.2}$$

We notice that differential equation (1.1) is a specific case of the differential equation of more general form

$$y^{(n)} = \alpha_0 p(t) \varphi(y) \,, \tag{1.3}$$

where α_0 and p are the same as in the equation (1.1) and $\varphi\colon \Delta_{Y_0} \to]0, +\infty[$ is a continuous and regularly varying function when $y \to Y_0$ of the order γ, Y_0 is equal to either zero or $\pm\infty$, Δ_{Y_0} is some one-sided neighborhood of Y_0. Here, by virtue of the definition of the properly varying function (see, for example, the monograph of E. Seneta [6, Ch. 1]), the function φ has the following representation

$$\varphi(y) = |y|^{\gamma} L(y) \,,$$

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[1] We suppose that $a > 1$ when $\omega = +\infty$ and $\omega - a < 1$ when $\omega < +\infty$.

where L is a slowly-varying function when $y \to Y_0$. The function $L(y) = |\ln|y||^\sigma$ might be the specific case of it.

In the paper [4], for the equation (1.3) the conditions for the existence and the asymptotics when $t \uparrow \omega$ for all possible types of the $P_\omega(\lambda_0)$-solutions were obtained. However, the results of that paper do not cover the situation when $\gamma = 1$, particularly, the differential equation (1.1).

For the equation (1.1), in the paper [2], the conditions for the existence of the $P_\omega(\lambda_0)$-solutions when $\lambda_0 \in \mathbb{R} \setminus \left\{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}\right\}$ are established, as well as the asymptotics for such solutions and their derivatives up to the order $n-1$ inclusive. Also the quantity of such solutions was established.

In specific cases, when $\lambda_0 = \frac{n-i-1}{n-i}$ $(i = \overline{1, n-1})$, by means of the results of V.M. Evtukhov [1, Ch.3, §10, pp. 142–144], the a priori asymptotic properties of the $P_\omega(\lambda_0)$-solutions of the equation (1.1) can be obtained. In the aim of that, we need the following notation

$$\pi_\omega(t) = \begin{cases} t\,, & \text{if} \quad \omega = +\infty, \\ t - \omega\,, & \text{if} \quad \omega < +\infty\,. \end{cases}$$

Lemma 1.1. 1. *If $n > 2$, then each $P_\omega\left(\frac{n-2}{n-1}\right)$-solution of the differential equation* (1.1) *when $t \uparrow \omega$ admits asymptotic representations*

$$y'(t) = o\Big(\frac{y(t)}{\pi_\omega(t)}\Big), \quad y^{(k)}(t) \sim (-1)^{k-1} \frac{(k-1)!}{[\pi_\omega(t)]^{k-1}}\, y'(t) \quad (k = 2, \dots, n)\,.$$

2. *If $i \in \{2, \dots, n-2\}$ and $n > i+1$, then each $P_\omega\left(\frac{n-i-1}{n-i}\right)$-solution of the differential equation* (1.1) *when $t \uparrow \omega$ admits asymptotic representations*

(1.4)
$$y^{(k-1)}(t) \sim \frac{[\pi_\omega(t)]^{i-k}}{(i-k)!}\, y^{(i-1)}(t) \quad (k = 1, \dots, i-1), \quad y^{(i)}(t) = o\Big(\frac{y^{(i-1)}(t)}{\pi_\omega(t)}\Big)\,,$$

(1.5) $$y^{(k)}(t) \sim (-1)^{k-i} \frac{(k-i)!}{[\pi_\omega(t)]^{k-i}}\, y^{(i)}(t) \quad (k = i+1, \dots, n)\,.$$

3. *If $n \geq 2$, then each $P_\omega(0)$-solution of the differential equation* (1.1) *when $t \uparrow \omega$ admits asymptotic representations*

$$y^{(k-1)}(t) \sim \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}\, y^{(n-2)}(t) \quad (k = 1, \dots, n-2), \quad y^{(n-1)}(t) = o\Big(\frac{y^{(n-2)}(t)}{\pi_\omega(t)}\Big)\,,$$

and in case of the existence of the limit $\lim\limits_{t \uparrow \omega} \frac{\pi_\omega(t) y^{(n)}(t)}{y^{(n-1)}(t)}$ *(finite or equal to $\pm\infty$), the following representation is valid*

$$y^{(n)}(t) \sim -\frac{y^{(n-1)}(t)}{\pi_\omega(t)} \quad \textit{when} \quad t \uparrow \omega\,.$$

From this lemma follows that while considering asymptotic behavior of the $P_\omega\left(\frac{n-i-1}{n-i}\right)$ $(i = \overline{1, n-1})$-solutions of the equation (1.1), we must separately consider the situations when: 1) $n > 2$ and $i = 1$ ($\lambda_0 = \frac{n-2}{n-1}$); 2) $1 < i < n-1$; 3) $n \geq 2$ and $i = n-1$ ($\lambda_0 = 0$).

The aim of the present paper is to derive necessary and sufficient conditions for the existence of $P_\omega\left(\frac{n-i-1}{n-i}\right)$-solutions of the equation (1.1) when $i \in \{2, \dots, n-2\}$ and $n > i+1$. When $t \uparrow \omega$, we also establish asymptotic representations for all such solutions and their derivatives up to the order $n-1$ inclusive.

2. Auxiliary notation

Besides the first statement of the Lemma 1.1, we also require one known result for the system of quasilinear differential equations about the existence of the solutions that tend to zero when $t \uparrow \omega$

$$\left\{\begin{aligned} v_k' &= h(t)\Big[f_k(t, v_1, \dots, v_n) + \sum_{i=1}^{n} c_{ki} v_i\Big] \;\; (k = \overline{1, n-1}), \\ v_n' &= H(t)\Big[f_n(t, v_1, \dots, v_n) + \sum_{i=1}^{n} c_{ni} v_i\Big], \end{aligned}\right. \tag{2.1}$$

where $c_{ki} \in \mathbb{R}$ $(k, i = \overline{1, n})$, h, $H\colon\ [t_0, \omega[\to \mathbb{R} \setminus \{0\}$ are continuously differentiable functions, $f_k\colon\ [t_0, \omega[\times \mathbb{R}^n$ $(k = \overline{1, n})$ are continuous functions that satisfy the conditions

$$\lim_{t \uparrow \omega} f_k(t, v_1, \dots, v_n) = 0 \quad \text{uniformly in} \quad (v_1, \dots, v_n) \in \mathbb{R}^n_{\frac{1}{2}}, \tag{2.2}$$

where

$$\mathbb{R}^n_{\frac{1}{2}} = \Big\{(v_1, \dots, v_n) \in \mathbb{R}^n \,:\, |v_i| \leq \frac{1}{2} \;\; (i = \overline{1, n})\Big\}.$$

By virtue of the Theorem 2.6 from the paper of V.M. Evtukhov and A.M. Samoilenko [3], the following statement is valid for the system of equations (2.1).

Lemma 2.1. *Let functions h and H satisfy the conditions*

$$\lim_{t \uparrow \omega} \frac{H(t)}{h(t)} = 0, \quad \int_{t_0}^{\omega} H(\tau)\, d\tau = \pm\infty, \quad \lim_{t \uparrow \omega} \frac{1}{H(t)} \left(\frac{H(t)}{h(t)}\right)' = 0. \tag{2.3}$$

Moreover, let for the matrixes $C_n = (c_{ki})_{k,i=1}^{n}$ and $C_{n-1} = (c_{ki})_{k,i=1}^{n-1}$, $\det C_n \neq 0$, and C_{n-1} does not have eigenvalues with zero real part. Then the system of differential equations (2.1) has at least one solution $(v_k)_{k=1}^{n}\colon\ [t_1, \omega[\to \mathbb{R}^n_{\frac{1}{2}}$ $(t_1 \in [t_0, \omega[)$ that tends to zero when $t \uparrow \omega$. Moreover, if inequality $H(t)\,(\det C_n)\,(\det C_{n-1}) > 0$ is valid in the interval $[t_0, \omega[$, there exists m-parametric family of such solutions if among the eigenvalues of the matrix C_{n-1}, there exists m eigenvalues (with regard of multiplicities) whose real parts have sign opposite to the sign of the function $h(t)$; if opposite inequality is valid, there exists $m+1$-parametric family.

3. Main results

In order to formulate the main result, let us introduce auxiliary functions

$$J_A(t)=\int_A^t p(\tau)\pi_\omega^{n-2}(\tau)\left|\ln|\pi_\omega(\tau)|\right|^\sigma\,d\tau,\qquad I(t)=\int_a^t J_A(\tau)\,d\tau\,,$$

where

$$A=\begin{cases} a\,, & \text{if} \quad \int\limits_a^\omega p(\tau)|\pi_\omega(\tau)|^{n-2}\left|\ln|\pi_\omega(\tau)|\right|^\sigma\,d\tau=+\infty\,,\\ \omega\,, & \text{if} \quad \int\limits_a^\omega p(\tau)|\pi_\omega(\tau)|^{n-2}\left|\ln|\pi_\omega(\tau)|\right|^\sigma\,d\tau<+\infty\,. \end{cases}$$

Theorem 3.1. *Let $i\in\{2,\dots,n-2\}$ $(n>i+1\geq 3)$. Then, for the existence of $P_\omega\left(\frac{n-i-1}{n-i}\right)$-solutions of the differential equation* (1.1)*, it is necessary and sufficient that the following conditions are valid*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_A'(t)}{J_A(t)}=-1\,,\quad \lim_{t\uparrow\omega}I(t)=\pm\infty\,,\quad \lim_{t\uparrow\omega}\pi_\omega(t)J_A(t)=0\,. \tag{3.1}$$

Moreover, when $t\uparrow\omega$, each solution of that kind admits the asymptotic representations

$$\frac{y^{(k-1)}(t)}{y^{(i-1)}(t)}=\frac{[\pi_\omega(t)]^{i-k}}{(i-k)!}[1+o(1)]\quad (k=\overline{1,i-1}), \tag{3.2}$$

$$\ln|y^{(i-1)}(t)|=\frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!}I(t)[1+o(1)] \tag{3.3}$$

$$\frac{y^{(k)}(t)}{y^{(i-1)}(t)}=\frac{\alpha_0(-1)^{n-k-1}(k-i)!|i-1|^\sigma}{(i-1)!(n-i)!}\,\frac{J_A(t)}{\pi_\omega^{k-i}(t)}[1+o(1)] \qquad (k=\overline{i,n-1}). \tag{3.4}$$

Moreover, if the conditions (3.1) *are valid, for the differential equation* (1.1) *in case $\omega=+\infty$, there exists i-parametric family of solutions that admit asymptotic representations* (3.2)–(3.4) *as $t\uparrow\omega$, and in case $\omega<+\infty$, there exists* $(n-i+1)$*-parametric family of solutions with such representations.*

Proof of the Theorem 3.1. *Necessity.* Let $y\colon [t_y,\omega[\ \to\mathbb{R}$ be an arbitrary $P_\omega\left(\frac{n-i-1}{n-i}\right)$-solution of the equation (1.1). Then, by virtue of the definition of $P_\omega(\lambda_0)$-solution, there exists $t_0\in[t_y,\omega[$ so that $\ln|y(t)|\neq 0$ in the interval $[t_0,\omega[$, and, by virtue of the second statement of the Lemma 1.1, the asymptotic representations (1.4), (1.5) are valid. According to the first asymptotic representations from (1.4)

$$y(t)\sim\frac{\pi_\omega^{i-1}(t)}{(i-1)!}y^{(i-1)}(t)\,,\qquad y'(t)\sim\frac{\pi_\omega^{i-2}(t)}{(i-2)!}y^{(i-1)}(t)\quad\text{as}\quad t\uparrow\omega\,.$$

This implies that

$$\frac{y'(t)}{y(t)}\sim\frac{i-1}{\pi_\omega(t)}\quad\text{as}\quad t\uparrow\omega\,,$$

and therefore

$$\ln|y(t)| \sim (i-1)\ln|\pi_\omega(t)| \quad \text{as} \quad t \uparrow \omega\,.$$

By virtue of the asymptotic representations from (1.1), we have

(3.5)

$$y^{(n)}(t) = \frac{\alpha_0}{(i-1)!} p(t)\pi_\omega^{i-1}(t)\,|(i-1)\ln|\pi_\omega(t)||^\sigma\, y^{(i-1)}(t)\,[1+o(1)] \quad \text{as} \quad t \uparrow \omega\,.$$

In its turn, according to asymptotic representations (1.5)

$$y^{(n)}(t) \sim (-1)^{n-i}\frac{(n-i)!}{\pi_\omega^{n-i}(t)} y^{(i)}(t)\,, \quad y^{(i+1)}(t) \sim -\frac{y^{(i)}(t)}{\pi_\omega(t)} \quad \text{as} \quad t \uparrow \omega\,. \tag{3.6}$$

Thus, from (3.5) follows that

$$\frac{y^{(i)}(t)}{y^{(i-1)}(t)} = \frac{\alpha_0(-1)^{n-i}}{(i-1)!(n-i)!}\, p(t)\pi_\omega^{n-1}(t)\,|(i-1)\ln|\pi_\omega(t)||^\sigma\,[1+o(1)] \quad \text{as} \quad t \uparrow \omega \tag{3.7}$$

and

$$\frac{y^{(i+1)}(t)}{y^{(i-1)}(t)} = \frac{\alpha_0(-1)^{n-i-1}}{(i-1)!(n-i)!}\, p(t)\pi_\omega^{n-2}(t)\,|(i-1)\ln|\pi_\omega(t)||^\sigma\,[1+o(1)] \quad \text{as} \quad t \uparrow \omega\,. \tag{3.8}$$

Now, considering the second relation from (1.4) and the second relation from (3.6), we obtain

$$\Big(\frac{y^{(i)}(t)}{y^{(i-1)}(t)}\Big)' = \frac{y^{(i+1)}(t)}{y^{(i-1)}(t)}\Big[1 - \frac{[y^{(i)}(t)]^2}{y^{(i+1)}(t)y^{(i-1)}(t)}\Big]$$
$$= \frac{y^{(i+1)}(t)}{y^{(i-1)}(t)}\Big[1 - \frac{\pi_\omega(t)y^{(i)}(t)}{y^{(i-1)}(t)}\,\frac{y^{(i)}(t)}{\pi_\omega(t)y^{(i+1)}(t)}\Big] \sim \frac{y^{(i+1)}(t)}{y^{(i-1)}(t)} \quad \text{as} \quad t \uparrow \omega\,.$$

Consequently, asymptotic representation (3.8) can be written in the form

$$\Big(\frac{y^{(i)}(t)}{y^{(i-1)}(t)}\Big)' = \frac{\alpha_0(-1)^{n-i-1}}{(i-1)!(n-i)!} p(t)\pi_\omega^{n-2}(t)\,|(i-1)\ln|\pi_\omega(t)||^\sigma\,[1+o(1)] \quad \text{as} \quad t \uparrow \omega\,.$$

By integrating this relation in the interval from t_0 to t, we obtain

$$\frac{y^{(i)}(t)}{y^{(i-1)}(t)} = c_0 + \frac{\alpha_0(-1)^{n-i-1}}{(i-1)!(n-i)!}\int_{t_0}^{t} p(\tau)\pi_\omega^{n-2}(\tau)\,|(i-1)\ln|\pi_\omega(\tau)||^\sigma\,[1+o(1)]\,d\tau\,,$$

where c_0 is some real constant. Or, by means of choosing the integration limit A in the function J_A,

$$\frac{y^{(i)}(t)}{y^{(i-1)}(t)} = c + \frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!} J_A(t)[1+o(1)] \quad \text{as} \quad t \uparrow \omega\,, \tag{3.9}$$

where

$$c = c_0 + \frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!}\int_{t_0}^{A} p(\tau)\pi_\omega^{n-2}(\tau)\,|\ln|\pi_\omega(\tau)||^\sigma\,[1+o(1)]d\tau\,.$$

When $A = a$, the integral from the right part of the relation tends to $\pm\infty$ when $t \uparrow \omega$, (3.9) can be written in the form

$$\frac{y^{(i)}(t)}{y^{(i-1)}(t)} = \frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!} J_A(t)[1+o(1)] \quad \text{as} \quad t \uparrow \omega \tag{3.10}$$

Let us show that in case $A = \omega$, when integral from the right part of the relation (3.9) tends to zero as $t \uparrow \omega$, the representation (3.10) is also valid, i.e. $c = 0$. Indeed, if $c \neq 0$, then from (3.9) follows

$$\frac{y^{(i)}(t)}{y^{(i-1)}(t)} = c + o(1) \quad \text{as} \quad t \uparrow \omega\,.$$

This representation when $\omega = +\infty$, i.e. when $\pi_\omega(t) = t$, contradicts to the last relation from (1.4). When $\omega < +\infty$, we obtain from it after integrating

$$\ln|y^{(i-1)}(t)| = c_1 + o(1) \quad \text{as} \quad t \uparrow \omega \quad (c_1 = \text{const})\,.$$

This contradicts to the first relations from (1.2) (as $k = i-1$).

Consequently, for each case from these two cases considered, the asymptotic representation (3.10) is valid.

From this representation, by virtue of the last relation from (1.4), we obtain the validity of the last condition from (3.1). Besides, from (3.10) and (3.7), it follows that

$$\lim_{t\uparrow\omega} \frac{p(t)\pi_\omega^{n-1}(t)\,|\ln|\pi_\omega(t)||^\sigma}{J_A(t)} = -1\,,$$

i.e. the first condition from (3.1) is valid.

By integrating (3.10) from t_0 to t, we obtain

$$\ln|y^{(i-1)}(t)| = c + \frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!}\int_{t_0}^{t} J_A(\tau)[1+o(1)]\,d\tau\,,$$

where c is some real constant. By virtue of the first relation from (1.2) from the definition of $P_\omega(\lambda_0)$-solution, $\ln|y^{(i-1)}(t)| \to \pm\infty$ as $t \uparrow \omega$. From this relation the validity of the second relation from (3.1) and the validity of (3.3) follows.

When $k = \overline{i, n-1}$, due to the asymptotic representations (1.5) from Lemma 1.1 and asymptotic representation (3.10), we obtain

$$\frac{y^{(k)}(t)}{y^{(i-1)}(t)} = \frac{y^{(k)}(t)}{y^{(i)}(t)}\,\frac{y^{(i)}(t)}{y^{(i-1)}(t)} \sim (-1)^{k-i}\frac{(k-i)!}{\pi_\omega^{k-i}(t)}\,\frac{y^{(i)}(t)}{y^{(i-1)}(t)}$$

$$\sim \frac{\alpha_0(-1)^{n-k-1}(k-i)!|i-1|^\sigma}{(i-1)!(n-i)!}\,\frac{J_A(t)}{\pi_\omega^{k-i}(t)} \quad \text{as} \quad t \uparrow \omega\,.$$

Consequently, when $t \uparrow \omega$, the asymptotic representations (3.4) are valid. The verity of (3.2) follows directly from the Lemma 1.1.

Sufficiency. Let $i \in \{2, \dots, n-2\}$, $n > i+1$ and let the conditions (3.1) be satisfied. We will show that in this case, the differential equation (1.1) has

$P_\omega\left(\frac{n-i-1}{n-i}\right)$-solutions that admit asymptotic representations (3.2)–(3.4) when $t \uparrow \omega$. We will also consider the question about the quantity of solutions with such representations.

Since

$$\pi_\omega(t)J_A(t) = \frac{\pi_\omega(t)J_A(t)}{I(t)} I(t)\,,$$

then from the conditions (3.1) follows that

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_A(t)}{I(t)} = 0\,. \tag{3.11}$$

Besides, by virtue of the l'Hospital rule

$$\lim_{t\uparrow\omega} \frac{I(t)}{\ln|\pi_\omega(t)|} = \lim_{t\uparrow\omega} \pi_\omega(t)J_A(t) = 0\,. \tag{3.12}$$

By introducing the transformation to the equation (1.1)

$$\begin{gathered}
\frac{y^{(k-1)}(t)}{y^{(i-1)}(t)} = \frac{[\pi_\omega(t)]^{i-k}}{(i-k)!}[1+v_k(t)] \qquad (k=\overline{1,i-1})\,, \\
\frac{y^{(k)}(t)}{y^{(i-1)}(t)} = \frac{\alpha_0(-1)^{n-k-1}(k-i)!|i-1|^\sigma}{(i-1)!(n-i)!}\frac{J_A(t)}{\pi_\omega^{k-i}(t)}[1+v_k(t)] \\
(k=\overline{i,n-1})\,, \\
\ln|y^{(i-1)}(t)| = \frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!} I(t)[1+v_n(t)]\,,
\end{gathered} \tag{3.13}$$

we obtain the system of differential equations

$$\begin{aligned}
v_k' &= \frac{i-k}{\pi_\omega(t)}(v_{k+1}-v_k) - \frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!}J_A(t)(1+v_k)(1+v_i) \\
&\qquad (k=\overline{1,i-2})\,, \\
v_{i-1}' &= -\frac{v_{i-1}}{\pi_\omega(t)} - \frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!}J_A(t)(1+v_{i-1})(1+v_i)\,, \\
v_k' &= -\frac{k+1-i}{\pi_\omega(t)}(1+v_{k+1}) + \frac{k-i}{\pi_\omega(t)}(1+v_k) - \frac{J_A'(t)}{J_A(t)}(1+v_k) \\
&\quad - \frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!}J_A(t)(1+v_k)(1+v_i) \quad (k=\overline{i,n-2})\,, \\
v_{n-1}' &= \frac{n-i-1}{\pi_\omega(t)}(1+v_{n-1}) - \frac{J_A'(t)}{J_A(t)}(1+v_{n-1}) \\
&\quad - \frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!}J_A(t)(1+v_{n-1})(1+v_i)
\end{aligned}$$

$$+\frac{(n-i)J_A'(t)}{J_A(t)}(1+v_1)\frac{\left|\ln\left|\frac{\pi_\omega^{i-1}(t)}{(i-1)!}(1+v_1)\right|\right|^\sigma}{|i-1|^\sigma\left|\ln|\pi_\omega(t)|\right|^\sigma}$$
$$\times\left|1+\frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!}\frac{I(t)(1+v_n)}{\ln\left|\frac{\pi_\omega^{i-1}(t)}{(i-1)!}(1+v_1)\right|}\right|^\sigma,$$
$$v_n'=\frac{J_A(t)}{I(t)}(1+v_i)-\frac{J_A(t)}{I(t)}(1+v_n)\,.$$

Assuming that

$$h(t)=\frac{1}{\pi_\omega(t)}\,,\qquad H(t)=\frac{J_A(t)}{I(t)}\,,$$
$$\delta_1(t)=\frac{\alpha_0(-1)^{n-i}|i-1|^\sigma}{(i-1)!(n-i)!}\pi_\omega(t)J_A(t)\,,\qquad \delta_2(t)=\frac{\pi_\omega(t)J_A'(t)}{J_A(t)}+1\,,$$
$$\delta_3(t)=\frac{\alpha_0(-1)^{n-i-1}|i-1|^\sigma}{(i-1)!(n-i)!(i-1)}\frac{I(t)}{\ln|\pi_\omega(t)|}\,,\quad \delta_4(t,v_1)=\frac{\ln\left|\frac{1+v_1}{(i-1)!}\right|}{(i-1)\ln|\pi_\omega(t)|}\,,$$

we can rewrite this system

$$\begin{cases} v_k'=h(t)\left[f_k(t,v_1,\dots,v_n)-(i-k)v_k+(i-k)v_{k+1}\right] \\ \qquad (k=\overline{1,i-2})\,, \\ v_{i-1}'=h(t)\left[f_{i-1}(t,v_1,\dots,v_n)-v_{i-1}\right]\,, \\ v_k'=h(t)\left[f_k(t,v_1,\dots,v_n)+(k-i+1)v_k-(k-i+1)v_{k+1}\right] \\ \qquad (k=\overline{i,n-2})\,, \\ v_{n-1}'=h(t)\left[f_{n-1}(t,v_1,\dots,v_n)-(n-i)v_1+(n-i)v_{n-1}\right]\,, \\ v_n'=H(t)\left[v_i-v_n\right]\,, \end{cases} \tag{3.14}$$

where

$$f_k(t,v_1,\dots,v_n)=\delta_1(t)(1+v_k)(1+v_i)\quad (k=\overline{1,i-1})\,,$$
$$f_k(t,v_1,\dots,v_n)=\delta_1(t)(1+v_k)(1+v_i)-\delta_2(t)(1+v_k)\quad (k=\overline{i,n-2})\,,$$
$$f_{n-1}(t,v_1,\dots,v_n)=\delta_1(t)(1+v_{n-1})(1+v_i)-\delta_2(t)(1+v_{n-1})$$
$$+(n-i)(1+v_1)\left[1+\frac{\pi_\omega(t)J_A'(t)}{J_A(t)}|1+\delta_4(t,v_1)|^\sigma\left|1+\frac{\delta_3(t)(1+v_n)}{1+\delta_4(t,v_1)}\right|^\sigma\right].$$

Here, by virtue of the conditions (3.1) and (3.12),

$$\lim_{t\uparrow\omega}\delta_i(t)=0\qquad (i=1,2,3) \tag{3.15}$$

and

$$\lim_{t\uparrow\omega} \delta_4(t, v_1) = 0 \quad \text{uniformly in} \quad v_1 \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{3.16}$$

Considering these relations, we select the number $t_0 \in\,]a, \omega[$ so that when $t \in [t_0, \omega[$ and $|v_1| \le \frac{1}{2}$, $|v_n| \le \frac{1}{2}$ the following inequalities are valid

$$|\delta_4(t, v_1)| \le \frac{1}{2}\,, \qquad \Big|\frac{\delta_3(t)(1+v_n)}{1+\delta_4(t,v_1)}\Big| \le \frac{1}{2}\,.$$

Further, we consider the system (3.14) on the set

$$\Omega = [t_0, \omega[\times \mathbb{R}^n_{\frac{1}{2}}\,, \quad \text{where} \quad \mathbb{R}^n_{\frac{1}{2}} = \Big\{(v_1, \dots, v_n) \in \mathbb{R}^n : |v_i| \le \frac{1}{2},\ i = \overline{1,n}\Big\}.$$

On this set, the right parts of the system are continuous, functions h, H are continuously differentiable in the interval $[t_0, \omega[$. Due to the conditions (3.15), (3.16),

$$\lim_{t\uparrow\omega} f_k(t, v_1, \dots, v_n) = 0 \quad \text{uniformly in} \quad (v_1, \dots, v_n) \in \mathbb{R}^n_{\frac{1}{2}}\,.$$

Consequently, the system of differential equations (3.14) is the system of quasilinear differential equations of the type (2.1).

Let us show that all conditions of the Lemma 2.1 are true for this system.

By virtue of the form of the functions I and J_A,

$$\int_{t_0}^{t} H(\tau)\, d\tau \sim \ln|I(t)| \to +\infty \quad \text{as} \quad t \uparrow \omega\,.$$

Moreover,

$$\frac{H(t)}{h(t)} = \frac{\pi_\omega(t) J_A(t)}{I(t)}\,, \qquad \frac{1}{H(t)}\Big(\frac{H(t)}{h(t)}\Big)' = 1 + \frac{\pi_\omega(t) J_A'(t)}{J_A(t)} - \frac{\pi_\omega(t) J_A(t)}{I(t)}$$

therefore, due to the validity of the first condition from (3.1) and the validity of the condition (3.11),

$$\lim_{t\uparrow\omega} \frac{H(t)}{h(t)} = 0\,, \qquad \lim_{t\uparrow\omega} \frac{1}{H(t)}\Big(\frac{H(t)}{h(t)}\Big)' = 0\,.$$

Consequently, the conditions (2.3) of the Lemma 2.1 are true for te system (3.14). It is also obvious that, for the system of differential equations (3.14), the matrixes C_{n-1} and C_n of dimensions $(n-1) \times (n-1)$ and $n \times n$ (subsequently) from Lemma 2.1 are following

$$\det C_{n-1} = (-1)^{i-1}(i-1)!(n-i)!\,, \quad \det C_n = (-1)^{i}(i-1)!(n-i)!$$

and

$$\det\,[C_{n-1} - \rho E_{n-1}] = \\ (-1)^{n-1}(\rho+i-1)(\rho+i-2)\dots(\rho+1)(\rho-1)(\rho-2)\dots(\rho-n+i)\,,$$

where E_{n-1} is the identity matrix of the dimension $(n-1) \times (n-1)$. Hence, in particular, we obtain that the matrix C_{n-1} has $n-1$ non-zero real eigenvalues, and there are $(i-1)$-negative and $(n-i)$-positive eigenvalues among these eigenvalues.

Thus, all conditions of Lemma 2.1 are true for the system of differential equations (3.14). According to this Lemma, the system of equations (3.14) has at least one solution $(v_k)_{k=1}^n : [t_1, \omega[\to \mathbb{R}^n$ ($t_1 \in [t_0, \omega[$) that tends to zero when $t \uparrow \omega$. Moreover, according to this lemma, since there are $(i-1)$ negative eigenvalues and there are $(n-i)$ positive eigenvalues among the eigenvalues of the matrix C_{n-1}, in case of the validity of the inequality $h(t) > 0$ ($h(t) < 0$) in the interval $[t_0, \omega[$, the system of differential equations (3.14) has $(i-1)$-parameter (subsequently, $(n-i)$-parameter) family of solutions that vanish in the point ω when $H(t) < 0$ in the interval $[t_0, \omega[$, and i-parameter ($(n-i+1)$-parameter) family of such solutions when $H(t) > 0$ in $[t_0, \omega[$.

In order to finally answer the question about the quantity of solutions that vanish when $t \uparrow \omega$, it is necessary to define the signs of the functions h and H in the interval $[t_0, \omega[$.

Since $h(t) = \pi_\omega^{-1}(t)$, according to the form of the function π_ω, we obtain

$$\operatorname{sign} h(t) = \begin{cases} 1\,, & \text{if} \quad \omega = +\infty\,, \\ -1\,, & \text{if} \quad \omega < +\infty\,. \end{cases}$$

For the function H, according to the definition of the function I, we have

$$H(t) = \frac{J_A(t)}{I(t)} = \frac{|J_A(t)|}{\int_a^t |J_A(\tau)|\, d\tau} > 0 \quad \text{as} \quad t \in [t_0, \omega[\ .$$

Considering the obtained sign conditions for the functions h and H, we can make the following conclusion about the quantity of solutions of the system (3.14) that tend to zero when $t \uparrow \omega$:

1) if $\omega = +\infty$, then the system of differential equations (3.14) has i-parameter family of vanishing solutions when $t \uparrow \omega$;

2) if $\omega < +\infty$, then the system of differential equations (3.14) has $(n-i+1)$-parameter family of vanishing solutions when $t \uparrow \omega$.

By virtue of the transformation (3.13), each of the vanishing solutions $(v_k)_{k=1}^n$: $[t_1, \omega[\to \mathbb{R}^n$ of the system (3.14) corresponds to the solution $y\colon\ [t_1, \omega[\to \mathbb{R}$ of the differential equation (1.1), that admits the asymptotic representations (3.2)–(3.4) as $t \uparrow \omega$. Considering these representations and the conditions (3.1), it is easy to ascertain that each such solution is $P_\omega\left(\frac{n-i-1}{n-i}\right)$-solution of the differential equation (1.1).

Theorem is proved. □

Remark 3.1. While establishing the validity of the conditions (3.1), it is possible to notice that, in virtue of the first condition, the second and the third conditions are subsequently equivalent to the conditions

$$\int_a^\omega p(t)|\pi_\omega(t)|^{n-1} \left|\ln|\pi_\omega(t)|\right|^\sigma\, dt = +\infty\,, \qquad \lim_{t\uparrow\omega} p(t)\pi_\omega^n(t) \left|\ln|\pi_\omega(t)|\right|^\sigma = 0\,.$$

Remark 3.2. Notice that conditions (3.1) do not depend on the index i. This indicates that when $n \geq 4$ and the conditions (3.1) are satisfied, for each $i \in \{2, \dots, n-2\}$ and for $\omega = +\infty$, resp. for $\omega < +\infty$, the differential equation (1.1) has i-parametric, resp. $(n-i+1)$-parametric, family of $P_\omega\left(\frac{n-i-1}{n-i}\right)$-solutions that admit the asymptotic representations (3.2)–(3.4) as $t\uparrow\omega$.

When $\sigma=0$, i.e. equation (1.1) is linear differential equation of the type

$$y^{(n)}=\alpha_0 p(t)y\,, \tag{3.17}$$

where $\alpha_0\in\{-1;1\}$ and $p\colon\ [a,\omega[\ \to\ ]0,+\infty[$ is a continuous function, from the Theorem 3.1 and the Remarks 3.1 and 3.2, the following statement directly follows.

Corollary 3.1. *If $n\geq 4$ and the following conditions are satisfied*

$$\begin{gathered}\lim_{t\uparrow\omega}\frac{\pi_\omega^{n-1}(t)p(t)}{\int_A^t\pi_\omega^{n-2}(\tau)p(\tau)\,d\tau}=-1\,,\\ \int_a^\omega|\pi_\omega(\tau)|^{n-1}p(\tau)\,d\tau=+\infty\,,\qquad \lim_{t\uparrow\omega}\pi_\omega^n(t)p(t)=0\,,\end{gathered}\tag{3.18}$$

then the linear differential equation (3.17) *has $n-3$ linearly independent $P_\omega\left(\frac{n-i-1}{n-i}\right)$-solutions $y_i\colon\ [t_1,\omega[\ \to\mathbb{R}$ $\ (i=\overline{2,n-2})$ that admit asymptotic representations when $t\uparrow\omega$*

$$\begin{gathered}\frac{y_i^{(k-1)}(t)}{y_i^{(i-1)}(t)}=\frac{[\pi_\omega(t)]^{i-k}}{(i-k)!}[1+o(1)]\qquad(k=\overline{1,i-1})\,,\\ \ln|y_i^{(i-1)}(t)|=\frac{\alpha_0(-1)^{n-i}}{(i-1)!(n-i)!}\int_a^t p(\tau)\pi_\omega^{n-1}(\tau)\,d\tau[1+o(1)]\,,\\ \frac{y_i^{(k)}(t)}{y_i^{(i-1)}(t)}=\frac{\alpha_0(-1)^{n-k}(k-i)!}{(i-1)!(n-i)!}p(t)\pi_\omega^{n-k+i-1}(t)[1+o(1)]\quad(k=\overline{i,n-1})\,,\end{gathered}$$

and, moreover, equation (3.17) *does not have any other $P_\omega\left(\frac{n-i-1}{n-i}\right)$-solutions for $i\in\{2,\dots,n-2\}$ that are different from mentioned above.*

Remark 3.3. The linear independence of the solutions mentioned in the corollary follows from the fact that the matrix of the dimension $n\times(n-3)$ with the columns consisting of the solutions y_i $(i=\overline{2,n-2})$ and their derivatives up to the order $n-1$ inclusive, contains minor of the dimension $(n-3)\times(n-3)$, that is different from zero in some left neighborhood of ω.

This corollary for $\omega=+\infty$ complements the results established in the monograph of I.T. Kiguradze and T.A. Chanturia (see Ch.1, §6, item 6.5, pp. 184–186) for linear differential equations with asymptotically small coefficients.

References

[1] Evtukhov, V.M., *Asymptotic representations of solutions of nonautonomous ordinary differential equations*, Doctoral (Phys.-Math.) Dissertation, Kiev (1998).

[2] Evtukhov, V.M., Elshour, Mousa Jaber Abu, *Asymptotic behaviour of solutions of the nonautonomous ordinary differential n-th order equations*, Nonlinear Oscillations **19** (1) (2016), 22–31.

[3] Evtukhov, V.M., Samoilenko, A.M., *Conditions for the existence of solutions of real nonautonomous systems of quasilinear differential equations vanishing at a singular point*, Ukrainian Math. J. **60** (1) (2010), 56–86.

[4] Evtukhov, V.M., Samoilenko, A.M., *Asymptotic representations of solutions of nonautonomous ordinary differential equations with regularly varying nonlinearities*, Differ. Uravn. **47** (5) (2011), 628–650.

[5] Kiguradze, I.T., Chanturia, T.A., *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Academic Publishers Group Dordrecht, 1993.

[6] Seneta, E., *Regularly Varying Functions*, Berlin Springer-Verlag, 1976, Translated under the title Pravil'no menyayushchiesya funktsii, Moscow Nauka, 1985.

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