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CHARACTERIZATIONS OF z-LINDELÖF SPACES

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Abstract. A topological space (X,τ) is said to be z-Lindelöf [1] if every cover of X by cozero sets of (X,τ) admits a countable subcover. In this paper, we obtain new characterizations and preservation theorems of z-Lindelöf spaces.

1. Introduction

A subset H of a topological space (X,τ) is called a cozero set if there is a continuous real-valued function g on X such that $H=\{x\in X : g(x)\neq 0\}$. The complement of a cozero set is called a zero set. Recently papers [2, 3, 4, 5, 8, 9] have introduced some new classes of functions via cozero sets. It is well known [6] that the countable union of cozero sets is a cozero set and the intersection of two cozero sets is a cozero set, so the collection of all cozero subsets of (X,τ) is a base for a topology τ_z on X, called the complete regularization of τ. It is clear that $\tau_z\subseteq\tau$ in general. Furthermore, the space (X,τ) is completely regular if and only if $\tau_z=\tau$. In general for any topological space τ, we note that (X,τ_z) is completely regular.

Throughout this paper, (X,τ) and (Y,σ) stand for topological spaces on which no separation axiom is assumed, unless otherwise stated. For a subset A of X, the closure of A and the interior of A will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A point $x\in X$ is called a condensation point of A if for each $U\in\tau$ with $x\in U$, the set $U\cap A$ is uncountable. A is said to be ω-closed [7] if it contains all its condensation points. The complement of an ω-closed set is said to be ω-open. It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x\in W$, there exists $U\in\tau$ such that $x\in U$ and $U-W$ is countable.

2. ω-cozero sets

In this section we introduce the following notion:

Definition 2.1. A subset A of (X,τ) is said to be ω-cozero if for each $x\in A$ there exists a cozero set U_x containing x such that U_x-A is a countable set. The complement of an ω-cozero is said to be ω-zero.

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The family of all ω-cozero (resp. cozero, zero) subsets of a space (X,τ) is denoted by $\omega ZO(X)$ (resp. $ZO(X)$, $ZC(X)$).

Lemma 2.1. *For a subset of a topological space (X,τ), every cozero set is ω-cozero and every ω-cozero set is ω-open.*

Proof. (1) Let A be a cozero set. For each $x \in A$, there exists a cozero set $U_x = A$ such that $x \in U_x$ and $U_x - A = \phi$. Therefore, A is ω-cozero.
(2) Assume A is an ω-cozero set. Then for each $x \in A$, there is a cozero set U_x containing x such that $U_x - A$ is a countable set. Since every cozero set is open, A is ω-open. □

For a subset of a topological space, the following implications hold and none of these implications is reversible.

$$\begin{tikzcd}
\text{cozero} \arrow[r] \arrow[d] & \omega\text{-cozero} \arrow[d] \\
\text{open} \arrow[r] & \omega\text{-open}
\end{tikzcd}$$

Diagram I.

Example 2.1. Let $\mathbb{R}$ be the set of all real numbers with the usual topology and $\mathbb{Q}$ the set of all rational numbers. Then $A = \mathbb{R} - \mathbb{Q}$ is an ω-cozero set but it is not open.

Example 2.2. Let X be a set and A be a subset of X such that A and $X - A$ are uncountable. Let τ=$\{\phi, X, A\}$. Then $\{A\}$ is an open set but it is not ω-cozero set.

Theorem 2.1. *Let (X,τ) be a topological space. Then $(X, \omega ZO(X))$ is a topological space.*

Proof.

(1) We have $\phi, X \in \omega ZO(X)$.

(2) Let $U, V \in \omega ZO(X)$ and $x \in U \cap V$. Then there exist cozero sets G, H of X containing x such that $G \setminus U$ and $H \setminus V$ are countable. And $(G \cap H) \setminus (U \cap V) = (G \cap H) \cap ((X \setminus U) \cup (X \setminus V)) \subseteq (G \cap (X \setminus U)) \cup (H \cap (X \setminus V))$. Thus $(G \cap H) \setminus (U \cap V)$ is countable. Since the intersection of two cozero sets is cozero, $U \cap V \in \omega ZO(X)$.

(3) Let $\{U_i : i \in I\}$ be a family of ω-cozero sets of X and $x \in \underset{i\in I}{\cup} U_i$. Then $x \in U_j$ for some $j \in I$. This implies that there exists a cozero set V of X containing x such that $V \setminus U_j$ is countable. Since $V \setminus \underset{i\in I}{\cup} U_i \subseteq V \setminus U_j$, then $V \setminus \underset{i\in I}{\cup} U_i$ is countable. Thus $\underset{i\in I}{\cup} U_i \in \omega ZO(X)$.

□

Lemma 2.2. *A subset A of a space X is ω-cozero if and only if for every $x \in A$, there exist a cozero set U_x containing x and a countable subset C such that $U_x - C \subseteq A$.*

Proof. Let A be ω-cozero and $x \in A$, then there exists a cozero set U_x containing x such that $U_x - A$ is countable. Let $C = U_x - A = U_x \cap (X - A)$. Then $U_x - C \subseteq A$. Conversely, let $x \in A$. Then there exist a cozero set U_x containing x and a countable subset C such that $U_x - C \subseteq A$. Thus $U_x - A \subseteq C$ and $U_x - A$ is a countable set. □

Theorem 2.2. *Let X be a space and $F \subseteq X$. If F is an ω-zero set, then $F \subseteq K \cup C$ for some zero subset K and a countable subset C.*

Proof. If F is an ω-zero set, then $X - F$ is an ω-cozero set and hence for each $x \in X - F$, there exist a cozero set U_x containing x and a countable set C_x such that $U_x - C_x \subseteq X - F$. Thus $F \subseteq X - (U_x - C_x) = X - (U_x \cap (X - C_x)) = (X - U_x) \cup C_x$. Let $K = X - U_x$. Then K is a zero set such that $F \subseteq K \cup C_x$. □

3. z-Lindelöf spaces

Definition 3.1.

(1) A topological space X is said to be *z-Lindelöf* [1] if every cover of X by cozero sets admits a countable subcover.

(2) A subset A of a space X is said to be *z-Lindelöf relative to X* if every cover of A by cozero sets of X admits a countable subcover.

Theorem 3.1. *For any space X, the following properties are equivalent:*

(1) *X is z-Lindelöf;*

(2) *Every cover of X by ω-cozero sets of X admits a countable subcover.*

Proof. (1) ⇒ (2): Let $\{U_\alpha : \alpha \in \Lambda\}$ be any cover of X by ω-cozero sets of X. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is ω-cozero, there exists a cozero set $V_{\alpha(x)}$ such that $x \in V_{\alpha(x)}$ and $V_{\alpha(x)} \backslash U_{\alpha(x)}$ is countable. The family $\{V_{\alpha(x)} : x \in X\}$ is a cover of X by cozero sets of X. Since X is z-Lindelöf, there exist $\{x_i : i < \omega\} \subseteq X$ such that $X = \cup\{V_{\alpha(x_i)} : i < \omega\}$. Now, we have

$$\begin{aligned} X &= \bigcup_{i<\omega} \left((V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}) \cup U_{\alpha(x_i)} \right) \\ &= \Big(\bigcup_{i<\omega} (V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}) \Big) \cup \Big(\bigcup_{i<\omega} U_{\alpha(x_i)} \Big). \end{aligned}$$

For each $\alpha(x_i)$, $V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}$ is a countable set and there exists a countable subset $\Lambda_{\alpha(x_i)}$ of Λ such that $V_{\alpha(x_i)} \backslash U_{\alpha(x_i)} \subseteq \cup\{U_\alpha : \alpha \in \Lambda_{\alpha(x_i)}\}$. Therefore, we have

$$X \subseteq \Big(\bigcup_{i<\omega} (\cup\{U_\alpha : \alpha \in \Lambda_{\alpha(x_i)}\}) \Big) \cup \Big(\bigcup_{i<\omega} U_{\alpha(x_i)} \Big).$$

(2) ⇒ (1): Since every cozero set is ω-cozero, the proof is obvious. □

We state the following proposition without proof.

Proposition 3.1. *A topological space X is z-Lindelöf if and only if for every family of ω-zero sets $\{F_\alpha : \alpha \in \Lambda\}$ of X, $\cap_{\alpha\in\Lambda} F_\alpha = \phi$ implies that there exists a countable subset $\Lambda_0 \subseteq \Lambda$ such that $\cap_{\alpha\in\Lambda_0} F_\alpha = \phi$.*

Proposition 3.2. *A topological space X is z-Lindelöf if and only if for every family $\{F_\alpha : \alpha \in \Lambda\}$ of ω-zero sets with countable intersection property, $\cap_{\alpha\in\Lambda} F_\alpha \neq \phi$.*

Proof. *Necessity.* Let X be a z-Lindelöf space and suppose that $\{F_\alpha : \alpha \in \Lambda\}$ be a family of ω-zero subsets of X with countable intersection property such that $\cap_{\alpha\in\Lambda} F_\alpha = \phi$. Let us consider the ω-cozero sets $U_\alpha = X \setminus F_\alpha$, the family $\{U_\alpha : \alpha \in \Lambda\}$ is a cover of X by ω-cozero sets of X. Since X is z-Lindelöf, the cover $\{U_\alpha : \alpha \in \Lambda\}$ has a countable subcover $\{U_{\alpha_i} : \alpha_i \in \mathbb{N}\}$. Therefore $X = \cup\{U_{\alpha_i} : \alpha_i \in \mathbb{N}\} = \cup\{(X \setminus F_{\alpha_i}) : \alpha_i \in \mathbb{N}\} = X \setminus \cap\{F_{\alpha_i} : \alpha_i \in \mathbb{N}\}$ and hence $\cap\{F_{\alpha_i} : \alpha_i \in \mathbb{N}\} = \phi$. Thus, if the family $\{F_\alpha : \alpha \in \Lambda\}$ of ω-zero sets with countable intersection property, then $\cap_{\alpha\in\Lambda} F_\alpha \neq \phi$.

Sufficiency. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of X by ω-cozero sets of X and suppose that for every family $\{F_\alpha : \alpha \in \Lambda\}$ of ω-zero sets with countable intersection property, $\cap_{\alpha\in\Lambda} F_\alpha \neq \phi$. Then $X = \cup\{U_\alpha : \alpha \in \Lambda\}$. Therefore, $\phi = X \setminus X = \cap\{(X \setminus U_\alpha) : \alpha \in \Lambda\}$ and $\{X \setminus U_\alpha : \alpha \in \Lambda\}$ is a family of ω-zero sets with an empty intersection. By the hypothesis, there exists a countable subset $\{(X \setminus U_{\alpha_i}) : i \in \mathbb{N}\}$ such that $\cap\{(X \setminus U_{\alpha_i}) : i \in \mathbb{N}\} = \phi$; hence $X \setminus \cap\{(X \setminus U_{\alpha_i}) : i \in \mathbb{N}\} = X = \cup\{U_{\alpha_i} : i \in \mathbb{N}\}$. Thus, X is z-Lindelöf. $\square$

Theorem 3.2. *Every ω-zero set of a z-Lindelöf space X is z-Lindelöf relative to X.*

Proof. Let A be an ω-zero set of X. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of A by cozero sets of X. Now for each $x \in X - A$, there is a cozero set V_x such that $V_x \cap A$ is countable. Since X is z-Lindelöf and the collection $\{U_\alpha : \alpha \in \Lambda\} \cup \{V_x : x \in X - A\}$ is a cover of X by cozero sets of X, there exists a countable subcover $\{U_{\alpha_i} : i \in \mathbb{N}\} \cup \{V_{x_i} : i \in \mathbb{N}\}$. Since $\bigcup_{i\in N}(V_{x_i} \cap A)$ is countable, so for each $x_j \in \cup(V_{x_i} \cap A)$, there is $U_{\alpha(x_j)} \in \{U_\alpha : \alpha \in \Lambda\}$ such that $x_j \in U_{\alpha(x_j)}$ and $j \in \mathbb{N}$. Hence $\{U_{\alpha_i} : i \in \mathbb{N}\} \cup \{U_{\alpha(x_j)} : j \in \mathbb{N}\}$ is a countable subcover of $\{U_\alpha : \alpha \in \Lambda\}$ and it covers A. Therefore, A is z-Lindelöf relative to X. $\square$

Corollary 3.1. *Every zero set of a z-Lindelöf space X is z-Lindelöf relative to X.*

The topology generated by the cozero sets of the space X is denoted by τ_z.

Definition 3.2. A topological space (X, τ) is said to be completely ω-regular if for each $x \in X$ and each open set U_x containing x, there exists an ω-cozero set H_x such that $x \in H_x \subseteq U_x$.

Proposition 3.3. *A completely ω-regular is z-Lindelöf if and only if it is Lindelöf.*

Proof. Let X be completely ω-regular. Suppose that X is a z-Lindelöf space and let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be any open cover of X. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since X is completely ω-regular, there exists an ω-cozero set $H_{\alpha(x)}$ such that $x \in H_{\alpha(x)} \subseteq U_{\alpha(x)}$. Then $\{H_{\alpha(x)} : x \in X\}$ is a cover of X by ω-cozero sets of X. By Theorem 3.1, there exists a countable subcover $\{H_{\alpha(x_i)} : i \in \mathbb{N}\}$. Therefore, $\{U_{\alpha(x_i)} : i \in \mathbb{N}\}$ is a countable subcover of $\mathcal{U}$. Hence X is Lindelöf. The converse is obvious. $\square$

Definition 3.3. A topological space (X, τ) is said to be almost ω-regular if for each $x \in X$ and each ω-cozero set U_x containing x, there exists a cozero set V_x such that $x \in V_x \subseteq \mathrm{Cl}(V_x) \subseteq U_x$.

Theorem 3.3. *Let X be an almost ω-regular and z-Lindelöf space. Then for every disjoint ω-zero sets C_1 and C_2, there exist two open sets U and V such that $C_1 \subseteq U$, $C_2 \subseteq V$ and $U \cap V = \phi$.*

Proof. Since X is an almost ω-regular space, for each $x \in C_1$ there exists a cozero set U_x containing x such that $\mathrm{Cl}(U_x) \cap C_2 = \phi$. Then the family $\{U_x : x \in C_1\} \cup \{X - C_1\}$ is an ω-cozero cover of X. Since X is z-Lindelöf, by Theorem 3.1 there exists $\{x_i : i < \omega\} \subseteq X$ such that $X = \left(\underset{i<\omega}{\cup} U_{x_i}\right) \cup (X - C_1)$. It follows that for each $i < \omega$, $C_1 \subseteq \left(\underset{i<\omega}{\cup} U_{x_i}\right)$ and $\mathrm{Cl}(U_{x_i}) \cap C_2 = \phi$. Analogously there exists a family of cozero sets V_{y_i} such that $C_2 \subseteq \left(\underset{i<\omega}{\cup} V_{y_i}\right)$ and $\mathrm{Cl}(V_{y_i}) \cap C_1 = \phi$. Let $G_k = U_{x_k} \setminus \left(\overset{k}{\underset{i=1}{\cup}} \mathrm{Cl}(V_{y_i})\right)$, $H_k = V_{y_k} \setminus \left(\overset{k}{\underset{i=1}{\cup}} \mathrm{Cl}(U_{x_i})\right)$ and $U = \underset{i<\omega}{\cup} G_i$, $V = \underset{i<\omega}{\cup} H_i$ such that U and V are open, $U \cap V = \phi$ and $C_1 \subseteq U$, $C_2 \subseteq V$. □

4. Preservation theorems

Definition 4.1. A function $f\colon X \to Y$ is said to be cozero-irresolute if for each $x \in X$ and each cozero set V of Y containing $f(x)$, there exists a cozero set U of X containing x such that $f(U) \subseteq V$.

Definition 4.2. A function $f\colon X \to Y$ is said to be ω-cozero-continuous if for each $x \in X$ and each cozero set V of Y containing $f(x)$, there exists an ω-cozero set U of X containing x such that $f(U) \subseteq V$.

It is clear that every cozero-irresolute function is ω-cozero-continuous.

Theorem 4.1. *Let $f\colon X \to Y$ be a ω-cozero-continuous surjection. If X is z-Lindelöf, then Y is z-Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of Y by cozero sets of Y. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is ω-cozero–continuous, there exists an ω-cozero set $U_{\alpha(x)}$ of X containing x such that $f(U_{\alpha(x)}) \subseteq V_{\alpha(x)}$. So $\{U_{\alpha(x)} : x \in X\}$ is a cover of the z-Lindelöf space X by ω-cozero sets of X, by Theorem 3.1 there exists a countable subset $\{x_k : k < \omega\} \subseteq X$ such that $X = \underset{k<\omega}{\cup} U_{\alpha(x_k)}$. Therefore $Y = f(X) = f(\underset{k<\omega}{\cup} U_{\alpha(x_k)}) = \underset{k<\omega}{\cup} f\left(U_{\alpha(x_k)}\right) \subseteq \underset{k<\omega}{\cup} V_{\alpha(x_k)}$. This shows that Y is z-Lindelöf. □

Corollary 4.1. *Let $f\colon X \to Y$ be a cozero-irresolute surjection. If X is z-Lindelöf, then Y is z-Lindelöf.*

Definition 4.3. A function $f\colon X \to Y$ is said to be almost cozero, if the image of each cozero set U of X is an open set in Y.

Proposition 4.1. *If $f: X \to Y$ is almost cozero, then the image of an ω-cozero set of X is ω-open in Y.*

Proof. Let $f: X \to Y$ be almost cozero and W an ω-cozero set of X. Let $y \in f(W)$, there exists $x \in W$ such that $f(x) = y$. Since W is an ω-cozero set, there exists a cozero set U such that $x \in U$ and $U - W = C$ is countable. Since f is almost cozero, $f(U)$ is an open set in Y such that $y = f(x) \in f(U)$ and $f(U) - f(W) \subseteq f(U - W) = f(C)$. Moreover, $f(C)$ is countable. Therefore, $f(W)$ is ω-open in Y. $\square$

Definition 4.4. A function $f: X \to Y$ is said to be ω^*-cozero-continuous if $f^{-1}(V)$ is ω-cozero in X for each open set V in Y.

Theorem 4.2. *Let $f: X \to Y$ be an ω^*-cozero–continuous surjection. If X is z-Lindelöf, then Y is Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is a cover of X by ω-cozero sets of X. Since X is z-Lindelöf, by Theorem 3.1, X has a countable subcover, say $\{f^{-1}(V_\alpha) : \alpha \in \Lambda_0\}$, where Λ_0 is a countable subset of Λ. Hence $\{V_\alpha : \alpha \in \Lambda_0\}$ is a countable subcover of Y. Hence Y is Lindelöf. $\square$

Definition 4.5. A function $f: X \to Y$ is said to be *ω-zero* if $f(A)$ is ω-zero in Y for each zero set A of X.

Theorem 4.3. *If $f: X \to Y$ is an ω-zero surjection such that $f^{-1}(y)$ is z-Lindelöf relative to X for each $y \in Y$, and Y is z-Lindelöf, then X is z-Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be any cover of X by cozero sets of X. For each $y \in Y$, $f^{-1}(y)$ is z-Lindelöf relative to X and there exists a countable subset $\Lambda(y)$ of Λ such that $f^{-1}(y) \subset \cup\{U_\alpha : \alpha \in \Lambda(y)\}$. Now we put $U(y) = \cup\{U_\alpha : \alpha \in \Lambda(y)\}$ which is a cozero set and $V(y) = Y - f(X - U(y))$. Then, since f is ω-zero, $V(y)$ is an ω-cozero set in Y containing y such that $f^{-1}(V(y)) \subset U(y)$. Since $\{V(y) : y \in Y\}$ is a cover of Y by ω-cozero sets of Y, by Theorem 3.1 there exists a countable set $\{y_k : k < \omega\} \subseteq Y$ such that $Y = \cup\{V(y_k) : k < \omega\}$. Therefore, $X = f^{-1}(Y) = \cup\{f^{-1}(V(y_k)) : k < \omega\} \subseteq \cup\{U(y_k) : k < \omega\} = \cup\{U_\alpha : \alpha \in \Lambda(y_k), k < \omega\}$. This shows that X is z-Lindelöf. $\square$

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