ARCHIVUM MATHEMATICUM (BRNO)
Tomus 53 (2017), 19–33

NEW TECHNIQUE FOR SOLVING UNIVARIATE GLOBAL OPTIMIZATION

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Abstract. In this paper, a new global optimization method is proposed for an optimization problem with twice differentiable objective function a single variable with box constraint. The method employs a difference of linear interpolant of the objective and a concave function, where the former is a continuous piecewise convex quadratic function underestimator. The main objectives of this research are to determine the value of the lower bound that does not need an iterative local optimizer. The proposed method is proven to have a finite convergence to locate the global optimum point. The numerical experiments indicate that the proposed method competes with another covering methods.

1. Introduction

In the convex optimization, we seek a local solution widely enough to determine the optimal solution [1, 2, 22], while the objective of global optimization is to find the globally best solution of possibly nonlinear models, in the possible or known presence of multiple local optima. Formally, global optimization seeks global solutions of a constrained optimization model [23]. Nonlinear models are ubiquitous in many applications, e.g., in advanced engineering design, biotechnology, data analysis, environmental management, financial planning, process control, risk management, scientific modeling, and etc. Their solution often requires a global search approach [3, 4, 13, 20, 21, 25, 24, 26].

A variety of adaptive partition strategies have been proposed to solve global optimization models. These are based upon partition, sampling, and subsequent lower and upper bounding procedures. These operations are applied iteratively to the collection of active subsets within the feasible set. In this connection several works have been proposed among others. Piyavskii [28] described a general algorithm for finding the absolute minimum of a function to a given accuracy, and illustrated a special aspects of its application by examples involving functions of one or more variables, satisfying a Lipschitz condition. Shpak [32] proposed a sequential deterministic algorithm with adaptive estimating of the Lipschitz constant for

2010 *Mathematics Subject Classification*: primary 90C26; secondary 90C30.

Key words and phrases: global optimization, Branch and Bound method, convex underestimation, piecewise quadratic, explicit solution.

Received October 21, 2016, revised December 2016. Editor M. Feistauer.

DOI: 10.5817/AM2017-1-19

solving unconstrained one-dimensional global optimization problems. Sergeyev et al [17, 19, 30, 31] proposed several efficient algorithms and improvements to solve the unconstrained optimization problem where the objective function is univariate and has Lipschitzean first derivatives. Cartis et al [10] presented branching and bounding improvements for global optimization algorithms with Lipschitz continuity properties. Adjiman et al [5] presented the detailed implementation of the alpha BB approach and computational studies in process design problems such as heat exchanger networks, reactor-separator networks, and batch design under uncertainty. Akrotirianakis and Floudas [6] presented computational results of the new class of convex underestimators embedded in a branch-and-bound framework for box-constrained NLPs. They also proposed a hybrid global optimization method that includes the random-linkage stochastic approach with the aim of improving the computational performance. Caratzoulas and Floudas [9] proposed novel convex underestimators for trigonometric functions. Recently, years univariate global optimization problems have attracted common attention because they arise in many real-life applications and the obtained results can be easily generalized to the multivariate case [7, 8, 11, 14, 15, 16, 27, 29].

In this paper, we propose an approach to find a global minimum of a univariate objective function. In the following we will present our technique.

A piecewise quadratics underestimations ($KBB_{\mathbf{m}}$)

The main idea consists in constructing piecewise quadratic underestimation functions closer to the given nonconvex f in a successive reduced intervals $[a_k, b_k]$ and their minima are explicitly given. Instead of using a single large quadratic away from the objective function [19], the determination of its minimum implies a local method [5]. We propose an explicit method of quadratic relaxation of building global optimization problems with bounded variables. This construction is based on the work of authors in [18], using the quadratic splines, the generated quadratic programs have exactly explicit optimal solutions, In each interval, the target underestimated by several quadratic splines reliable to calculate the lower bounds. The structure of the rest of the paper is as follows: Section 2 presents the two underestimators proposed in [5, 18]. Section 3 discusses the construction of a new lower bound on the objective function, and describes a proposed algorithm (KBB_m) to solve the univariate global optimization problem with box constrained. Section 4 presents some numerical examples of different nonconvex objective functions while we conclude the paper in Section 5.

2. Background

Consider the following global minimization problem:

$$(P)\begin{cases} \alpha = \min f(x) \\ x \in X = [a,b] \end{cases}, \tag{1}$$

with f is a nonconvex twice differentiable function on X.

In what follows we give two underestimators developed by the authors, respectively in [5, 18].

2.1. **Underestimator in (αBB) method** [5]**.** The underestimator in αBB method on the interval $[a, b]$ is as follows

$$L(x) = f(x) - \frac{\alpha}{2}(x-a)(b-x)\,, \tag{2}$$

where $\alpha \geq \max\{0, -f^{''}(x)\}$, for all $x \in [a, b]$.

This underestimator satisfies the following properties:

(1) It is convex (i.e. $L^{''}(x) = f^{''}(x) + \alpha \geq f^{''}(x) + \max\{0, -f^{''}(x)\} \geq 0$, for all $x \in [a, b]$).

(2) It coincides with the function $f(x)$ at the endpoint of the interval $[a, b]$.

(3) It is an underestimator of the objective function $f(x)$.

(4) Requires solving the convex problem $\min L(x)$, for all $x \in [a, b]$ to determine the values of the lower bound of the objective function $f(x)$. For more details, see [5].

2.2. **Quadratic underestimator in (KBB) method** [18]**.** The quadratic underestimator developed in [18] on the interval $[a, b]$ is :

$$q(x) = f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a} - \frac{K}{2}(x-a)(b-x)\,, \tag{3}$$

where $|f''(x)| \leq K$, for all $x \in [a, b]$.

This quadratic underestimator satisfies the following properties:

(1) It is convex (i.e. $q^{''}(x) = K \geq 0$, for all $x \in [a, b]$.

(2) It coincides with the function $f(x)$ at the endpoint of the interval $[a, b]$.

(3) It is an underestimator of the objective function $f(x)$.

(4) The values of the lower bound are given explicitly. For more details, see [18].

2.3. **Advantages and disadvantages of two methods.**

(1) The advantage of αBB is that the best initial lower bound is obtained, also the underestimator is close to the objective function see Table (2), and Table (3).

(2) The disadvantage of (αBB) is in a local method for determining the values of the lower bounds.

(3) The advantage of KBB is that values of the lower bounds are given explicitly.

(4) The disadvantage of (KBB) is that initial lower bound is very far from the optimal solution, also the underestimator is far away from objective function see Figure (1), and Figure (3).

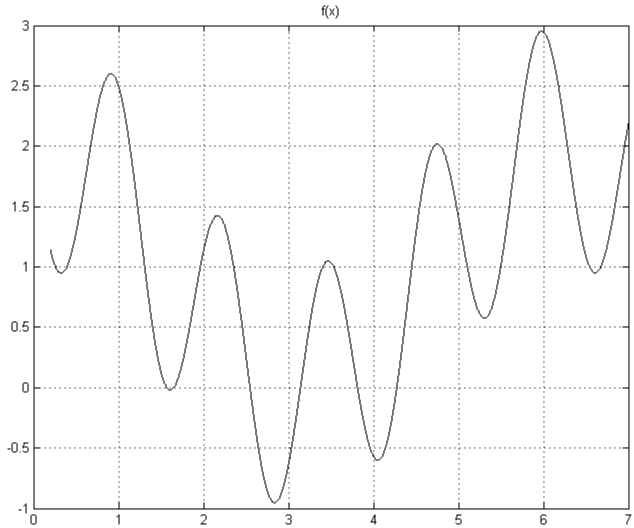

FIG. 1: **The graph of a multi extremal function** $f(x) = \cos(x) - \sin(5x) + 1$ **on** $[0.2, 7]$ **where** $\min f(x) = -0.952896$

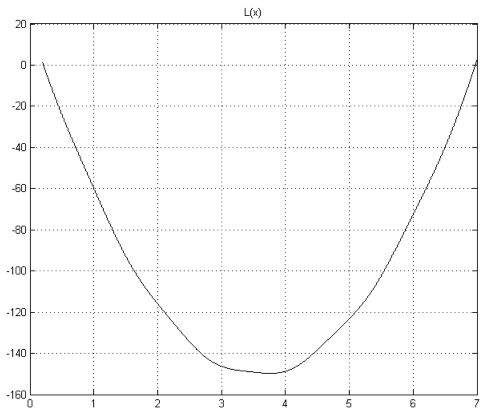

FIG. 2: **The underestimator** αBB **for** $f(x)$ **where** $\min L(x) = -149.71371$

3. THE PROPOSED UNDERESTIMATOR (KBB_m)

In this section we present a new lower bound. In this lower bound we merge the advantages of KBB and αBB.

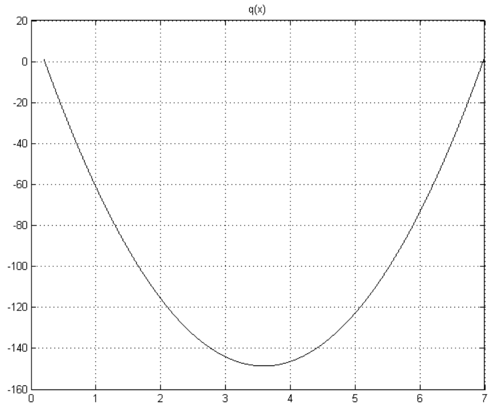

FIG. 3: **The underestimator** KBB **for** $f(x)$ **where** $\min q(x) = -148.620102$

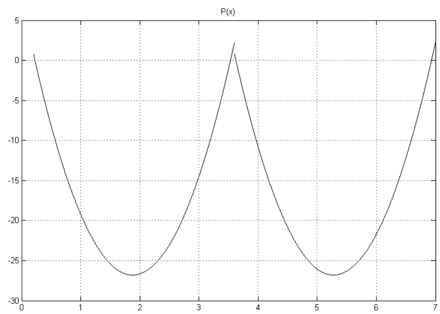

FIG. 4: **The underestimator** KBB_m **for** $f(x)$ **where** $n = 2$**, and** $\min p(x) = -28.626255$

Let $X = [a, b]$ be a bounded closed interval in R. Let f be a continuously twice differentiable function on X. Let $x^0$and x^1 be two real numbers in $[a, b]$ such that $x^0 \le x^1$. Let l_0 and l_1 be real valued functions defined in [13,15,29] by

$$l_0(x) = \frac{x^1 - x}{x^1 - x^0} \text{ if } x^0 \le x \le x^1, \quad l_1(x) = \frac{x - x^0}{x^1 - x^0} \text{ if } x^0 \le x \le x^1 . \tag{4}$$

For all x in the interval $[x^0, x^1]$, we have $l_0(x) + l_1(x) = 1$. We have also that $l_i(x^j)$ is equal to 0 if $i \neq j$, and 1 otherwise, i, $j = 0, 1$. Let $h = x^1 - x^0$ and $L_h f$ be the linear interpolant to f at points x^0, x^1, such that

$$L_h f(x) = \sum_{i=0}^{1} l_i(x) f(x^i) . \tag{5}$$

Let $f(x)$ be a univariate function that needs to be underestimated in the interval $[a,b]$. Suppose that the nodes are chosen to be equally spaced in $[a,b]$, so that $x_i = a + ih$, $h = \frac{b-a}{n}$, $i = 0, \dots, n$.

On each interval $[x_i, x_{i+1}]$ we construct the corresponding local quadratic underestimator as follows

$$p_i(x) = L_{h_i} f(x) - Q_i(x)\,, \quad i = 0, \dots, n-1\,, \tag{6}$$

where $Q_i(x) = \frac{1}{2} K_i (x - x_i)(x_{i+1} - x)$, where K_i is an upper bound of the second derivative which is valid for $[x_i, x_{i+1}]$. Instead of considering one quadratic lower bound over $[a,b]$, we construct a piecewise quadratic lower bound.

Remark 1. The upper bounds K_i are computed with interval analysis see [32].

In the following theorem we will show that the new lower bound is tighter than the lower bound constructed in [18] see Figure (5).

Theorem 1. *We have*

$$q(x) \le p(x) \le f(x)\,, \quad \forall x \in [a,b]\,, \tag{7}$$

where $p(x) = p_i(x)$, $\forall x \in [\ x_i, x_{i+1}]$, $i = 0, \dots, n-1$.

The function $p(x)$ is a continuous piecewise convex valid underestimator of $f(x)$ for all x in $[a,b]$, and it is tighter than the underestimator $q(x)$ introduced in [18].

Proof. For every interval $[x_i, x_{i+1}]$, $i = 0, \dots, n-1$

$$E(x) = q(x) - p_i(x) = \frac{1}{2}(K_i - K)(x - x_i)(x_{i+1} - x)\,. \tag{8}$$

On the other hand $E''(x) = K - K_i \ge 0$ for all $x \in [x_i, x_{i+1}]$, hence E is a convex function, and therefore for all $x \in [x_i, x_{i+1}]$ we have:

$$E(x) \le \max\left\{E(x), x \in [x_i, x_{i+1}]\right\} = E(x_i) = E(x_{i+1}) = 0 \tag{9}$$

and the first inequality of (7) is verified. To justify the second inequality, consider now the function ϕ defined on $[x_i, x_{i+1}]$ by

$$\phi(x) = f(x) - p_i(x) = f(x) - L_{h_i} f(x) + \frac{1}{2} K_i (x - x_i)(x_{i+1} - x)\,. \tag{10}$$

Clearly, that $\phi''(x) = f''(x) - K_i \le 0$ for all x in $[x_i, x_{i+1}]$ hence ϕ is a concave function, and therefore we have

$$\phi(x) \ge \min\{\phi(x), x \in [x_i, x_{i+1}]\} = \phi(x_i) = \phi(x_{i+1}) = 0\,. \tag{11}$$

The second inequality of (7) is also proved. □

In each sub-interval $[x_i, x_{i+1}]$, one has to compute a lower bound of the objective function f.

$$x_i^* = \begin{cases} \dfrac{1}{2}(x_i + x_{i+1}) - \dfrac{1}{K_i} \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} & \text{if} \quad x \in [x_i, x_{i+1}] \\ x_i & \text{if} \quad x \le x_i \\ x_{i+1} & \text{if} \quad x \ge x_{i+1} \end{cases} \tag{12}$$

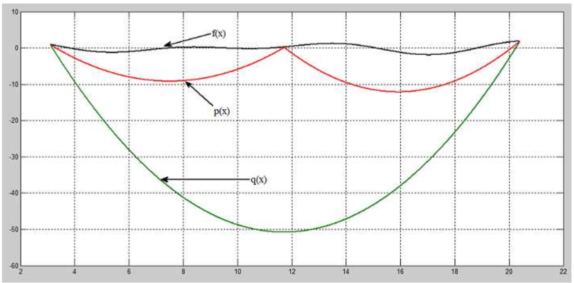

FIG. 5: **The tightness of our underestimator $p(x)$ than the $q(x)$ for $f(x) = \sin(x)$ where $n = 2$**

Now, we compute the values of $p_i(x_i^*)$ to select the best as follows:

$$LB_i = \min p_i(x_i^*) \,. \tag{13}$$

The objective function is evaluated at the different trials points to determine the upper bound.

$$UB_i = \min \left\{ f(x_i^*),\ f(x_i) \right\} \tag{14}$$

Remark 2. The proposed underestimator KBB_m verifies the following properties:

(1) It is continues piecewise convex on $[a, b]$.
(2) It coincides with the function $f(x)$ at the endpoint of the interval $[x_i, x_{i+1}]$ for all $i = 0, \ldots, n-1$.
(3) It is an underestimator of the objective function $f(x)$.
(4) The values of the lower bound are given explicitly.
(5) When we double the quadratic, we obtained good lower bounds see Table (5).

To solve the problem (P), we use an iterative process that converts the set $X = [a, b]$ into several smaller and smaller subsets. To each subset of X we construct a lower bound of the objective function in order to eliminate the parts which do not contain the global optimum and to select the subset that must be expressed. Our algorithm consists in generating two convergent sequences $\{UB_k\}$and $\{LB_k\}$ of the upper and lower bounds respectively, of the minimum value of the objective function of the problem (P).

An initial subdivision of $X = [a, b]$ into $n > 1$ subinterval of the same length will be made such that, $M^k = \bigcup_{i=0}^{n-1} \{[x_i, x_{i+1}]\}$ the set of subintervals not explored. At each iteration k and on each subinterval $X_i^k = [x_i, x_{i+1}]$ the lower and upper bounds LB_i^k and UB_i^k will be computed respectively by:

$$\begin{cases} LB_i^k = p_i(x_i^*) \\ UB_i^k = \min\{f(x_i), f(x_i^*)\} \end{cases} \tag{15}$$

Our method uses the "best first" strategy.

In effect, the final upper and lower bound for the iteration k will be given respectively by:

$$\begin{cases} UB^k = \min\{UB^{k-1}, \min UB_i^k\} \\ LB^k = \min LB_j^k, \quad j = 1, \dots, m \end{cases}, \tag{16}$$

with m is the cardinal of M^k, and any subset on which the lower bound exceeds UB^k will be eliminated from M^{k+1}, since $\min f$ will not be reached on such a subset.

In fact, our method can be represented schematically by a tree structure whose root $X = [a, b]$, and for the vertex the subset X_i^k which are obtained by the successive subdivisions, and two or more vertices will be connected if the second subset is obtained by the direct partitioning from the first. And at each level of the tree created the lower and upper bounds will be obtained by applying the relations (15), (16). Let x^* be the optimal solution of the problem (P) for the following:
The different steps for solving the problem (P) are summarized in the following proposed algorithm:

Algorithm

Input:

- $[a, b]$: A real interval.
- ε : The accuracy.
- f : The objective function.
- n : The number of quadratic.

Output:

- x^* : The global minimum of f.

(1) **Initialization step** $k = 0$

  (a) **for all** $i = 0, \dots, n$
  compute $x_i = a + \frac{b-a}{n} i$, and set $M^0 = \bigcup_{i=0}^{n-1} \{[x_i, x_{i+1}]\}$
  (b) Compute K_i such that $|f''(x)| \leq K_i$ on each $[x_i, x_{i+1}]$
  for all $i = 0, \dots, n-1$
  (c) Compute x_i^* by using **(12)** **for all** $i = 0, \dots, n-1$
  (d) Compute $UB^k = \min\{\min f(x_i^*), \ \min f(x_i)\}$
  (e) Set $LB^k = \min LB_i^k$ with $LB_i^k = p_i(x_i^*)$
  (f) $\bar{i} \longleftarrow$ the index corresponding to $\min LB_i^k$

(2) **Iteration step**
  While $(UB^k - LB^k > \varepsilon$ and $M^k \neq \emptyset)$ **do**

  (a) $a \longleftarrow x_{\bar{i}}$, $b \longleftarrow x_{\bar{i}+1}$ and **apply** step (a), (b), (c), and (d)
  (b) **Update** UB^k
  (c) **For all** $i = 1, \dots, m$; $(m = \mathrm{card}(M^k))$

– **Elimination step:**
if $(UB^k - LB_i^k < \varepsilon)$ **then** remove $[x_i, x_{i+1}]$ from M
– **Selection step:**
if $(UB^k - LB_i^k \geq \varepsilon)$ **then** $\min LB_i^k$; $\bar{i} \longleftarrow$ the index corresponding to $\min LB_i^k$

(d) $k = k + 1$

end While

(3) $x^* = x^k \in \left\{x : f(x) = UB^k,\ x \in X = [a, b]\right\}$is the optimal solution corresponding to the best UB^k found.

end algorithm

Theorem 2 (Convergence of the algorithm)**.** *Either the algorithm is finite or it generates a bounded sequence* $\{x_k\}$*. Any accumulation point of the sequence is a global optimal solution of* (P)*. We have:* $UB^k \searrow \alpha$*,* $LB^k \nearrow \alpha$*.*

Proof. Let us consider an infinite sequence of intervals $\{T^k\}$ generated by our algorithm, whose lengths h_i with $i = 1, \ldots, n$ decreases to zero, then the whole sequence $\{T^k\}$ shrinks to a singleton. Since the values of his UB^k obtained by evaluating $f(x)$ at the different trials points of $[a, b]$, then the sequence $\{UB^k\}$ is bounded below by $\alpha = \min f(x)$. On the other hand, the values of LB^k are the lower bounds of the objective function, which can not exceed $\alpha = \min f(x)$, then the sequence $\{LB^k\}$ is bounded above by α. Subsequently $LB^k \leq \alpha \leq UB^k$. It suffices to prove that $\{UB^k\}$ is a decreasing sequence, and $\{LB^k\}$ is a increasing sequence.

First, from the description of the algorithm we see that, at each iteration $k + 1$, $k \geq 0$ the value of UB^{k+1} will be selected as the lesser between the current UB^k and the new value to be determined see (16), which always results $UB^{k+1} \leq UB^k$, $\forall k \geq 0$, so, $\{UB^k\}$ is a decreasing sequence. Similarly, at each iteration $k + 1$, $k \geq 0$ the value of the lower bound LB^{k+1} will be selected as the minimum of a certain quadratic located in the interior of a big quadratic underestimate the objective on the current interval $[a_{k+1}, b_{k+1}]$ see figure (5), which automatically leads $LB^{k+1} \geq LB^k$, for all $k \geq 0$, then the sequence $\{LB^k\}$ is increasing on $[a, b]$. The theorem is proved. □

4. Computational aspects and results

To measure the performances of our KBB_m algorithm, we perform a comparative study with KBB and αBB. These algorithms are implemented in $C-$programming language with double precision floating point, and run on a computer with an Intel (R) core (TM) i3-311MCP4 with CPU 2.40GHz. Numerical tests are performed in tow parts on a set of test functions [12]. In the first experiment, we compare the performances of the KBB, αBB and the KBB_m algorithms on a set of 10 functions. Here, we include a method that computes the positive numbers α and K [32]. The number of the quadratic functions used in KBB_m at each iteration as

Exp	$f(x)$	$[x^L, x^U]$	LM	GM	opt
1	$e^{-3x} - \sin^3 x$	$[0, 20]$	4	1	-1
2	$\cos x - \sin(5x) + 1$	$[0.2, 7]$	6	1	-0.952897
3	$x + \sin(5x)$	$[0.2, 7]$	7	1	-0.077590
4	$e^{-x} - \sin(2\pi x)$	$[0.2, 7]$	7	1	-0.478362
5	$\ln(3x)\ln(2x) - 0.1$	$[0.2, 7]$	1	1	-0.141100
6	$\sqrt{x}\sin^2 x$	$[0.2, 7]$	3	2	0
7	$2\sin x e^{-x}$	$[0.2, 7]$	2	1	-0.027864
8	$2\cos x + \cos(2x) + 5$	$[0.2, 7]$	3	2	3.5
9	$\sin x$	$[0, 20]$	4	3	-1
10	$\sin x \cos x - 1.5\sin^2 x + 1.2$	$[0.2, 7]$	3	2	-0.451388
11	$\left(x - x^2\right)^2 + (x-1)^2$	$[-10, 10]$	1	1	0
12	$\frac{x^2}{20} - \cos x + 2$	$[-20, 20]$	7	1	1
13	$x^2 - \cos(18x)$	$[-5, 5]$	29	1	-1
14	e^{x^2}	$[-10, 10]$	1	1	1
15	$(x + \sin x)\, e^{-x^2}$	$[-10, 10]$	1	1	-0.824239
16	$x^4 - 12x^3 + 47x^2 - 60x - 20e^{-x}$	$[-1, 7]$	1	1	-32.78126
17	$x^6 - 15x^4 + 27x^2 + 250$	$[-4, 4]$	2	2	7
18	$x^4 - 10x^3 + 35x^2 - 50x + 24$	$[-10, 20]$	2	2	-1
19	$24x^4 - 142x^3 + 303x^2 - 276x + 3$	$[0, 3]$	2	1	-89
20	$\cos x + 2\cos(2x)\, e^{-x}$	$[0.2, 7]$	2	1	-0.918397

TAB. 1: **Test functions**

fixed to $n = 16$. and the accuracy fixed to $\varepsilon = 10^{-6}$. In the second experiment, we were tested the KBB_m algorithm according to the initial lower bound obtained for different numbers of quadratic function used on a set of 20 functions see Table (1).

In our results, we consider the following notations as table anterior:

- $f^*$is the optimum obtained.
- LB_0 is the initial lower bound.
- T_{CPU} is the execution time in seconds.
- n_{It} is the number of iterations.
- m is the number of interval.
- m_e is the number of intervals eliminated.
- LM is the number of local minimum.
- GM is the number of global minimum.
- $*$ An asterisk denotes that the lower bound is equal to the known global optimum f^*, within six decimal digits of accuracy.

To determine the lower bound, αBB uses a local method at each iteration, making it is more expensive than it in KBB method in which the lower bounds are

Exp			αBB			
	n_{It}	T_{CPU}	m	m_e	LB_0	f^*
1	23	0	47	24	−273.76041	−0.99999
2	8	0	17	9	−65.9109	−0.95203
3	15	0	31	16	−118.96147	−0.07759
4	7	0	15	8	−121.60896	−0.47797
5	5	0	11	6	−527.67986	−0.14099
6	6	0	13	7	−2733.29510	0.00199
7	6	0	13	7	−18.57601	−0.02761
8	5	0	11	6	−28.84495	3.56245
9	6	0	13	7	−46.40909	−0.99997
10	8	0	17	9	−29.62761	−0.45138

TAB. 2: **Computational results for 10 functions by αBB algorithm.**

Exp			KBB			
	n_{It}	T_{CPU}	m	m_e	LB_0	f^*
1	27	2.211	55	28	−564.01754	−1
2	12	10.645	25	13	−116.08120	−0.95289
3	12	4.604	25	13	−117.80163	−0.07758
4	28	3.576	57	29	−121.20354	−0.47834
5	9	3.312	19	10	−528.13263	−0.14110
6	14	2.854	29	15	−6664.14641	0
7	33	3.132	67	34	−5.50269	−0.02786
8	8	3.012	21	11	−14.03655	3.5
9	9	2.293	15	8	−45.19193	−1
10	13	3.460	27	14	−29.66644	−0.45139

TAB. 3: **Computational results for 10 functions by KBB algorithm.**

given explicitly. So our comparison will not be based on the number of iterations required to achieve the optimum. The execution time required to achieve the optimal value is considered as a reliable criterion to the algorithm's performances. According to the numerical results summarized in Table (3) and Table (4), the performances of the proposed method is clearly better than the performance of the KBB method. The best initial lower bound obtained remains an important criterion for measuring the validity of the underestimator. In Table (2), Table (3) and Table (4), the comparative study of the quality of the initial lower bound found

Exp			KBB_m			
	n_{It}	T_{CPU}	m	m_e	LB_0	f^*
1	8	0	144	136	-2.26691	-1
2	1	0	32	31	-0.97784	-0.9529
3	2	0	48	46	-0.09467	-0.07759
4	10	0	176	166	-0.65053	-0.47820
5	2	0	48	46	-1.73375	-0.14110
6	2	0	48	46	-0.23383	0
7	6	0	112	106	-0.04618	-0.02786
8	2	0	48	46	3.49276	3.50001
9	3	0	64	61	-1.00563	-1
10	3	0	64	61	-0.45957	-0.45139

TAB. 4: **Computational results for 10 functions by KBB_m algorithm with $n = 16$.**

by the three algorithms show that our method is better than the two methods. In Table 5 confirmes the competence of our method by doubling the number of quadratics, we can notice that the values of the lower bound are improved.

5. Conclusion

We presented a method of underestimation of nonconvex objective based on piecewise quadratic functions which have explicit minima. A comparison of the lower bounds favors such quadratic against others guaranteeing the underestimation of the objective. This approach is validated by considering a deterministic branch and bound methods which is fully detailed and allows certifying still coaching the value of the global minimum at the end of the performance. Many digital exprements are performed, that confirm the effectiveness of this new acceleration technique. The performance of the proposed procedure depends on the quality of the chosen lower bound of f. Our piecewise quadratics lower bounding functions is better than the two underestimators introduced in [5, 21].

Acknowledgement. We appreciate the time and efforts by the editor and referees in reviewing this manuscript.

n	2	4	8	16	32	64	128
1	−152.72	−45.16	−7.15	−2.26	*	*	*
2	−28.43	−7.92	−2.16	−0.97	*	*	*
3	−28.45	−6.17	−1.34	−0.094	*	*	*
4	−30.018	−8.85	−2.207	−0.65	−0.49	*	*
5	−121.3	−29.66	−7.17	−1.73	−0.40	−0.149	−0.1417
6	−448.19	−41.56	−3.542	−0.23	−0.0019	−0.0002	−0.00003
7	−1.307	−0.340	−0.104	−0.04	−0.03	−0.02	−0.028
8	0.33	2.54	3.394	3.49	*	*	*
9	−11.23	−3.751	−1.141	−1.005	*	*	*
10	−5.85	−1.98	−0.598	−0.459	−0.453	−0.452	−0.4515
11	−16118.1	−1297.05	−107.4	−9.34	−0.67	−0.09	−0.01113
12	1	*	*	*	*	*	*
13	−20.42	−2.2	*	*	*	*	*
14	*	*	*	*	*	*	*
15	−173493.6	−27703.7	−7855.9	−769.2	−575.3	−48.21	−46.4
16	−19351.54	−576.321	−45.152	−33.67	−32.84	−32.80	−32.789
17	−14875.91	−2957.18	−362.63	−21.88	4.71	6.82	6.98
18	−93572.1	−9016.69	−1032.09	−142.7	−27.31	−7.07	−2.4
19	−578.6	−141.98	−95.79	−89.8	−89.1	−89.01	−89.001
20	−6.906	−2.59	*	*	*	*	*

Tab. 5: LB_0 **values obtained by** KBB_m**.**

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