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ON THE EXISTENCE OF NON-LINEAR FRAMES

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Abstract. A stronger version of the notion of frame in Banach space called Strong Retro Banach frame (SRBF) is defined and studied. It has been proved that if $\mathcal{X}$ is a Banach space such that $\mathcal{X}^*$ has a SRBF, then $\mathcal{X}$ has a Bi-Banach frame with some geometric property. Also, it has been proved that if a Banach space $\mathcal{X}$ has an approximative Schauder frame, then $\mathcal{X}^*$ has a SRBF. Finally, the existence of a non-linear SRBF in the conjugate of a separable Banach space has been proved.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [5] in the context of nonharmonic Fourier series. Frames now a days are widely used in various branches of mathematics and engineering. Feichtinger and Grochenig [6] generalized the notion of frame to Banach spaces and introduced the concept of atomic decomposition in a Banach space. Also, Grochenig [7] introduced a more general concept namely Banach frame for Banach spaces. For a nice and comprehensive survey of frames and related concepts one may refer to [1, 4].

Various other generalizations of frames for Banach spaces were defined and studied by many authors namely Schauder frames by Han and Larson [8] and also studied by Casazza et al. [2, 3], frames by Terekhin [17]. Banach frames in conjugate Banach spaces, called retro Banach frames, were introduced and studied by Jain et al. [9] and further studied in [13]. Approximative atomic decompositions in Banach spaces were studied in [10]. Schauder frames in conjugate Banach spaces were defined and studied in [12] while approximative Schauder frames were studied in [11]. The notion of Bi-Banach frame in a Banach space was defined and studied in [14] wherein they noted that a Schauder frame for a Banach space is a Bi-Banach frame but the converse is not true.

In the present paper, we shall consider a stronger notion of frame in a Banach space called strong Retro Banach frame (SRBF). It has been proved that if $\mathcal{X}$ is a Banach space such that $\mathcal{X}^*$ has a SRBF, then $\mathcal{X}$ has a Bi- Banach frame with some geometric property. Also, it has been proved that if a Banach space $\mathcal{X}$ has an approximative Schauder frame, then $\mathcal{X}^*$ has a SRBF. Finally, a result related to

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the existence of a non-linear SRBF in the conjugate of a separable Banach space has been proved.

Throughout this paper $\mathcal{X}$ will denotes an infinite dimensional Banach space over the scalar field $\mathbb{K}(\mathbb{R}, \mathbb{C})$, $\mathcal{X}^*$ denotes the conjugate space of $\mathcal{X}$ and $L(\mathcal{X}, \mathcal{X})$ denote the Banach space of all continuous linear mappings of $\mathcal{X}$ into $\mathcal{X}$. For a sequence $\{x_n\} \in \mathcal{X}$ and $\{f_n\} \in \mathcal{X}^*$, $[x_n]$ denotes the closed linear span of $\{x_n\}$ in the norm topology of $\mathcal{X}$ and $\overline{[f_n]}$ the closed linear span of $\{f_n\}$ in the weak star topology of $\mathcal{X}^*$. A sequence space S is called a *BK-space* if it is a Banach space and the co-ordinate functionals are continuous on S. That is the relations $x_n = \{{\alpha_j}^{(n)}\}$, $x = \{\alpha_j\} \in S$, $\lim\limits_{n\to\infty} x_n = x$ imply $\lim\limits_{n\to\infty} \alpha_j^{(n)} = \alpha_j$ $(j = 1, 2, 3, \dots)$. Also, if $V \subseteq \mathcal{X}^*$, then we define $\gamma(v) = \inf\limits_{\substack{x\in\mathcal{X}\\ x\neq 0}} \sup\limits_{\substack{f\in v\\ \|f\|\leq I}} \left|f\left(\frac{x}{\|x\|}\right)\right|$.

A sequence $\{x_n\} \subset \mathcal{X}$ is said to be a *Markusevic basis* (M-basis) for $\mathcal{X}$ if $\{x_n\}$ is complete in $\mathcal{X}$ and there exists a sequence $\{f_n\}$ in $\mathcal{X}^*$ biorthogonal to $\{x_n\}$, called an associated sequence of coefficient functional (a.s.c.f.), which is total on $\mathcal{X}$.

Definition 1.1 ([9])**.** Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. Let $\{x_n\} \subset \mathcal{X}$ and $J\colon \mathcal{X}_d^* \to \mathcal{X}^*$ be given. The pair $(\{x_n\}, J)$ is called a *retro Banach frame* for $\mathcal{X}^*$ with respect to $\mathcal{X}_d^*$ if

(a) $\{f(x_n)\} \in \mathcal{X}_d^*$, for all $f \in \mathcal{X}^*$.

(b) There exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{\mathcal{X}^*} \leq \|\{f(x_n)\}\|_{\mathcal{X}_d^*} \leq B\|f\|_{\mathcal{X}^*}, \quad \text{for all} \quad f \in \mathcal{X}^*. \tag{1.1}$$

(c) J is a bounded linear operator such that

$$J(\{f(x_n)\}) = f, \quad \text{for all} \quad f \in \mathcal{X}^*.$$

The constants A and B are called lower and upper bounds of the retro Banach frame $(\{x_n\}, J)$. The inequality (1.1) is called the retro Banach frame inequality.

A retro Banach frame $(\{x_n\}, J)$ is said to be exact if there exists a sequence $\{f_n\} \subset \mathcal{X}^*$ such that $f_i(x_j) = \delta_{i,j}$, for all $i, j \in \mathbb{N}$.

Definition 1.2. Let $\mathcal{X}$ be a Banach space with dual $\mathcal{X}^*$. A pair $(\{x_n\}, \{f_n\})$ (where $\{f_n\} \subset \mathcal{X}^*$ and $\{x_n\} \subset \mathcal{X}$) is called a *Bi-Banach frame* for $\mathcal{X}$ if there exist associated Banach spaces $\mathcal{X}_d$ and $(\mathcal{X}^*)_d$ and bounded linear operators $S\colon \mathcal{X}_d \to \mathcal{X}$, $T\colon (\mathcal{X}^*)_d \to \mathcal{X}^*$ such that $(\{f_n\}, S)$ is a Banach frame for $\mathcal{X}$ and $(\{x_n\}, T)$ is retro Banach frame for $\mathcal{X}^*$.

A Bi-Banach frame $(\{x_n\}, \{f_n\})$ is called *tight* if both the retro Banach frame $(\{x_n\}, T)$ and the Banach frame $(\{f_n\}, S)$ are tight.

The following results are stated in the form of lemmas which will be used in the subsequent work.

Lemma 1.3 ([16])**.** *Let $\mathcal{X}$ be a Banach space and $\{f_n\} \subset \mathcal{X}^*$ be a sequence such that $\{x \in \mathcal{X} : f_n(x) = 0, \forall\, n \in \mathbb{N}\} = \{0\}$. Then $\mathcal{X}$ is linearly isometric to the Banach space $\mathcal{X}_d = \{\{f_n(x)\} : x \in \mathcal{X}\}$, where the norm is given by $\|\{f_n(x)\}\|_{\mathcal{X}_d} = \|x\|_{\mathcal{X}}, x \in \mathcal{X}$.*

Lemma 1.4 ([16]). *Let $\mathcal{X}$ be a separable normed linear space and let $\{x_n^*\}$ be a sequence in $\mathcal{X}^*$ such that $\frac{x_n^*}{\|x_n^*\|} \xrightarrow{w^*} 0$ and that for the linear subspace $[x_n^*]$ of $\mathcal{X}^*$, $\gamma([x_n^*]) \geq 0$. Then there exist a norm $|\cdot|$ on $\mathcal{X}$ equivalent to the initial norm on $\mathcal{X}$ such that $(\mathcal{X}, |\cdot|)$ is strictly convex and satisfies the following property*

$$\text{(1.2)} \qquad \text{If} \quad \lim_{n\to\infty} f_k(x_n) = f_k(x_0)(k = 1, 2, \dots), \quad \text{then} \quad \lim_{n\to\infty} |x_n| \geq |x_0|.$$

2. Main result

Approximative Schauder frames in Banach spaces were studied in [11] and the notion of Bi-Banach frame was studied in [14]. In the following definition, we gave a stronger notion called Strong Retro Banach frame (SRBF). The idea of defining this notion is to correlate this notion with the existing notions like approximative Schauder frames and Bi-Banach frames.

Definition 2.1. Let $\{x_n\} \subset \mathcal{X}$ be an exact RBF for $\mathcal{X}^*$ with admissible sequence $\{f_n\} \subset \mathcal{X}^*$. Let $X_n = [x_1, x_2, \dots, x_n]$, $n \in \mathbb{N}$. If there exists a sequence $\{v_n\}$, where each $v_n \colon X_n \to X_n$ is a continuous linear mapping, such that $x = \lim\limits_{n\to\infty} v_n \sum\limits_{i=1}^{n} f_i(x)x_i$, $x \in \mathcal{X}$, then $(\{x_n\}, \{f_n\}, \{v_n\})$ is called a strong RBF (or SRBF) for $\mathcal{X}^*$.

Remark 2.2. If we define $u_n \colon \mathcal{X} \to \mathcal{X}$, $n \in \mathbb{N}$ by

$$u_n(x) = v_n \sum_{i=1}^{n} f_i(x)x_i\,, \quad n \in \mathbb{N}.$$

Then one may observe that if $(\{x_n\}, \{f_n\}, \{v_n\})$ is a SRBF, then $\lim\limits_{n\to\infty} u_n(x) = x$ and $\dim u_n(\mathcal{X}) = \dim (v_n \sum\limits_{i=1}^{n} f_i(x)x_i)(\mathcal{X}) \leq n < \infty$ and so $\{x_n\}$ is an approximative basis of $\mathcal{X}$.

In the following result, we prove that the existence of SRBF in the conjugate of a Banach space guarantees the existence of a Bi-Banach frame in the Banach space along with some geometric property.

Theorem 2.3. *Let X be a Banach space and $(\{x_n\}, \{f_n\}, \{v_n\})$ be a SRBF for $\mathcal{X}^*$ with admissible sequence $\{f_n\} \subset \mathcal{X}^*$. Then $(\{x_n\}, \{f_n\})$ is a Bi-Banach frame for $\mathcal{X}$ such that $\gamma([f_n]) > 0$.*

Proof. Clearly, by definition of SRBF, $\{x \in \mathcal{X} : f_n(x) = 0, \text{for all } n \in \mathbb{N}\} = \{0\}$. Therefore, by Lemma 1.3, there exists an associated Banach space $\mathcal{X}_d = \{\{f_n(x)\}; x \in \mathcal{X}\}$ with norm given by $\|\{f_n(x)\}\|_{\mathcal{X}_d} = \|x\|_{\mathcal{X}}$, $x \in \mathcal{X}$. Define $J \colon \mathcal{X}_d \to \mathcal{X}$ by $J(\{f_n(x)\}) = x$, $x \in \mathcal{X}$. Then J is a bounded linear operator such that $(\{f_n\}, J)$ is a Banach frame for $\mathcal{X}$. Hence $(\{x_n\}, \{f_n\})$ is a Bi-Banach frame for $\mathcal{X}$. Let for each $n \in \mathbb{N}$, $v_n \colon X_n \to X_n$ be a continuous linear mapping given by $\lim\limits_{n\to\infty} v_n \sum\limits_{i=1}^{n} f_i(x)x_i = x$. Write $v_n(x_j) = \sum_{i=1}^{n} a_{ji}^{(n)} x_i$, $j = 1, 2, \dots, n$, $n \in \mathbb{N}$, where

$a_{ji}^{(n)} = f_i(v_n(x_j))$, for all $i, j = 1, 2, 3, \dots, n$, $n \in \mathbb{N}$. Thus

$$v_n\Big(\sum_{i=1}^{n} f_i(x)x_i\Big) = \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} a_{ij}^{(n)} f_i(x)\Big)x_j\,, \quad x \in \mathcal{X},\ n \in \mathbb{N}\,.$$

Define

$$h_{n,j} = \sum_{i=1}^{n} a_{ij}^{(n)} f_i\,, \quad j = 1, 2, 3, \dots, n,\ n \in \mathbb{N}\,.$$

Then $h_{n,i} \in [f_i]_{i=1}^{n}$ $(j = 1, 2, \dots, n;\ n \in \mathbb{N})$. Hence, we conclude that $\gamma([f_n]) > 0$. □

In the following result, we prove a weak duality type result.

Theorem 2.4. *Let $(\{x_n\}, \{f_n\}, \{v_n\})$ be a SRBF for $\mathcal{X}^*$ with admissible sequence $\{f_n\} \subset \mathcal{X}^*$. Then there exists a sequence of continuous linear mappings $\{\tau_n\}$ ($\tau_n \colon V_n \to V_n$, where $V_n = [f_1, f_2, \dots, f_n], n \in \mathbb{N}$) such that*

$$f(x) = \lim_{n\to\infty}\Big(\tau_n \sum_{i=1}^{n} f(x_i)f_i\Big)(x)$$

Proof. For each $k = 1, 2, 3, \dots, n$, $n \in \mathbb{N}$, define $\tau_n \colon \operatorname{span}\{f_1, f_2, \dots, f_n\} \to \operatorname{span}\{f_1, f_2, \dots, f_n\}$ by

$$\tau_n(f_k) = \sum_{i=1}^{n} a_{ik}^{(n)} f_i = \sum_{i=1}^{n} f_k(v_n(x_i))f_i\,.$$

Extend each τ_n to $[f_1, f_2, \dots, f_n]$. Then

$$\begin{aligned}\Big(\tau_n \sum_{i=1}^{n} f(x_i)f_i\Big)(x) &= \sum_{i=1}^{n} f(x_i)\big(\tau_n(f_i)\big)(x)\\ &= \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} a_{ji}^{(n)} f_j(x)\Big)f(x_i)\\ &= \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} f_i\big(v_n(x_j)\big)f_j(x)\Big)f(x_i)\\ &= \sum_{i=1}^{n} v_n\Big(\sum_{j=1}^{n} f_i(x_j)f_j(x)\Big)f(x_i)\\ &= f\Big(v_n\Big(\sum_{i=1}^{n} f_i(x)x_i\Big)\Big)\\ &=\longrightarrow f(x) \quad \text{as} \quad n \to \infty\end{aligned}$$

□

Approximative Schauder frames were defined and studied in [11]. In the following result, we prove that if a Banach space $\mathcal{X}$ has an approximative Schauder frame, then its dual space has a SRBF.

Theorem 2.5. *If a Banach space $\mathcal{X}$ has an approximative Schauder frame, then $\mathcal{X}^*$ has a SRBF.*

Proof. Let $\{u_n\}$ be a sequence of finite rank continuous linear mapping from $\mathcal{X}$ to $\mathcal{X}$ such that $\lim\limits_{n\to\infty} u_n(x) = x$, $x \in \mathcal{X}$. Let $\{x_n\}$ be a Markusevic basis for $\mathcal{X}$ with a.s.c.f. $\{f_n\} \subset \mathcal{X}^*$ such that

$$\bigcup_n u_n^*(\mathcal{X}^*) \subset [f_n]\,. \tag{2.1}$$

Since each u_n is finite dimensional, we may write

$$u_n(x) = \sum_{i=1}^{p_n} \psi_{ni}(x)\phi_{ni}\,, \qquad x \in \mathcal{X},\ n \in \mathbb{N}\,,$$

where $\{\phi_{ni}\}_{i=1}^{p_n}$ is a basis for $\{u_n(\mathcal{X})\}$ with associated sequence $\{\psi_{ni}\}_{i=1}^{p_n} \subset \mathcal{X}^*$. Let $\{g_{ni}\}_{i=1}^{p_n}$ be a sequence in $\mathcal{X}^*$ that is biorthogonal to $\{\phi_{ni}\}_{i=1}^{p_n}$. Then

$$\begin{aligned} u_n^*(g_{nj})(x) &= g_{nj}\big(u_n(x)\big) \\ &= g_{nj}\Big(\sum_{i=1}^{p_n} \psi_{ni}(x)\phi_{n_i}\Big) \\ &= \sum_{i=1}^{p_n} \psi_{ni}(x) g_{nj}(\phi_{n_i}) \\ &= \psi_{nj}(x)\,, \qquad j = 1, 2, \ldots, p_n\,. \end{aligned}$$

Hence, $\psi_{nj} \in [f_n]$, $j = 1, 2, \ldots, p_n$, $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be given. Then, for any $\epsilon > 0$, there exists an integer $m_n(\epsilon)$ such that for each $i = 1, 2, \ldots, p_n$ one can find $\overline{\phi}_{ni} \in [x_1, x_2, \ldots, x_{m_n}]$ and $\overline{\psi}_{ni} \in [f_1, f_2, \ldots, f_{m_n}]$ such that

$$\|\phi_{ni} - \overline{\phi}_{ni}\| < \epsilon \quad \text{and} \quad \|\psi_{ni} - \overline{\psi}_{ni}\| < \epsilon\,, \qquad i = 1, 2, 3, \ldots, p_n\,. \tag{2.2}$$

Write

$$\overline{v}_{m_n}(x) = \sum_{i=1}^{p_n} \overline{\psi}_{ni}(x)\overline{\phi}_{ni}\,, \qquad x \in \mathcal{X}\,.$$

Then

$$\begin{aligned} \|\overline{v}_{m_n}(x) - u_n(x)\| &= \Big\| \sum_{i=1}^{p_n} \big(\overline{\psi}_{ni}(x) - \psi_{ni}(x)\big)\overline{\phi}_{ni} + \sum_{i=1}^{p_n} \psi_{ni}(x)(\overline{\phi}_{ni} - \phi_{ni}) \Big\| \\ &\leq \Big(\sum_{i=1}^{p_n} \|\overline{\psi}_{ni} - \psi_{ni}\| \|\overline{\phi}_{ni}\| + \sum_{i=1}^{p_n} \|\psi_{ni}\| \|\overline{\phi}_{ni} - \phi_{ni}\| \Big) \|x\|\,,\ x \in \mathcal{X}\,. \end{aligned}$$

Therefore, by (2.2), taking $\epsilon = \frac{1}{n}$ and $\{m_n\}$ to be an increasing sequence, we obtain

$$\|\overline{v}_{m_n} - u_n\| < \frac{1}{n}\,. \tag{2.3}$$

Now, observe that

$$f_i\Big(x-\sum_{j=1}^{k} f_j(x)x_j\Big)=0\,, \tag{2.4}$$
$$\text{for all}\quad x\in\mathcal{X},\ i=1,2,\dots,m_n\,;\quad k=m_n,m_n+1,\dots.$$

Also, $\overline{\psi}_{ni}\in[f_1,f_2,\dots,f_{m_n}]$. So, for $x\in\mathcal{X}$, we have

$$\begin{aligned}\overline{v}_{m_n}\Big(\sum_{j=1}^{k} f_j(x)x_j\Big)&=\sum_{i=1}^{p_n}\overline{\psi}_{ni}\Big(\sum_{j=1}^{k} f_j(x)x_j\Big)\overline{\phi}_{ni}\\&=\sum_{i=1}^{p_n}\overline{\psi}_{ni}(x)\overline{\phi}_{ni}\\&=\overline{v}_{m_n}(x)\,,\quad\text{for all}\quad x\in\mathcal{X}\quad\text{and}\quad k\geq m_n\,.\end{aligned}\tag{2.5}$$

Therefore

$$\begin{aligned}\lim_{n\to\infty}\overline{v}_{m_n}\Big(\sum_{j=1}^{m_n} f_j(x)x_j\Big)&=\lim_{n\to\infty}\overline{v}_{m_n}(x)\\&=\lim_{n\to\infty}u_n(x)\\&=x\,,\qquad x\in\mathcal{X}\,.\end{aligned}\tag{2.6}$$

Define a sequence $\{T_n\}$ by $T_n(x)=\sum_{i=1}^{n}f_i(x)x_i,\ \ n\in\mathbb{N}$. Write

$$v_k=T_k|_{[x_1,x_2,\dots,x_k]}\,,\quad k=1,2,\dots,m_1-1$$

and

$$v_k=\overline{v}_{m_n}|_{[x_1,x_2,\dots,x_k]}\,,\quad k=m_n,m_n+1,\dots,m_n-1\,,\qquad n=1,2,3\dots\ .$$

Then, each v_k is a continuous linear mapping defined on $[x_1,x_2,\dots,x_k]$ with range given by

$$v_k\big([x_1,x_2,\dots,x_k]\big)=[x_1,x_2,\dots,x_k]\,;\quad k=1,2,3,\dots,m_1-1$$

$$v_k\big([x_1,x_2,...x_k]\big)\subset[x_1,x_2,\dots,x_k]\,,\quad k=m_n,m_n+1,\dots,m_{n+1}-1\,,\quad n\in\mathbb{N}\,.$$

Using (2.5) and (2.6), we obtain $\lim\limits_{n\to\infty}\overline{v}_{m_n}(x)=x$. Hence $\lim\limits_{n\to\infty}v_n\big(\sum_{i=1}^{n}f_i(x)x_i\big)=x$. □

In view of the proof of Theorem 2.5, one may observe that existence of approximative Schauder frame in a Hilbert space is a sufficient condition for the space having Markusevic basis to have a SRBF. More precisely we have

Corollary 2.6. *Let $\mathcal{H}$ be a Hilbert space with an approximative Schauder frame. Then every Markusevic basis of $\mathcal{H}$ give rise to a SRBF for $\mathcal{H}$.*

In the following example, we show that in general, a SRBF do not have strong duality

Example 2.7. Let $\mathcal{X}$ be a Banach space with a Schauder basis and such that $\mathcal{X}^*$ is separable but fails to have approximative property. Let $\{x_n\}$ be a shrinking Markusevic basis of $\mathcal{X}$ with associated sequence of coefficient functional $\{f_n\} \subset \mathcal{X}^*$. Define

$$u_n(x) = \sum_{i=1}^{n} f_i(x)x_i\,, \qquad x \in \mathbb{N}\,.$$

Then $\{x_n\}$ is an approximative Schauder frame for $\mathcal{X}$ satisfying $\bigcup_{n=1}^{\infty} {u_n}^*(\mathcal{X}^*) \subset [f_n]$. Therefore $(\{x_n\}, \{f_n\}, \{v_n\})$ is a SRBF for $\mathcal{X}^*$. However, $\mathcal{X}^{**}$ has no SRBF.

One may observe that in Definition 2.1, each v_n is linear. Now, we would like to drop this condition of linearity and in the process define non-linear SRBF.

Definition 2.8. A SRBF $(\{x_n\}, \{f_n\}, \{v_n\})$ is called non-linear SRBF if each v_n is continuous but not necessarily linear.

Finally, we prove the following result related to the existence of a non-linear SRBF.

Theorem 2.9. *If $\mathcal{X}$ is a separable Banach Space, then $\mathcal{X}^*$ has a non-linear SRBF.*

Proof. Let $\{x_n\}$ be a Markusevic basis with a sequence of coefficient functional $\{f_n\} \subset \mathcal{X}^*$ such that $\gamma([f_n]) > 0$. Then, by Lemma 1.4, there is a norm $|\cdot|$ on $\mathcal{X}$ that is equivalent to the original norm $\|\cdot\|$ on $\mathcal{X}$ such that $\mathcal{X}$ with this new norm $|\cdot|$ is strictly convex. Therefore, by [15, Corollary 3.3, page 110], for every finite dimensional subspace $\mathcal{G}$ of $\mathcal{X}$ and for every $x \in \mathcal{X} \setminus \mathcal{G}$, there is a unique $\pi_{\mathcal{G}}(x) \in \mathcal{G}$ such that $|x - \pi_{\mathcal{G}}(x)| = \operatorname{dist}(x, \mathcal{G}) = \min_{x \in \mathcal{G}} |x - g|$ and such that the mapping $\pi_{\mathcal{G}} \colon \mathcal{X} \to \mathcal{G}$ is continuous (here note that, in general, $\pi_{\mathcal{G}}$ is non-linear). Let $\mathfrak{N}(a, b)$ denote a positive integer depending on a and b. For each n, choose an increasing sequence of positive integers $\{m_n\}$ with $m_1 = \mathfrak{N}(1, 1)$, $m_2 = \mathfrak{N}(m_1, \frac{1}{2})$, $m_3 = \mathfrak{N}(m_2, \frac{1}{3})$, $\ldots$, $m_n = \mathfrak{N}(m_{n-1}, \frac{1}{n})$, for all $n \geq 2$ and satisfying

$$\operatorname{dist}\,\left(a, [x_i]_{i=m_{n-1}+1}^{m_n}\right) \leq \left(1 + \frac{1}{n}\right) \operatorname{dist}\,\left(a, [x_i]_{i=m_{n-1}+1}^{\infty}\right),$$

where $a \in [x_i]_{i=1}^{m_{n-1}}$. Define $\{v_n\}$ by $v_k = T_k|_{[x_1,\ldots,x_k]}$, $k = 1, 2, \ldots, m_1 - 1$, where $T_k(x) = \sum_{i=1}^{k} f_i(x)x_i$ and for any $b = \sum_{i=1}^{k} a_i x_i \in [x_i]_{i=1}^{k}$, $(k = m_n, m_n + 1, \ldots, m_{n+1} - 1;\ n \in \mathbb{N})$

$$v_k(b) = \sum_{i=1}^{m_{n-1}} a_i x_i - \pi_{\mathcal{G}}\Big(\sum_{i=1}^{m_{n-1}} a_i x_i\Big),$$

where $\mathcal{G} = [x_i]_{i=m_{n-1}+1}^{m_n}$. Then each v_n is continuous (in general, non-linear) with range given by

$$v_k\big([x_1, \ldots, x_k]\big) = [x_1, \ldots, x_k]\,, \quad k = 1, 2, 3, \ldots, m_1 - 1$$

$$v_k\big([x_1, \ldots, x_k]\big) \subset [x_1, \ldots, x_k]\,, \quad (k = m_n, m_n + 1, \ldots, m_{n+1} - 1;\ n \in \mathbb{N})\,.$$

Let $x \in \mathcal{X}$ be any element. Then

$$f_i\Big(v_k\Big(\sum_{j=1}^{k} f_j(x)x_j\Big)\Big) = f_i(x)\,,\ i = 1, 2, \dots, m_n\,;\ k = m_n, m_n{+}1, \dots, m_{n+1}{-}1; n \in \mathbb{N}\,.$$

This gives

$$\lim_{k\to\infty} f_i\Big(v_k\Big(\sum_{j=1}^{k} f_j(x)x_j\Big)\Big) = f_i(x)\,, \quad i = 1, 2, \dots\,. \tag{2.7}$$

In view of Lemma 1.4, we have

$$\lim_{k\to\infty} \Big|v_k\Big(\sum_{i=1}^{k} f_i(x)x_i\Big)\Big| \ge |x|\,. \tag{2.8}$$

Also, we have

$$\begin{aligned}
v_k\Big(\sum_{i=1}^{k} f_i(x)x_i\Big) &= \Big|\sum_{i=1}^{m_{n-1}} f_i(x)x_i - \pi_{\mathcal{G}} \sum_{i=1}^{m_{n-1}} f_i(x)x_i\Big| \\
&= \operatorname{dist}\Big(\sum_{i=1}^{m_{n-1}} f_i(x)x_i, \mathcal{G}\Big) \\
&\le \Big(1 + \frac{1}{n}\Big) \operatorname{dist}\Big(\sum_{i=1}^{m_{n-1}} f_i(x)x_i, [x_i]_{i=m_{n-1}+1}^{\infty}\Big) \\
&\le \Big(1 + \frac{1}{n}\Big)\Big|\sum_{i=1}^{m_{n-1}} f_i(x)x_i + \Big(x - \sum_{i=1}^{m_{n-1}} f_i(x)x_i\Big)\Big| \\
&= \Big(1 + \frac{1}{n}\Big)|x|\,, \quad k = m_n, m_n + 1, \quad m_{n+1} - 1;\ n \in \mathbb{N}\,.
\end{aligned}$$

Thus, by (2.8), we have

$$\lim_{k\to\infty} \Big|v_k\Big(\sum_{i=1}^{k} f_i(x)x_i\Big)\Big| = |x|\,. \tag{2.9}$$

Hence, we conclude that

$$\lim_{k\to\infty} \Big|v_k\Big(\sum_{i=1}^{k} f_i(x)x_i\Big) - x\Big| = 0\,.$$

Since $|\cdot|$ is equivalent to the initial norm of $\mathcal{X}$, we obtain

$$\lim_{n\to\infty} v_n\Big(\sum_{i=1}^{n} f_i(x)x_i\Big) = x\,, \quad x \in \mathcal{X}\,.$$

□

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