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EXISTENCE RESULTS FOR A CLASS OF HIGH ORDER DIFFERENTIAL EQUATION ASSOCIATED WITH INTEGRAL BOUNDARY CONDITIONS AT RESONANCE

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Abstract. By using Mawhin's continuation theorem, we provide some sufficient conditions for the existence of solution for a class of high order differential equations of the form

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}), \quad t \in [0,1],$$

associated with the integral boundary conditions at resonance. The interesting point is that we shall deal with the case of nontrivial kernel of arbitrary dimension corresponding to high order differential operator which will cause some difficulties in constructing the generalized inverse operator.

1. Introduction

In this paper, we consider the n^{th} order differential equation

$$x^{(n)}(t) = f(t, x, x', \dots, x^{(n-1)}), \quad t \in (0,1), \tag{1.1}$$

subjected to the integral boundary conditions

$$\alpha_i x^{(i-1)}(0) + \beta_i x^{(i-1)}(1) = \gamma_i \int_0^1 x(s)ds, \quad i = 1, 2, \dots, n, \tag{1.2}$$

where α_i, β_i and γ_i, $i = 1, 2, \dots, n$ are real constants.

The problem (1.1)–(1.2), as we will see in the next sections, can be written in operator form

$$Lx = Nx, \tag{1.3}$$

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(called semilinear) where L is a linear and N is a nonlinear operator in appropriate function spaces. If L has a trivial null space, Ker L, the problem (1.3) is said to be at nonresonance case. Otherwise, we call the problem (1.3) at resonance. In the resonance case, the problem (1.3) can be studied by various methods including the alternative method, the continuation method of Mawhin and perturbation method (see [19]).

In topological approach, the boundary value problems (BVPs) for the second order ordinary differential equations has been studied by many authors with different boundary conditions for both cases non-resonance [8, 9] and resonance [3, 2, 7, 11, 12, 15, 10, 6, 4, 21].

However, there is rarely works that have been done for higher order BVPs particularly at resonance case. In the nonresonance case, we refer to [17, 18, 20] and references therein. For the resonance case, W. Ge at al., [1, 14, 13] studied the high order ordinary differential equation

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}) + e(t)\,,$$

associated with multi-point conditions

$$x'(0) = 0,\ x''(0) = 0, \dots, x^{(n-1)}(0) = 0\,, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i)\,, \quad \text{or}$$

$$x'(0) = 0,\ x''(0) = 0, \dots, x^{(n-1)}(0) = 0\,, \quad x'(1) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i)\,, \ \text{or}$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i)\,, \quad x'(0) = 0, \dots, x^{(n-2)}(0) = 0\,, \quad x(1) = x(\eta)\,,$$

where $f\colon [0,1] \times \mathbb{R}^{n-1} \to \mathbb{R}$ is a continuous function and $e \in L^1(0,1)$. In these setting, the set of nontrivial solutions of the associated homeogeneous problem $Lx = x^{(n)} = 0$, $\operatorname{Ker} L$, is isomorphic to $\mathbb{R}$. Then the authors use the coincidence degree theory of Mawhin [5] in order to prove the existence of solutions of high-order multi-point BVPs. However, to our best knowledge, the high-order BVPs with the higher dimension of null space has not been studied broadly.

Motivated by these works, in this paper, we discuss the existence of solutions of problem (1.1)–(1.2) at resonance, in which the dimension of null space $\operatorname{Ker} L$ might be $1, 2, \dots, n$. It is noticed that in the way of Mawhin's approach, the higher dimension of kernel, the more difficult to construct the projections P, Q, and this paper will contribute a slightly general way to construct such projectors.

Our paper is organized as follows. In Section 2, we first recall some abstract results from the coincidence degree theory for L-compact operator and then give the conditions on the nonlinear term $f(t, x, x', \dots, x^{(n-1)})$ to be such operator. Finally, in Section 3, we prove the existence result of the problem (1.1)–(1.2) and give an example to demonstrate the results.

2. Preliminaries

2.1. **A continuation theorem.** We start this section by recalling some definitions and abstract results from the coincidence degree theory. For more details we refer the readers to [5, 16].

Assume that X and Z are two real Banach spaces.

Definition 2.1 (see [5])**.** Let $L\colon \operatorname{dom}(L) \subset X \to Z$ be a linear operator. The operator L is said to be a Fredholm operator of index zero if the following conditions hold:

(i) $\operatorname{Im} L$ is a closed subset of Z,

(ii) $\operatorname{codim} \operatorname{Im} L = \dim \operatorname{Ker} L < +\infty$.

It follows from the Definition 2.1 that if L is a Fredholm operator of index zero, then there exist continuous projectors $P\colon X \to X$ and $Q\colon Z \to Z$ such that

$$\operatorname{Im} P = \operatorname{Ker} L\,, \quad \operatorname{Ker} Q = \operatorname{Im} L\,, \quad X = \operatorname{Ker} L \oplus \operatorname{Ker} P\,, \quad Z = \operatorname{Im} L \oplus, \operatorname{Im} Q\,.$$

Further, the restriction of L on $\operatorname{dom} L \cap \operatorname{Ker} P$, which is $L_P\colon \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$, is invertible. We denote by K_P the inverse of L_P and $K_{P,Q} := K_P(I-Q)$ the generalized inverse of L. In addition, if L has index zero (i.e., $\operatorname{Im} Q$ and $\operatorname{Ker} L$ are isomorphic) then the operator $JQ + K_{P,Q}\colon Z \to \operatorname{dom} L$ is isomorphism, and

$$(JQ + K_{P,Q})^{-1} = \left(L + J^{-1}P\right)\big|_{\operatorname{dom} L}\,,$$

for every isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$. It follows from Mawhin's equivalent theorem that $x \in \overline{\Omega}$ is a solution to equation $Lx = Nx$ if and only if it is a fixed point of Mawhin's operator

$$\Phi := P + (JQ + K_{P,Q})\,N\,,$$

where Ω is an given open bounded subset of X such that $\operatorname{dom}(L) \cap \Omega \neq \emptyset$.

Definition 2.2 (see [5])**.** Let $L\colon \operatorname{dom}(L) \subset X \to Z$ be a Fredholm mapping of index zero. The operator $N\colon \overline{\Omega} \to Z$ is said to be L-compact operator on $\overline{\Omega}$ if:

a) the map $QN\colon \overline{\Omega} \to Z$ is continuous and $QN\left(\overline{\Omega}\right)$ is bounded in Z,

b) the map $K_{P,Q}N\colon \overline{\Omega} \to X$ is completely continuous.

In addition, we say that N is L-completely continuous if it is L-compact on every bounded subset in X.

The following continuation theorem due to Mawhin [5] will be used for our main purpose.

Theorem 2.3. *Let $\Omega \subset X$ be open and bounded, L be a Fredholm operator of index zero and N be L-compact operator on Ω. Assume that the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in (\mathrm{dom}\,(L) \setminus \mathrm{Ker}\ L) \cap \partial\Omega \times (0,1)$;*

(2) *$QNx \neq 0$ for every $x \in \mathrm{Ker}\ L \cap \partial\Omega$;*

(3) *for some isomorphism $J\colon \mathrm{Im}\ Q \to \mathrm{Ker}\ L$, we have*
$$\deg\,(JQN; \Omega \cap \mathrm{Ker}\ L, 0) \neq 0\,,$$
where $Q\colon Z \to Z$ is a projector given as above.

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{dom}\,(L) \cap \overline{\Omega}$.

2.2. **Some preliminary results.** In order to obtain the existence of solutions for (1.1)–(1.2) by applying the Theorem 2.3, we first formulate the problem (1.1)–(1.2) as a semilinear equation in Banach spaces and then offer some certain conditions on the nonlinearity $f(t,x,\dots,x^{(n-1)})$ so that it is L-completely continuous. Let $\nu \in \mathbb{Z}^+$, we denote
$$I^\nu z(t) := \frac{1}{(\nu-1)!}\int_0^t (t-s)^{\nu-1} z(s)\,ds\,,$$
for convenient reason. Let X be the Banach space $C^{n-1}[0,1]$ with the norm
$$\|x\| = \max\left\{\|x^{(i)}\|_\infty : 1 = 1,\dots,n-1\right\},$$
and Z be the Banach space $L^1(0,1)$ with its usual norm $\|\cdot\|_1$.

We now define L the linear operation from $\mathrm{dom}\,(L) \subset X$ to Z by $Lx = x^{(n)}$, for $x \in \mathrm{dom}\,(L)$, where
$$\mathrm{dom}\,(L) =$$
$$\left\{x \in AC^n[0,1] : \alpha_i x^{(i-1)}(0) + \beta_i x^{(i-1)}(1) = \gamma_i \int_0^1 x(s)\,ds\,,\ i = 1,2,\dots,n\right\}.$$
We also define the nonlinear operator $N\colon X \to Z$ by
$$Nx(t) = f\big(t, x(t), x'(t), \dots, x^{(n-1)}(t)\big)\,,\quad t \in (0,1)\,. \tag{2.1}$$

Then the problem (1.1)–(1.2) is equivalent to the abstract equation

$$Lx = Nx\,.$$

We shall show that under certain conditions then L is a Fredholm operator of index zero and N is L-completely continuous.

First, we shall to look for the kernel and the range of L. To determine the Ker L, we suppose $Lx = 0$ for $x \in \mathrm{dom}(L)$ which implies

$$x(t) = c_1 + c_2 t + \cdots + \frac{c_n}{(n-1)!} t^{n-1}\,.$$

Using the boundary conditions (1.2), we derive

$$\mathcal{A}\begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}^T = 0\,,$$

where $\mathcal{A} = (a_{ij})$ is a square matrix of order n with

$$a_{1j} = \begin{cases} \alpha_1 + \beta_1 - \gamma_1 & \text{if} \quad j = 1 \\ ]\frac{\beta_1}{(j-1)!} - \frac{\gamma_1}{j!} & \text{if } 2 \le j \le n\,, \end{cases}$$

and

$$a_{ij} = \begin{cases} -\frac{\gamma_i}{j!} & \text{if} \quad 1 \le j \le i-1 \\ \alpha_i + \beta_i - \frac{\gamma_i}{i!} & \text{if} \quad j = i \\ \frac{\beta_i}{(j-i)!} - \frac{\gamma_i}{j!} & \text{if} \quad i+1 \le j \le n\,, \end{cases} \qquad (2 \le i \le n)\,.$$

This follows

$$\mathrm{Ker}\ L = \Big\{ x(t) = c_1 + c_2 t + \cdots + \frac{c_n}{(n-1)!} t^{n-1} : (c_1, c_2, \ldots, c_n) \in \mathrm{Ker}\ \mathcal{A} \Big\} \cong \mathrm{Ker}\ \mathcal{A}\,.$$

Hence we get $\dim \mathrm{Ker}\ L = \dim \mathrm{Ker}\ A < +\infty$.

Now, we determine the range of operator L, Im L. Consider the function $\phi \colon Z \to \mathbb{R}^n$ given by

$$\phi(z) = \mathcal{D}\begin{bmatrix} I^{n+1}z(1) & I^n z(1) & \cdots & Iz(1) \end{bmatrix}^T\,, \tag{2.2}$$

where $\mathcal{D}$ is a $n \times (n+1)$ matrix defined by

$$\mathcal{D} = \begin{bmatrix} \gamma_1 & -\beta_1 & 0 & \cdots & 0 \\ \gamma_2 & 0 & -\beta_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_n & 0 & 0 & \cdots & -\beta_n \end{bmatrix}.$$

Then we claim that

$$\mathrm{Im}\ L = \{z \in Z : \phi(z) \in \mathrm{Im}\ \mathcal{A}\}\,. \tag{2.3}$$

Indeed, for $z \in \mathrm{Im}\, L$, there exists $x \in \mathrm{dom}\,(L)$ such that $x^{(n)}(t) = z(t)$ and then

$$x(t) = c_1 + c_2 t + \cdots + \frac{c_n}{(n-1)!} t^{n-1} + I^n z(t)\,.$$

Since x satisfies boundary conditions

$$\alpha_i x^{(i-1)}(0) + \beta_i x^{(i-1)}(1) = \gamma_i \int_0^1 x(s)ds\,, \quad i = 1, 2, \ldots, n\,,$$

we can deduce that

$$\phi(z) = \mathcal{A}\begin{bmatrix} x(0) & x'(0) & \ldots & x^{(n-1)}(0)\end{bmatrix}^T \in \mathrm{Im}\ \mathcal{A}\,.$$

Conversely, if $z \in L^1[0,1]$ and holds $\phi(z) \in \mathrm{Im}\ \mathcal{A}$, then there exists $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ such that

$$\mathcal{A}\begin{bmatrix} c_1 & c_2 & \ldots & c_n\end{bmatrix}^T = \phi(z)\,.$$

Then by setting

$$x(t) = c_1 + c_2 t + \cdots + \frac{c_n}{(n-1)!} t^{n-1} + I^n z(t)\,,$$

it calculates straightforwardly that $x \in \mathrm{dom}\,(L)$ and $Lx = z \in \mathrm{Im}\, L$. Thus, the claim (2.3) is valid.

The following lemma gives the properties of the operator ϕ.

Lemma 2.4. *Let $\phi\colon Z \to \mathbb{R}^n$ be the linear operator defined by* (2.2). *Then the following statements are hold:*

i/ $|\phi(z)|_{\mathbb{R}^n} \le \|\mathcal{D}\|_* \|z\|_1$, *for all $z \in Z$, and*

ii/ $\mathrm{Im}\ \phi = \mathrm{Im}\ \mathcal{D}$,

where $|\cdot|_{\mathbb{R}^n}$ and $\|\cdot\|_*$ are the max-norms on $\mathbb{R}^n$ and $\mathbb{M}_{n\times(n+1)}(\mathbb{R})$, respectively.

Proof. Setting the operator $\mathcal{I}\colon Z \to \mathbb{R}^{n+1}$ by

$$\mathcal{I}z = \begin{bmatrix} I^{n+1}z(1) & I^n z(1) & \ldots & Iz(1)\end{bmatrix}^T, \quad z \in Z\,.$$

We derive that $\phi = \mathcal{D} \circ \mathcal{I}$. Thanks to the linearity of $\mathcal{I}$, we obtain the linearity of the operator ϕ. Furthermore, for $\nu \ge 1$, $\nu \in \mathbb{N}$, we have

$$|I^\nu z(1)| \le \frac{1}{(\nu-1)!}\int_0^1 (1-s)^{\nu-1}|z(s)|ds \le \frac{1}{(\nu-1)!}\|z\|_1\,.$$

It follows that

$$|\phi(z)|_{\mathbb{R}^n} \le \|\mathcal{D}\|_* \max\{|I^\nu z(1)| : \nu = 1, 2, \ldots, n+1\} \le \|\mathcal{D}\|_* \|z\|_1\,,$$

for $z \in Z$. Hence, ϕ is a linear continuous operator which satisfies i/. In order to prove ii/, it suffices to show that the operator $\mathcal{I}$ is surjective. In fact, it is obviously that $\mathrm{Im}\ \mathcal{I} \subset \mathbb{R}^{n+1}$. Conversely, for $\xi = (\xi_1, \xi_2, \dots, \xi_{n+1}) \in \mathbb{R}^{n+1}$, we shall show that there exists $z \in Z$ to be form of

$$z(t) = c_1 + c_2(1-t) + \cdots + c_{n+1}(1-t)^n \,,$$

which satisfies $\mathcal{I}z = \xi$. Notice that for $\nu = 1, 2, \dots, n+1$, we have

$$\begin{aligned} I^\nu z(1) &= \frac{1}{(\nu-1)!} \int_0^1 (1-s)^{\nu-1} \left[c_1 + c_2(1-s) + \cdots + c_{n+1}\,(1-s)^n \right] ds \\ &= \frac{1}{(\nu-1)!} \begin{bmatrix} \frac{1}{\nu} & \frac{1}{\nu+1} & \dots & \frac{1}{\nu+n} \end{bmatrix} \begin{bmatrix} c_1 & c_2 & \dots & c_{n+1} \end{bmatrix}^T . \end{aligned}$$

Hence, one has $\mathcal{I}z = \mathcal{C} \begin{bmatrix} c_1 & c_2 & \dots & c_{n+1} \end{bmatrix}^T$, where $\mathcal{C}$ denotes the following square matrix of order $n+1$

$$\mathcal{C} = \begin{bmatrix} \frac{1}{n!(n+1)} & \frac{1}{n!(n+2)} & \cdots & \frac{1}{n!(2n+1)} \\ \frac{1}{(n-1)!n} & \frac{1}{(n-1)!(n+1)} & \cdots & \frac{1}{(n-1)!2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1} & \frac{1}{2} & \cdots & \frac{1}{n+1} \end{bmatrix} .$$

On the other hand, it is not difficult to see that $\mathcal{C}$ is invertible matrix and denote its inverse as $\mathcal{C}^{-1}$. We now let $c = (c_1, c_2, \dots, c_{n+1}) = \mathcal{C}^{-1} \cdot \xi$ and $z(t) = c_1 + c_2(1-t) + \cdots + c_{n+1}(1-t)^n$, then $\mathcal{I}z = \xi$. The proposition is proved. $\square$

Next we note that this paper is merely interested in resonance case, that is to say that the dimension of Ker $\mathcal{A}$ is larger than or equal to 1. Therefore there exists an orthonormal basis of the orthorgonal complement of $\mathrm{Im}\,\mathcal{A} \cap \mathrm{Im}\,\mathcal{D}$ in $\mathrm{Im}\,\phi = \mathrm{Im}\,\mathcal{D}$ which is denoted by

$$\{\omega_k : k = 1, 2, \dots, m\}$$

for some $1 \le m \le n$. Then we could represent the range of L as follows

$$\mathrm{Im}\ L = \{ z \in Z : \langle \phi(z), \omega_k \rangle = 0 \,, \quad k = 1, 2, \dots, m \} \,, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

On the other hand, for each $\omega_k \in \mathrm{Im}\ \phi = \mathrm{Im}\ \mathcal{D}$, there exists $\xi_k = \left(\xi_1^k, \xi_2^k, \dots, \xi_{n+1}^k\right) \in \mathbb{R}^{n+1}$ to be a solution of the system linear equation $\mathcal{D}\mathcal{C}\xi = \omega_k$. Put

$$z_k(t) = \xi_1^k + \xi_2^k(1-t) + \cdots + \xi_{n+1}^k(1-t)^n \,,$$

then we obtain $z_k \in Z$ and $\phi(z_k) = \mathcal{DC}\xi_k = \omega_k$. Moreover, thanks to the linearity of the operator ϕ and the independence of system vector $\{\omega_k : k = 1, 2, \ldots, m\}$, we deduce that $\{z_k : k = 1, 2, \ldots, m\}$ is an independent system in Z.

We now offer a sufficient condition given by the following lemma to ensure that L is a Fredholm operator of index zero.

Lemma 2.5. *Suppose that* $\mathrm{Im}\ \mathcal{A} + \mathrm{Im}\ \mathcal{D} = \mathbb{R}^n$. *Then the operator* $L\colon \mathrm{dom}\,(L) \subset X \to Z$ *is a Fredholm operator of index zero.*

Proof. First, we notice that $\mathrm{Ker}\ L \cong \mathrm{Ker}\ \mathcal{A}$, so one has $\dim \mathrm{Ker}\ L < +\infty$. In addition, $\mathrm{Im}\ L$ is closed subset in Z because ϕ is a linear continuous operator. So, in order to prove that L is a Fredholm of index zero, we need only to prove that $\mathrm{codim}\ \mathrm{Im}\ L = \dim \mathrm{Ker}\ L$. In fact, we define the linear operator $Q\colon Z \to Z$ as follows

$$Qz(t) = \sum_{k=1}^{m} \langle \phi(z), \omega_k \rangle\, z_k(t)\,. \tag{2.5}$$

Since $\phi(z_k) = \omega_k$ and $\{\omega_k : k = 1, \ldots, m\}$ being an othornormal basis we deduce that

$$\langle \phi(Qz), \omega_k \rangle = \langle \phi(z), \omega_k \rangle$$

for all $k = 1, 2, \ldots, m$. This implies that Q is idempotent and therefore Q is a projector. In addition, using the continuity of the operator ϕ and inner product in $\mathbb{R}^n$, one gain Q is a continuous projector. Next, utilizing $\{z_k : k = 1, \ldots, m\}$ an independent system of Z, we argue that

$$\begin{aligned} z \in \mathrm{Ker}\ Q \Leftrightarrow \sum_{k=1}^{m} \langle \phi(z), \omega_k \rangle\, z_k = 0 \Leftrightarrow \langle \phi(z), \omega_k \rangle = 0\,, \\ \forall\, k \in \{1, 2, \ldots, m\} \Leftrightarrow z \in \mathrm{Im}\, L\,. \end{aligned}$$

Hence $\mathrm{Ker}\ Q = \mathrm{Im}\ L$. On the other hand, it is not difficult to show that $\mathrm{Im}\, Q = \mathrm{span}\, \{z_1, z_2, \ldots, z_m\}$. And therefore, we have

$$\begin{aligned} \mathrm{codim}\ \mathrm{Im}\ L &= \dim \mathrm{Im}\ Q \\ &= \dim \mathrm{Im}\ \mathcal{D} - \dim\,(\mathrm{Im}\ \mathcal{A} \cap \mathrm{Im}\ \mathcal{D}) \\ &= \dim\,(\mathrm{Im}\ \mathcal{A} + \mathrm{Im}\ \mathcal{D}) - \dim \mathrm{Im}\ \mathcal{A} \\ &= \dim(\mathbb{R}^n) - \dim \mathrm{Im}\ \mathcal{A} \\ &= \dim \mathrm{Ker}\ \mathcal{A} = \dim \mathrm{Ker}\ L\,, \end{aligned}$$

where we use the hypothesis $\mathrm{Im}\ \mathcal{A} + \mathrm{Im}\ \mathcal{D} = \mathbb{R}^n$. The proof is complete. $\square$

Let $P\colon X \to X$ be the operator defined by

$$Px(t) = \begin{bmatrix} 1 & t & \cdots & \frac{t^{n-1}}{(n-1)!} \end{bmatrix} (\mathbb{I}_n - \mathcal{A}^+\mathcal{A}) \begin{bmatrix} x(0) & x'(0) & \cdots & x^{(n-1)}(0) \end{bmatrix}^T ,$$

where $\mathcal{A}^+$ is Moore-Penrose pseudoinverse of $\mathcal{A}$ and $\mathbb{I}_n$ denotes the square matrix of n order. Since $P_{\mathcal{A}} = \mathbb{I}_n - \mathcal{A}^+\mathcal{A}$ is an orthorgonal projector onto Ker $\mathcal{A}$, it is not difficult to see that P is a projector onto Ker L and

$$\begin{aligned}\text{Ker } P = \Big\{ x \in X : & \begin{bmatrix} x(0) & x'(0) & \cdots & x^{(n-1)}(0) \end{bmatrix}^T \\ & = A^+A \begin{bmatrix} x(0) & x'(0) & \cdots & x^{(n-1)}(0) \end{bmatrix}^T \Big\} .\end{aligned}$$

The following lemma gives us the properties of pseudoinverse of L.

Lemma 2.6. *Let $K_P\colon \text{Im } L \to \text{dom}\,(L) \cap \text{Ker } P$ be a linear operator defined by*

$$(K_P z)(t) = \begin{bmatrix} 1 & t & \cdots & \frac{t^{n-1}}{(n-1)!} \end{bmatrix} \mathcal{A}^+\phi(z) + I^n z(t) ,$$

for $z \in \text{Im } L$. Then K_P is a pseudoinverse of L which means that

$$K_P = (L|_{\text{dom } L \cap \text{Ker } P})^{-1} .$$

Moreover, we have the following estimate

$$\|K_P z\| \le \left(1 + n \left\|\mathcal{A}^+\right\|_* \|\mathcal{D}\|_*\right) \|z\|_1$$

for every $z \in \text{Im } L$.

Proof. For each $z \in \text{Im}L$, it is not difficult to see that $K_P z \in AC^n[0,1]$ and

$$\begin{bmatrix} K_P z(0) & (K_P z)'(0) & \cdots & (K_P z)^{(n-1)}(0) \end{bmatrix}^T = \mathcal{A}^+\phi(z) .$$

It is straightforward to verify that $K_P z \in \text{dom } L \cap \text{Ker } P$. Hence K_P is well defined. On the other hand, it is clear that $LK_P z(t) = z(t)$ for all $t \in [0,1]$ and $z \in \text{Im } L$. Moreover, for each $x \in \text{dom } L \cap \text{Ker } P$, we have $x \in \text{dom}\,(L)$ which implies

$$\mathcal{A} \begin{bmatrix} x(0) & x'(0) & \cdots & x^{(n-1)}(0) \end{bmatrix}^T = \phi(Lx) .$$

It follows that

$$\begin{aligned}(K_P Lx)(t) &= \begin{bmatrix} 1 & t & \cdots & \frac{t^{n-1}}{(n-1)!} \end{bmatrix} \mathcal{A}^+\phi(Lx) + I^n Lx(t) \\ &= x(t) - \begin{bmatrix} 1 & t & \cdots & \frac{t^{n-1}}{(n-1)!} \end{bmatrix} \left(\mathbb{I}_n - \mathcal{A}^+\mathcal{A}\right) \\ &\quad \times \begin{bmatrix} x(0) & x'(0) & \cdots & x^{(n-1)}(0) \end{bmatrix}^T \\ &= x(t) - Px(t) = x(t) ,\end{aligned} \tag{2.6}$$

where we use the fact that $x \in \text{Ker } P$ in the last equality. Hence, $(K_P L x)(t) = x(t)$ for all $t \in [0,1]$ and for every $x \in \text{dom } L \cap \text{Ker } P$. Thus, $K_P = \left(L|_{\text{dom } L \cap \text{Ker } P}\right)^{-1}$. Furthermore, by the definition of the the pseudoinverse K_P of L, we get

$$(K_P z)^{(i)}(t) = \begin{bmatrix} 0 & \cdots & 1 & \cdots & \frac{t^{n-1-i}}{(n-1-i)!} \end{bmatrix} \mathcal{A}^+ \phi(z) + I^{n-i} z(t), \tag{2.7}$$

where 1 is $(i+1)^{th}$ position, for $i = 0, 1, \ldots, (n-1)$. It follows from (2.7) and Lemma 2.4 that

$$\begin{aligned} \left|(K_P z)^{(i)}(t)\right| &\le (n-i) \left\|\mathcal{A}^+\right\|_* |\phi(z)|_{\mathbb{R}^n} + \frac{1}{(n-1-i)!} \int_0^t (t-s)^{n-1-i} |z(s)|\, ds \\ &\le \left[1 + (n-i) \left\|\mathcal{A}^+\right\|_* \|\mathcal{D}\|_*\right] \|z\|_1 , \end{aligned}$$

for all $t \in [0,1]$ and for each $i = 0, 1, \ldots, (n-1)$, which implies

$$\|K_P z\| = \max\left\{\|(K_P z)^{(i)}\|_\infty : i = 0, 1, \ldots, n-1\right\} \le \left(1 + n\|\mathcal{A}^+\|_* \|\mathcal{D}\|_*\right) \|z\|_1 .$$

This results in the Lemma 2.6. $\square$

In what follows, we always assume that $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies Caretheódory conditions, that is:

(a) $f(\cdot, u)$ is measurable for $u \in \mathbb{R}^n$,

(b) $f(t, \cdot)$ is continuous on $\mathbb{R}^n$ for almost every $t \in [0,1]$,

(c) for each compact set $K \subset \mathbb{R}^n$, the function $h_K(t) = \sup\{|f(t,u)| : u \in K\}$ is Lebesgue integrable on $[0,1]$.

By these assumptions on f and dominated convergence theorem, it is well known that the Nemytskii operator associated with f, $N\colon X \to Z$ defined by (2.1) is continuous mapping and takes bounded sets into bounded sets. Furthermore, N is also a L-completely continuous on X, claimed by the following lemma.

Lemma 2.7. *Assume that $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies Caretheódory conditions as above. Then the operator $N\colon X \to Z$ defined by* (2.1) *is a L-completely continuous operator on X.*

Proof. Let Ω be an arbitrary open bounded subset in X. Then it is clearly seen that $QN\colon \overline{\Omega} \to Z$ is continuous and $QN\left(\overline{\Omega}\right)$ is bounded because N is continuous mapping and takes bounded sets into bounded sets. It now remains to verify that $K_{P,Q}N\colon \overline{\Omega} \to X$ is completely continuous on $\overline{\Omega}$. In fact, since $K_{P,Q}N$ is the composition of the continuous operators N, Q and K_P, so $K_{P,Q}N$ is also continuous operator. In addition, by the definition of the operator $K_{P,Q}$, we have

$$\begin{aligned}(K_{P,Q}Nx)^{(i)}(t) &= (K_P(I-Q)Nx)^{(i)}(t)\\ &= \begin{bmatrix}0 & \cdots & 1 & \cdots & \frac{t^{n-1-i}}{(n-1-i)!}\end{bmatrix}\mathcal{A}^+\phi((I-Q)Nx)\\ &\quad + I^{n-i}(I-Q)Nx(t)\,,\end{aligned} \tag{2.8}$$

where 1 is $(i+1)^{th}$ position, for $i=0,1,\ldots,(n-1)$ and $x\in\overline{\Omega}$. Set $z(t):=(I-Q)\,Nx(t)$ for $t\in[0,1]$, then there exists a positive constant M such that $\|z\|_1\le M$ for all $x\in\overline{\Omega}$. This implies that $K_{P,Q}N\left(\overline{\Omega}\right)$ is bounded by using the Lemma 2.6.

On the other hand, since $\left\{t^i : t\in[0,1]\right\}$ and $\left\{I^{n-i}z(t) : t\in[0,1]\right\}$, for each $i=0,1,\ldots,n-1$ are equicontinuous families, it follows from (2.8) that $\left\{(K_Pz)^{(i)}(t) : t\in[0,1]\right\}$ are equicontinuous on $[0,1]$. Hence, one obtains $(K_{P,Q}N)^{(i)}\left(\overline{\Omega}\right)$ is relatively compact, for $i=0,1,\ldots,n-1$ due to Arzela-Ascoli theorem. Thus, $K_{P,Q}N\left(\overline{\Omega}\right)$ is relatively compact in X which results in the proof. $\square$

3. Main results

In this section we use the Theorem 2.3 to prove the existence of the solutions of problem (1.1)–(1.2). For this purpose we assume that

$$\mathrm{Im}\ \mathcal{A}+\mathrm{Im}\ \mathcal{D}=\mathbb{R}^n\,,$$

the assumptions of Lemma 2.7 hold, and

(A_1) there exist the positive functions $a_0,a_1,\ldots,a_n\in Z$ with $C\sum_{i=0}^{n-1}\|a_i\|_1<1$ and $C=1+n\|\mathbb{I}_n-\mathcal{A}^+\mathcal{A}\|_*+n\|\mathcal{A}^+\|_*\|\mathcal{D}\|_*$, such that

$$\left|f\left(t,x_0,x_1,\ldots,x_{n-1}\right)\right|\le\sum_{i=0}^{n-1}a_i(t)\left|x_i\right|+a_n(t)\,,$$

for almost everywhere $t\in[0,1]$ and $x_i\in\mathbb{R}$, for $i=0,1,\ldots,(n-1)$;

(A_2) there exists a positive constant M_1 such that for each $x\in\mathrm{dom}\,(L)$, $\phi(Nx)\notin\mathrm{Im}\ \mathcal{A}$ provided that $\max\left\{\left|x^{(i)}(t)\right| : i=0,\ldots,(n-1)\right\}>M_1$ for all $t\in[0,1]$;

(A_3) there exists a positive constant M_2 such that for any $\{c_i\}_{i=1}^m \subset \mathbb{R}$ with $\sum\limits_{i=1}^m |c_i| > M_2$ then

$$c_i\Big\langle \phi \circ N\Big(\sum_{j=1}^m c_j x_j\Big), \omega_i\Big\rangle < 0\,, \tag{3.1}$$

or

$$c_i\Big\langle \phi \circ N\Big(\sum_{j=1}^m c_j x_j\Big), \omega_i\Big\rangle > 0\,, \tag{3.2}$$

for all $i \in \{1, 2, \dots, m\}$, where $\{x_j : j = 1, \dots, m\}$ is a basis of Ker L.

Lemma 3.1. *Let $\Omega_1 = \{x \in \mathrm{dom}\,(L) \setminus \mathrm{Ker}\ L : Lx = \lambda Nx, \lambda \in (0,1)\}$. Then Ω_1 is bounded subset in X.*

Proof. Let $x \in \Omega_1$, then there exists $\lambda \in (0,1)$ such that $Lx = \lambda Nx$. It follows that $Nx \in \mathrm{Im}\ L = \mathrm{Ker}\ Q$ which implies

$$QNx(t) = \sum_{k=1}^m \langle \phi(Nx), \omega_k\rangle z_k(t) = 0\,, \quad \forall t \in [0,1]\,.$$

Thanks to the linearly independent property of $\{z_k : k = 1, \dots, m\}$, we derive that $\langle \phi(Nx), \omega_k\rangle = 0$ for all $k = 1, \dots, m$. Therefore, we possess $\phi(Nx) \in \mathrm{Im}\ \mathcal{A} \cap \mathrm{Im}\ \mathcal{D}$ which implies $\phi(Nx) \in \mathrm{Im}\ \mathcal{A}$. By utilizing the assumption (A_2), there exists $t_0 \in [0,1]$ such that

$$\max\left\{\left|x^{(i)}(t_0)\right| : i = 0, 1, \dots, (n-1)\right\} \le M_1\,.$$

It follows from the identities

$$x^{(i)}(t) = x^{(i)}(t_0) + x^{(i+1)}(t_0)\,(t - t_0) + \cdots + \frac{1}{(n-1-i)!} \times \int_{t_0}^t (t-s)^{n-1-i}\, x^{(n)}(s)\, ds\,,$$

for all $t \in [0,1]$, $i = 1, \dots, n-1$, and

$$x(t) = x(t_0) + x'(t_0)\,(t-t_0) + \cdots + \frac{1}{(n-1)!}\int_{t_0}^t (t-s)^{n-1}\, x^{(n)}(s)\, ds\,,$$

that

$$|x^{(i)}(0)| \le (n-i)\, M_1 + \|x^{(n)}\|_1\,, \ \ i = 1, \dots, n-1 \ \text{ and } \ |x(0)| \le nM_1 + \|x^{(n)}\|_1\,.$$

Therefore, we get

$$\max\left\{|x^{(i)}(0)| : i = 0, 1, \dots, (n-1)\right\} \le nM_1 + \|x^{(n)}\|_1 \le nM_1 + \|Nx\|_1\,.$$

It follows from the definition of the projector P and the inequality above that

$$\begin{aligned} \|Px\| &\le n\left\|\mathbb{I}_n - \mathcal{A}^+\mathcal{A}\right\|_* \max\left\{\|x^{(i)}(0)| : i = 0, 1, \ldots, (n-1)\right\} \\ &\le n\|\mathbb{I}_n - \mathcal{A}^+\mathcal{A}\|_*\big(nM_1 + \|Nx\|_1\big)\,. \end{aligned} \tag{3.3}$$

On the other hand, since $(I - P)\,x \in \operatorname{dom}(L) \cap \operatorname{Ker} P$ and using Lemma 2.6, we achieve

$$\|(I-P)x\| = \|K_P L(I-P)x\| = \|K_P Lx\| \tag{3.4}$$

$$\le \left(1 + n\left\|\mathcal{A}^+\right\|_* \|\mathcal{D}\|_*\right)\|Nx\|_1\,. \tag{3.5}$$

Combining (3.3)-(3.5), we obtain

$$\begin{aligned} \|x\| &= \|Px + (I-P)x\| \le \|Px\| + \|(I-P)x\| \\ &\le C\,\|Nx\|_1 + n^2 M_1\left\|\mathbb{I}_n - \mathcal{A}^+\mathcal{A}\right\|_*\,, \end{aligned} \tag{3.6}$$

where $C = 1 + n\,\|\mathbb{I}_n - \mathcal{A}^+\mathcal{A}\|_* + n\,\|\mathcal{A}^+\|_*\,\|\mathcal{D}\|_*$.
Exploiting the assumptions of nonlinear term, (A_1) and the definition of the operator N, we gain

$$\begin{aligned} \|Nx\|_1 &\le \int_0^1 \left|f\big(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\big)\right| ds \\ &\le \sum_{i=1}^{n-1} \|a_i\|_1 \left\|x^{(i)}\right\|_\infty + \|a_n\|_1 \\ &\le \Big(\sum_{i=1}^{n-1} \|a_i\|_1\Big)\|x\| + \|a_n\|_1\,. \end{aligned} \tag{3.7}$$

It follows from (3.6)–(3.7) and $C\big(\sum\limits_{i=1}^{n-1}\|a_i\|_1\big) < 1$ that

$$\|x\| \le \frac{C\,\|a_n\|_1 + n^2 M_1\,\|I_n - \mathcal{A}^+\mathcal{A}\|_*}{1 - C\sum\limits_{i=1}^{n-1}\|a_i\|_1}\,.$$

Thus, Ω_1 is bounded in X. □

Lemma 3.2. *The set* $\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$ *is a bounded subset in* X.

Proof. Let $x \in \Omega_2$ and assume that $x(t) = c_0 + c_1 t + \cdots + c_{n-1}\frac{t^{n-1}}{(n-1)!}$, where $(c_0, c_1, \ldots, c_{n-1}) \in \operatorname{Ker} \mathcal{A}$. Also, we have $Nx \in \operatorname{Im} L$, by the same argument as in the proof of Lemma 3.1, one could show that

$$\max\left\{|x^{(i)}(t_0)|, i = 0, \ldots, (n-1)\right\} \le M_1\,,$$

for some $t_0 \in [0,1]$. As a result, for each $i = 0, 1, \ldots, (n-1)$, c_i is bounded in $\mathbb{R}$. And therefore the Lemma 3.2 is valid. $\square$

Lemma 3.3. *Let*

$$\Omega_3^- = \{x \in \text{Ker } L : -\lambda x + (1-\lambda)JQNx = 0, \lambda \in [0,1]\} \ ,$$

and

$$\Omega_3^+ = \{x \in \text{Ker } L : \lambda x + (1-\lambda)JQNx = 0, \lambda \in [0,1]\} \ ,$$

where $J\colon \text{Im } Q \to \text{Ker } L$ *is a linear isomorphism which is defined by*

$$J\Big(\sum_{i=1}^{m} c_i z_i\Big) = \sum_{i=1}^{m} c_i x_i \, ,$$

for $\sum\limits_{i=1}^{m} c_i z_i \in \text{Im } Q$. *Then* Ω_3^- *and* Ω_3^+ *are bounded subsets in* X *provided that* (3.1) *and* (3.2) *of the assumption* (A_3) *hold, respectively.*

Proof. Assume that (A_3)–(3.1) holds. Let $x \in \Omega_3^-$, then we might assume that $x = \sum\limits_{i=1}^{m} c_i x_i \in \text{Ker } L$, where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$ and

$$\lambda J^{-1}\Big(\sum_{i=1}^{m} c_i x_i\Big) = (1-\lambda)QN\Big(\sum_{i=1}^{m} c_i x_i\Big) \, ,$$

for $\lambda \in [0,1]$. It follows from the definitions of the operators J and Q that

$$\lambda \sum_{i=1}^{m} c_i z_i = (1-\lambda) \sum_{i=1}^{m} \Big\langle \phi \circ N\Big(\sum_{j=1}^{m} c_j x_j\Big), \omega_i \Big\rangle z_i \, .$$

This implies

$$\lambda c_i = (1-\lambda)\Big\langle \phi \circ N\Big(\sum_{j=1}^{m} c_j x_j\Big), \omega_i \Big\rangle \, ,$$

for all $i \in \{1, \ldots, m\}$. If $\lambda = 1$, then $c_i = 0$ for all $i \in \{1, 2, \ldots, m\}$. In this case, it is obvious that Ω_3^- is bounded. And if $\lambda \in [0,1)$ and $\sum\limits_{i=1}^{m} |c_i| > M_2$, then by assumption (A_3)–(3.1) we get a contradiction

$$0 \le \lambda c_i^2 = (1-\lambda)c_i\Big\langle \phi \circ N\Big(\sum_{j=1}^{m} c_j x_j\Big), \omega_i \Big\rangle < 0 \, ,$$

for some $i \in \{1, 2, \dots, m\}$. Thus Ω_3^- is bounded in X. If (A_3)–(3.2) holds, then by using the same arguments as in above we are also able to prove that Ω_3^+ is bounded in X. The Lemma 3.3 has been proved. □

Theorem 3.4. *Suppose that the assumptions* (A_1), (A_2) *and* (A_3)–(3.1) *hold. Then the problem* (1.1)–(1.2) *has at least one solution in* X.

Proof. We shall prove that all the conditions of Theorem 2.3 are satisfied, where Ω is open and bounded such that $\bigcup_{i=1}^{3} \Omega_i \subset \Omega$. It is clear that the operator L is a Fredholm operator of index zero by Lemma 2.5 and N is L-compact on $\overline{\Omega}$ by Lemma 2.7. Futhermore, the conditions (1) and (2) of the Theorem 2.3 are fulfilled by exploiting Lemma 3.1 and Lemma 3.2. So, it remains to verify the third condition of Theorem 2.3. For this purpose, we apply the degree property of invariance under a homotopy. Let us define

$$H(\lambda, x) = -\lambda x + (1-\lambda)\, JQNx\,,$$

where the isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$ is defined as in Lemma 3.3. By Lemma 3.3, we obtain $H(\lambda, x) \neq 0$ for all $(\lambda, x) \in [0,1] \times (\operatorname{Ker} L \cap \partial\Omega)$. Hence, we get

$$\begin{aligned} \deg\left(JQN; \Omega \cap \operatorname{Ker} L, 0\right) &= \deg\left(H(0,\cdot), \Omega \cap \operatorname{Ker} L, 0\right) \\ &= \deg\left(H(1,\cdot), \Omega \cap \operatorname{Ker} L, 0\right) \\ &= \deg\left(-I, \Omega \cap \operatorname{Ker} L, 0\right) \neq 0\,. \end{aligned}$$

Thus, Theorem 3.4 is proved. □

Remark 3.5. If we replace the assumption (A_3)–(3.1)in the Theorem 3.4 by (A_3)–(3.2), then by considering the homotopy

$$H(\lambda, x) = \lambda x + (1-\lambda)\, JQNx\,,$$

and using similar arguments above, we obtain

$$\begin{aligned} \deg\left(JQN; \Omega \cap \operatorname{Ker} L, 0\right) &= \deg\left(H(0,\cdot), \Omega \cap \operatorname{Ker} L, 0\right) \\ &= \deg\left(H(1,\cdot), \Omega \cap \operatorname{Ker} L, 0\right) \\ &= \deg\left(I, \Omega \cap \operatorname{Ker} L, 0\right) \neq 0\,. \end{aligned}$$

And therefore, the conlusion of Theorem 3.4 does not change.

An illustration for this will be given by following example.

Example. Consider the equation

$$x''(t) = f\left(t, x(t), x'(t)\right)\,, \quad t \in (0,1) \tag{3.8}$$

associated with the integral boundary condition

$$\begin{cases} x(0)+2x(1)=2\int_0^1 x(s)ds \\ \frac{1}{4}x'(0)+\frac{1}{4}x'(1)=-\int_0^1 x(s)ds \end{cases} \tag{3.9}$$

where

$$f(t,x_0,x_1)=\operatorname{sign}\left\{\frac{3}{2}t^2-t-\frac{1}{4}\right\}g(t,x_0,x_1)\,,$$
$$g(t,x_0,x_1)=\frac{t^3}{28}\sin x_0+\frac{1}{11}(1+t)|x_1|+t^2+2\,.$$

It is clear that the problem (3.8)–(3.9) is a special case of the problem (1.1)–(1.2) in which $\alpha_1=1$, $\beta_1=2$, $\gamma_1=2$, $\alpha_2=\frac{1}{4}$, $\beta_2=\frac{1}{4}$ and $\gamma_2=-1$. Therefore, in order to show that the problem (3.8)-(3.9) has at least one solution, it suffices to verify the conditions of the Theorem 3.4.

In this case, we have $L\colon \operatorname{dom}(L)\subset C^1[0,1]\to L^1(0,1)$ defined by $Lx(t)=x''(t)$ with

$$\operatorname{dom}(L)=\{x\in AC^2[0,1]: x \text{ satisfies the integral boundary condition (3.9)}\}$$

and the nonlinear operator $N\colon C^1[0,1]\to L^1(0,1)$ defined by

$$Nx(t)=f(t,x(t),x'(t))\,.$$

In the following, we need to show that

(1) L is a Fredholm operator of index zero;
(2) N is a L-completely continuous;
(3) the conditions of Theorem 3.4 hold.

Firstly, we have the matrix $\mathcal{A}=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, which has Ker $\mathcal{A}=\operatorname{span}\{(1;-1)\}$, Im $\mathcal{A}=\operatorname{span}\{(1;1)\}$ and the Moore-Penrose matrix $\mathcal{A}^+=\begin{bmatrix}\frac{1}{4}&\frac{1}{4}\\\frac{1}{4}&\frac{1}{4}\end{bmatrix}=\frac{1}{4}\mathcal{A}$. The kernel of L

$$\operatorname{Ker} L=\{x=c_1+c_2t:(c_1,c_2)\in\operatorname{Ker}\mathcal{A}\}=\operatorname{span}\{x_1\}\,,$$

where $x_1(t)=1-t$.

Moreover, we could see that the matrix $\mathcal{D}=\begin{bmatrix}2&-2&0\\-1&0&-\frac{1}{4}\end{bmatrix}$ and the operator $\phi\colon Z\to\mathbb{R}^2$ defined by

$$\phi(z)=\mathcal{D}\begin{bmatrix}I^3z(1) & I^2z(1) & Iz(1)\end{bmatrix}^T=\left(2I^3z(1)-2I^2z(1);-I^3z(1)-\frac{1}{4}Iz(1)\right).$$

So the image of L

$$\begin{aligned}\text{Im } L &= \left\{z \in L^1(0,1) : \phi(z) \in \text{Im } \mathcal{A}\right\} \\ &= \left\{z \in L^1(0,1) : \int_0^1 \left(\frac{3}{2}s^2 - s - \frac{1}{4}\right) z(s)\, ds = 0\right\}.\end{aligned}$$

It is clearly seen that $\text{Im } \mathcal{D} = \mathbb{R}^2$ and $\text{Im } \mathcal{A} \cap \text{Im } \mathcal{D} = \text{Im } \mathcal{A}$. Therefore, we have

$$\dim\left(\text{Im } \mathcal{A} + \text{Im } \mathcal{D}\right) = \dim \text{Im}\, \mathcal{A} + \dim \text{Im } \mathcal{D} - \dim\left(\text{Im}\, \mathcal{A} \cap \text{Im } \mathcal{D}\right) = \dim(\mathbb{R}^2).$$

It follows that $\text{Im } \mathcal{A} + \text{Im } \mathcal{D} = \mathbb{R}^2$. Hence, according to Lemma 2.5, L is a Fredholm operator of index zero.

Now, taking $\left\{\omega_1 = \frac{\sqrt{2}}{2}(1; -1)\right\}$ is an orthonormal basis of the orthogonal complement of $\text{Im } \mathcal{A} \cap \text{Im } \mathcal{D} = \text{Im } \mathcal{A}$ in $\mathbb{R}^2$ and setting

$$z_1(t) = \xi_1 + \xi_2(1-t) + \xi_3(1-t)^2,$$

where $\xi = (\xi_1, \xi_2, \xi_3) = \left(\frac{8\sqrt{2}}{21}, -\frac{38\sqrt{2}}{21}, 0\right) \in \mathbb{R}^3$ is a solution of equation $\mathcal{D}\mathcal{C}\xi = \omega_1$. Then one has $\phi(z_1) = \omega_1$.

We can now define the projectors $P\colon X \to X$ and $Q\colon Z \to Z$ by

$$Px(t) = \begin{bmatrix}1 & t\end{bmatrix}\left(\mathbb{I}_2 - \mathcal{A}^+\mathcal{A}\right)\begin{bmatrix}x(0) & x'(0)\end{bmatrix}^T = \frac{1}{2}\left(x(0) - x'(0)\right) x_1(t),$$

and

$$Qz(t) = \langle \phi(z), \omega_1 \rangle\, z_1(t) = \left[\frac{\sqrt{2}}{2}\int_0^1 \left(\frac{3}{2}s^2 - s - \frac{1}{4}\right) z(s) ds\right] z_1(t).$$

The pseudoinverse K_p is defined by

$$K_P(z)(t) = \begin{bmatrix}1 & t\end{bmatrix} \mathcal{A}^+\phi(z) + \int_0^t (t-s) z(s)\, ds$$

which implies $\|K_P z\| \le (1 + 2\|\mathcal{A}^+\|_* \|\mathcal{D}\|_*)\|z\|_1 = 2\|z\|_1$ for all $z \in Z$.

Secondly, it is not difficult to see that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory conditions. Thus, N is a L-completely continuous by Lemma 2.7.

Finally, we verify the conditions of Theorem 3.4. In fact, we have the growth condition on f

$$|f(t, x_0, x_1)| \le a_0(t)|x_0| + a_1(t)|x_1| + a_2(t),$$

for all $t \in [0,1]$ and $x_0, x_1 \in \mathbb{R}$, where $a_0(t) = \frac{t^3}{28}$, $a_1(t) = \frac{1+t}{11}$, $a_2(t) = t^2 + 2$. Furthermore, it is straightforward to calculate that

$$C = 1 + 2\left\|\mathbb{I}_2 - \mathcal{A}^+\mathcal{A}\right\|_* + 2\left\|\mathcal{A}^+\right\|_* \|\mathcal{D}\|_* = 3$$
$$\text{and} \quad C\left(\|a_0\|_\infty + \|a_1\|_\infty\right) = 3\Big(\frac{1}{28} + \frac{2}{11}\Big) < 1\,.$$

Therefore, the condition (A_1) holds.

Next, it is noticed that $\phi(Nx) \in \operatorname{Im} \mathcal{A}$ is equivalent to

$$\int_0^1 \Big(\frac{3}{2}s^2 - s - \frac{1}{4}\Big) Nx(s)\,ds = 0 \Leftrightarrow \int_0^1 \Big|\frac{3}{2}s^2 - s - \frac{1}{4}\Big| g\big(s, x(s), x'(s)\big)\,ds = 0\,.$$

On the other hand, if $|x_1| > 22$ then we get $g(t, x_0, x_1) > 1$ for all $t \in [0,1]$. Hence, taking $M_1 = 22$, then we obtain

$$\int_0^1 \Big|\frac{3}{2}s^2 - s - \frac{1}{4}\Big| g\big(s, x(s), x'(s)\big)\,ds > 0,$$

provided that $\max\left\{\left|x^{(i)}(t)\right| : i = 0,1\right\} > M_1$ for all $t \in [0,1]$. This results $\phi(Nx) \notin \operatorname{Im} \mathcal{A}$. The condition (A_2) holds.

For $x_1(t) = 1 - t$, we have

$$\langle \phi \circ N(c_1 x_1), \omega_1 \rangle = \frac{\sqrt{2}}{2}\int_0^1 \Big|\frac{3}{2}s^2 - s - \frac{1}{4}\Big| g\big(s, c_1(1-s), -c_1\big)\,ds.$$

Similarly, taking $M_2 = 22$, then we get $c_1 g(t, c_1(1-t), -c_1) > 1$ if $c_2 > M_2$ and $c_1 g(t, c_1(1-t), -c_1) < -1$ if $c_1 < -M_2$. Therefore, if $|c_1| > M_2$, then

$$c_1 \langle \phi \circ N(c_1 x_1), \omega_1 \rangle = \frac{\sqrt{2}}{2}\int_0^1 \Big|\frac{3}{2}s^2 - s - \frac{1}{4}\Big| c_1 g\big(s, c_1(1-s), -c_1\big)\,ds > 0\,,$$

or

$$c_1 \langle \phi \circ N(c_1 x_1), \omega_1 \rangle = \frac{\sqrt{2}}{2}\int_0^1 \Big|\frac{3}{2}s^2 - s - \frac{1}{4}\Big| c_1 g\big(s, c_1(1-s), -c_1\big)\,ds < 0\,.$$

Hence, the condition (A_3) holds. Thus, by the Theorem 3.4, the problem (3.8)–(3.9) has at least one solution.

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