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ON JACOBI FIELDS AND A CANONICAL CONNECTION IN SUB-RIEMANNIAN GEOMETRY

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Abstract. In sub-Riemannian geometry the coefficients of the Jacobi equation define curvature-like invariants. We show that these coefficients can be interpreted as the curvature of a canonical Ehresmann connection associated to the metric, first introduced in [15]. We show why this connection is naturally nonlinear, and we discuss some of its properties.

1. Introduction

A key tool for comparison theorems in Riemannian geometry is the Jacobi equation, i.e. the differential equation satisfied by Jacobi fields. Assume γ_ε is a one-parameter family of geodesics on a Riemannian manifold (M,g) satisfying

$$\ddot{\gamma}_\varepsilon^k + \Gamma_{ij}^k(\gamma_\varepsilon)\dot{\gamma}_\varepsilon^i\dot{\gamma}_\varepsilon^j = 0\,. \tag{1}$$

The corresponding Jacobi field $J = \frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}\gamma_\varepsilon$ is a vector field defined along $\gamma=\gamma_0$, and satisfies the equation

$$\ddot{J}^k + 2\Gamma_{ij}^k\dot{J}^i\dot{\gamma}^j + \frac{\partial\Gamma_{ij}^k}{\partial x^\ell}J^\ell\dot{\gamma}^i\dot{\gamma}^j = 0\,. \tag{2}$$

The Riemannian curvature is hidden in the coefficients of this equation. To make it appear explicitly, however, one has to write (2) in terms of a parallel transported frame $X_1(t),\dots,X_n(t)$ along $\gamma(t)$. Letting $J(t)=\sum_{i=1}^n J_i(t)X_i(t)$ one gets the following normal form:

$$\ddot{J}_i + R_{ij}(t)J_j = 0\,. \tag{3}$$

Indeed the coefficients R_{ij} are related with the curvature R^∇ of the unique linear, torsion free and metric connection ∇ (Levi-Civita) as follows

$$R_{ij} = g(R^\nabla(X_i,\dot{\gamma})\dot{\gamma},X_j)\,.$$

Eq. (3) is the starting point to prove many results in Riemannian geometry. In particular, bounds on the curvature (i.e. on the coefficients R, or its trace) have deep consequences on the analysis and the geometry of the underlying manifold.

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In the sub-Riemannian setting this construction cannot be directly generalized. Indeed, the analogous of the Jacobi equation is a first-order system on the cotangent bundle that cannot be written as a second-order equation on the manifold. Still one can put it in a normal form, analogous to (3), and study its coefficients [15]. These appear to be the correct objects to bound in order to control the behavior of the geodesic flow and get comparison-like results (see for instance [10, 7]). Nevertheless one can wonder if these coefficients can arise, as in the Riemannian case, as the curvature of a suitable connection. We answer to this question, by showing that these coefficients are part of the curvature of a nonlinear canonical Ehresmann connection associated with the sub-Riemannian structure. In the Riemannian case this reduces to the classical, linear, Levi-Civita connection.

1.1. **The general setting.** A sub-Riemannian structure is a triple $(M, \mathcal{D}, g)$ where M is smooth n-dimensional manifold, $\mathcal{D}$ is a smooth, completely non-integrable vector sub-bundle of TM and g is a smooth scalar product on $\mathcal{D}$. Riemannian structures are included in this definition, taking $\mathcal{D} = TM$. The sub-Riemannian distance is the infimum of the length of absolutely continuous admissible curves joining two points. Here admissible means that the curve is almost everywhere tangent to the distribution $\mathcal{D}$, in order to compute its length via the scalar product g. The totally non-holonomic assumption on $\mathcal{D}$ implies, by the Rashevskii-Chow theorem, that the distance is finite on every connected component of M, and the metric topology coincides with the one of M. A more detailed introduction on sub-Riemannian geometry can be found in [12, 2, 13, 8].

In Riemannian geometry, it is well-known that the geodesic flow can be seen as a Hamiltonian flow on the cotangent bundle T^*M, associated with the Hamiltonian

$$H(p, x) = \frac{1}{2}\sum_{i=1}^{n} \langle p, X_i(x)\rangle^2 , \qquad (p, x) \in T^*M ,$$

where $X_1, \dots, X_n$ is any local orthonormal frame for the Riemannian structure, and the notation $\langle p, v\rangle$ denotes the action of a covector $p \in T_x^*M$ on a vector $v \in T_xM$. In the sub-Riemannian case, the Hamiltonian is defined by the same formula, where the sum is taken over a local orthonormal frame $X_1, \dots, X_k$ for $\mathcal{D}$, with $k = \operatorname{rank} \mathcal{D}$. The restriction of H to each fiber is a degenerate quadratic form, but Hamilton's equations are still defined. These can be written as a flow on T^*M

$$\dot{\lambda} = \vec{H}(\lambda) , \qquad \lambda \in T^*M ,$$

where $\vec{H}$ is the Hamiltonian vector field associated with H. This system cannot be written as a second order equation on M as in (1). The projection $\pi \colon T^*M \to M$ of its integral curves are geodesics, i.e. locally minimizing curves. In the general case, some geodesics may not be recovered in this way. These are the so-called strictly abnormal geodesics [11], and they are related with hard open problems in sub-Riemannian geometry [1].

In what follows, with a slight abuse of notation, the term "geodesic" refers to the not strictly abnormal ones.

An integral line of the Hamiltonian vector field $\lambda(t) = e^{t\vec{H}}(\lambda) \in T^*M$, with initial covector λ is called *extremal*. Notice that the same geodesic may be the projection of two different extremals. For these reasons, it is convenient to see the Jacobi equation as a first order equation for vector fields on T^*M, associated with an extremal, rather then a second order system on M, associated with a geodesic.

2. Jacobi equation revisited

For any vector field $V(t)$ along an extremal $\lambda(t)$ of the sub-Riemannian Hamiltonian flow, a dot denotes the Lie derivative in the direction of $\vec{H}$:

$$\dot{V}(t) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} e_*^{-\varepsilon\vec{H}} V(t+\varepsilon)\,.$$

A vector field $\mathcal{J}(t)$ along $\lambda(t)$ is called a *sub-Riemannian Jacobi field* if it satisfies

$$\dot{\mathcal{J}} = 0\,. \tag{4}$$

The space of solutions of (4) is a $2n$-dimensional vector space. The projections $J = \pi_*\mathcal{J}$ are vector fields on M corresponding to one-parameter variations of $\gamma(t) = \pi(\lambda(t))$ through geodesics; in the Riemannian case, they coincide with the classical Jacobi fields.

We intend to write (4) using the natural symplectic structure σ of T^*M. First, observe that on T^*M there is a natural smooth sub-bundle of Lagrangian[1] spaces:

$$\mathcal{V}_\lambda := \ker \pi_*|_\lambda = T_\lambda\big(T^*_{\pi(\lambda)}M\big)\,.$$

We call this the *vertical subspace*. Then, pick a Darboux frame $\{E_i(t), F_i(t)\}_{i=1}^n$ along $\lambda(t)$. It is natural to assume that $E_1, \dots, E_n$ belong to the vertical subspace. To fix the ideas, one can think at the canonical basis $\{\partial_{p_i}|_{\lambda(t)}, \partial_{x_i}|_{\lambda(t)}\}$ induced by a choice of coordinates $(x_1, \dots, x_n)$ on M.

In terms of this frame, $\mathcal{J}(t)$ has components $(p(t), x(t)) \in \mathbb{R}^{2n}$:

$$\mathcal{J}(t) = \sum_{i=1}^n p_i(t)E_i(t) + x_i(t)F_i(t)\,.$$

The elements of the frame satisfy

$$\begin{pmatrix} \dot{E} \\ \dot{F} \end{pmatrix} = \begin{pmatrix} C_1(t)^* & -C_2(t) \\ R(t) & -C_1(t) \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix}, \tag{5}$$

for some smooth families of $n \times n$ matrices $C_1(t), C_2(t), R(t)$, where $C_2(t) = C_2(t)^*$ and $R(t) = R(t)^*$. We stress that the particular structure of the equations is implied solely by the fact that the frame is Darboux, that is

$$\sigma(E_i, E_j) = \sigma(F_i, F_j) = \sigma(E_i, F_j) - \delta_{ij} = 0\,, \qquad i, j = 1, \dots, n\,.$$

[1] A Lagrangian subspace $L \subset \Sigma$ of a symplectic vector space (Σ, σ) is a subspace with $\dim L = \dim \Sigma/2$ and $\sigma|_L = 0$.

Moreover, $C_2(t) \geq 0$ as a consequence of the non-negativity of the sub-Riemannian Hamiltonian. To see this, for a bilinear form $B : V \times V \to \mathbb{R}$ and n-tuples $v, w \in V$ let $B(v,w)$ denote the matrix $B(v_i, w_j)$. With this notation

$$C_2(t) = \sigma(\dot{E}, E)|_{\lambda(t)} = 2H(E,E)|_{\lambda(t)} \geq 0\,,$$

where we identified $\mathcal{V}_{\lambda(t)} \simeq T^*_{\gamma(t)}M$ and we see the Hamiltonian as a symmetric bilinear form on fibers. In the Riemannian case, $C_2(t) > 0$. In turn, the Jacobi equation, written in terms of the components $(p(t), x(t))$, becomes

$$\begin{pmatrix} \dot{p} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -C_1(t) & -R(t) \\ C_2(t) & C_1(t)^* \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}. \tag{6}$$

In the Riemannian case one can choose a suitable frame to simplify (6) as much as possible. Let $X_1, \dots, X_n$ be a parallel transported frame along the geodesic $\gamma(t)$. Let $h_i \colon T^*M \to \mathbb{R}$ be the fiber-wise linear functions, defined by $h_i(\lambda) := \langle \lambda, X_i \rangle$. Indeed $h_1, \dots, h_n$ define coordinates on each fiber, and the vectors ∂_{h_i}. We define a moving frame along the extremal $\lambda(t)$ as follows

$$E_i := \partial_{h_i}\,, \qquad F_i := -\dot{E}_i\,.$$

One can recover the original parallel transported frame by projection, namely $\pi_* F_i|_{\lambda(t)} = X_i|_{\gamma(t)}$. We state here the properties of the moving frame.

Proposition 2.1. *The smooth moving frame $\{E_i, F_i\}_{i=1}^n$ satisfies:*

(i) $\pi_* E_i|_{\lambda(t)} = 0$.

(ii) *It is a Darboux basis, namely*

$$\sigma(E_i, E_j) = \sigma(F_i, F_j) = \sigma(E_i, F_j) - \delta_{ij} = 0\,, \qquad i, j = 1, \dots, n\,.$$

(iii) *The frame satisfies the structural equations*

$$\dot{E}_i = -F_i\,, \qquad \dot{F}_i = \sum_{j=1}^n R_{ij}(t) E_j\,,$$

for some smooth family of $n \times n$ symmetric matrices $R(t)$.

If $\{\widetilde{E}_i, \widetilde{F}_j\}_{i=1}^n$ is another smooth moving frame along $\lambda(t)$ satisfying (i)–(iii)*, for some matrix $\widetilde{R}(t)$ then there exist a constant, orthogonal matrix O such that*

$$\widetilde{E}_i|_{\lambda(t)} = \sum_{j=1}^n O_{ij} E_j|_{\lambda(t)}, \qquad \widetilde{F}_i|_{\lambda(t)} = \sum_{j=1}^n O_{ij} F_j|_{\lambda(t)}\,, \qquad \widetilde{R}(t) = O R(t) O^*.$$

Thanks to this proposition, the symmetric matrix $R(t)$ induces a well defined quadratic form $\mathfrak{R}_{\lambda(t)} : T_{\gamma(t)}M \times T_{\gamma(t)}M \to \mathbb{R}$

$$\mathfrak{R}_{\lambda(t)}(v,v) := \sum_{i,j=1}^n R_{ij}(t) v_i v_j, \qquad v = \sum_{i=1}^n v_i X_i|_{\gamma(t)}.$$

Indeed one can prove that

$$\mathfrak{R}_{\lambda(t)}(v,v) = g(R^\nabla(v, \dot{\gamma})\dot{\gamma}, v), \qquad v \in T_{\gamma(t)}M. \tag{7}$$

The proof is a standard computation that can be found, for instance, in [7, Appendix C]. Then, in the Jacobi equation (6), one has $C_1(t) = 0$, $C_2(t) = \mathbb{I}$ (in particular, they are constant matrices), and the only non-trivial block $R(t)$ is the curvature operator along the geodesic:

$$\dot{x} = p \,, \qquad \dot{p} = -R(t)x \,,$$

3. The sub-Riemannian case

The problem of finding a set of Darboux frames normalizing the Jacobi equation has been first studied by Agrachev-Zelenko in [5, 6] and subsequently completed by Zelenko-Li in [15] in the general setting of curves in the Lagrange Grassmannian. A dramatic simplification, analogous to the Riemannian one, cannot be achieved in the general sub-Riemannian setting. Nevertheless, it is possible to find a normal form of (6) where the matrices C_1 and C_2 are constant. Moreover, the very block structure of these matrices depends on the geodesic and already contains important geometric invariants, that we now introduce.

3.1. **Geodesic flag and Young diagram.** Let $\gamma(t)$ be a geodesic. Recall that $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for every t. Consider a smooth admissible extension of the tangent vector, namely a vector field $\mathsf{T} \in \Gamma(\mathcal{D})$ such that $\mathsf{T}|_{\gamma(t)} = \dot{\gamma}(t)$.

Definition 3.1. The *flag of the geodesic* $\gamma(t)$ is the sequence of subspaces

$$\mathcal{F}^i_{\gamma(t)} := \operatorname{span}\{\mathcal{L}^j_{\mathsf{T}}(X)|_{\gamma(t)} \mid X \in \Gamma(\mathcal{D}),\, j \leq i-1\} \subseteq T_{\gamma(t)}M \,, \qquad \forall\, i \geq 1 \,,$$

where $\mathcal{L}_{\mathsf{T}}$ denotes the Lie derivative in the direction of T.

By definition, this is a filtration of $T_{\gamma(t)}M$, i.e. $\mathcal{F}^i_{\gamma(t)} \subseteq \mathcal{F}^{i+1}_{\gamma(t)}$, for all $i \geq 1$. Moreover, $\mathcal{F}^1_{\gamma(t)} = \mathcal{D}_{\gamma(t)}$. Definition 3.1 is well posed, namely does not depend on the choice of the admissible extension T (see [3, Sec. 3.4]). The *growth vector* of the geodesic $\gamma(t)$ is the sequence of integer numbers

$$\mathcal{G}_{\gamma(t)} := \{\dim \mathcal{F}^1_{\gamma(t)}, \dim \mathcal{F}^2_{\gamma(t)}, \ldots\} \,.$$

A geodesic $\gamma(t)$, with growth vector $\mathcal{G}_{\gamma(t)}$, is said

– *equiregular* if $\dim \mathcal{F}^i_{\gamma(t)}$ does not depend on t for all $i \geq 1$,

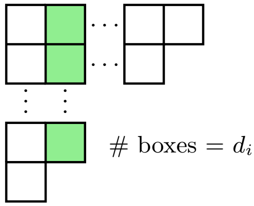
– *ample* if for all t there exists $m \geq 1$ such that $\dim \mathcal{F}^m_{\gamma(t)} = \dim T_{\gamma(t)}M$.

Equiregular (resp. ample) geodesics are the microlocal counterpart of equiregular (resp. bracket-generating) distributions. Let $d_i := \dim \mathcal{F}^i_\gamma - \dim \mathcal{F}^{i-1}_\gamma$, for $i \geq 1$, be the increment of dimension of the flag of the geodesic at each step (with the convention $\dim \mathcal{F}^0 = 0$).

Lemma 3.2 ([3])**.** *For an equiregular, ample geodesic, $d_1 \geq d_2 \geq \ldots \geq d_m$.*

The generic geodesic is ample and equiregular. More precisely, the set of points $x \in M$ such that there a exists non-empty Zariski open set $A_x \subseteq T^*_x M$ of initial covectors for which the associated geodesic is ample and equiregular with the same (maximal) growth vector, is open and dense in M. See [3, 15] for more details.

For an ample, equiregular geodesic we can build a tableau D with m columns of length d_i, for $i = 1, \ldots, m$, as follows:



Indeed $\sum_{i=1}^m d_i = n = \dim M$ is the total number of boxes in D.

Consider an ample, equiregular geodesic, with Young diagram D, with k rows, of length $n_1, \ldots, n_k$. Indeed $n_1 + \ldots + n_k = n$. The moving frame we are going to introduce is indexed by the boxes of the Young diagram. The notation $ai \in D$ denotes the generic box of the diagram, where $a = 1, \ldots, k$ is the row index, and $i = 1, \ldots, n_a$ is the progressive box number, starting from the left, in the specified row. We employ letters $a, b, c, \ldots$ for rows, and $i, j, h, \ldots$ for the position of the box in the row.

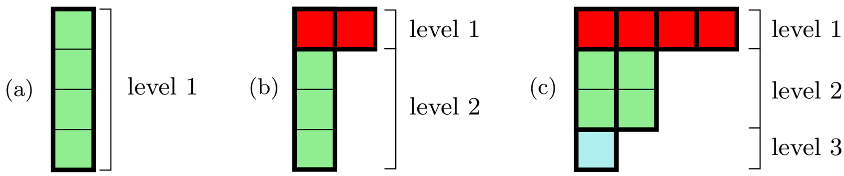


FIG. 1: Levels (shaded regions) and superboxes (delimited by bold lines) for the Young diagram of (a) Riemannian, (b) contact, (c) a more general structure. The Young diagram for any Riemannian geodesic has a single level and a single superbox.

We collect the rows with the same length in D, and we call them *levels* of the Young diagram. In particular, a level is the union of r rows $D_1, \ldots, D_r$, and r is called the *size* of the level. The set of all the boxes $ai \in D$ that belong to the same column and the same level of D is called *superbox*. We use Greek letters $\alpha, \beta, \ldots$ to denote superboxes. Notice that that two boxes ai, bj are in the same superbox if and only if ai and bj are in the same column of D and in possibly distinct row but with same length, i.e. if and only if $i = j$ and $n_a = n_b$ (see Fig. 1).

The following theorem is proved in [15].

Theorem 3.3. *Assume $\lambda(t)$ is the lift of an ample and equiregular geodesic $\gamma(t)$ with Young diagram D. Then there exists a smooth moving frame $\{E_{ai}, F_{ai}\}_{ai \in D}$ along $\lambda(t)$ such that*

(i) $\pi_* E_{ai}|_{\lambda(t)} = 0$.

(ii) *It is a Darboux basis, namely*

$$\sigma(E_{ai}, E_{bj}) = \sigma(F_{ai}, F_{bj}) = \sigma(E_{ai}, F_{bj}) = \delta_{ab}\delta_{ij}, \qquad ai, bj \in D.$$

(iii) *The frame satisfies structural equations*

$$
\begin{cases}
\dot{E}_{ai} = E_{a(i-1)} & a = 1,\dots,k, \quad i = 2,\dots,n_a, \\
\dot{E}_{a1} = -F_{a1} & a = 1,\dots,k, \\
\dot{F}_{ai} = \sum_{bj\in D} R_{ai,bj}(t)E_{bj} - F_{a(i+1)} & a = 1,\dots,k, \quad i = 1,\dots,n_a - 1, \\
\dot{F}_{an_a} = \sum_{bj\in D} R_{an_a,bj}(t)E_{bj} & a = 1,\dots,k,
\end{cases}
\tag{8}
$$

for some smooth family of $n \times n$ symmetric matrices $R(t)$, with components $R_{ai,bj}(t) = R_{bj,ai}(t)$, indexed by the boxes of the Young diagram D. The matrix $R(t)$ is normal *in the sense of* [15] *(see Appendix A).*

If $\{\widetilde{E}_{ai}, \widetilde{F}_{ai}\}_{ai\in D}$ is another smooth moving frame along $\lambda(t)$ satisfying (i)–(iii)*, with some normal matrix $\widetilde{R}(t)$, then for any superbox α of size r there exists an orthogonal constant $r \times r$ matrix O^α such that*

$$
\widetilde{E}_{ai} = \sum_{bj\in\alpha} O^\alpha_{ai,bj} E_{bj}\,, \qquad \widetilde{F}_{ai} = \sum_{bj\in\alpha} O^\alpha_{ai,bj} F_{bj}\,, \qquad ai \in \alpha\,.
$$

Remark 3.4. For $a = 1,\dots,k$, the symbol E_a denotes the n_a-dimensional column vector $E_a = (E_{a1}, E_{a2},\dots,E_{an_a})^*$, with analogous notation for F_a. Similarly, E denotes the n-dimensional column vector $E = (E_1,\dots,E_k)^*$, and similarly for F. Then, we rewrite the system (8) as follows (compare with (5))

$$
\begin{pmatrix} \dot{E} \\ \dot{F} \end{pmatrix} = \begin{pmatrix} C_1^* & -C_2 \\ R(t) & -C_1 \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix}, \tag{9}
$$

where $C_1 = C_1(D)$, $C_2 = C_2(D)$ are $n \times n$ matrices, depending on the Young diagram D, defined as follows: for $a, b = 1,\dots,k$, $i = 1,\dots,n_a$, $j = 1,\dots,n_b$:

$$
[C_1]_{ai,bj} := \delta_{ab}\delta_{i,j-1},\,, \qquad [C_2]_{ai,bj} := \delta_{ab}\delta_{i1}\delta_{j1}\,.
$$

It is convenient to see C_1 and C_2 as block diagonal matrices:

$$
C_i(D) := \begin{pmatrix} C_i(D_1) & & \\ & \ddots & \\ & & C_i(D_k) \end{pmatrix}, \qquad i = 1, 2,
$$

the a-th block being the $n_a \times n_a$ matrices

$$
C_1(D_a) := \begin{pmatrix} 0 & \mathbb{I}_{n_a-1} \\ 0 & 0 \end{pmatrix}, \qquad C_2(D_a) := \begin{pmatrix} 1 & 0 \\ 0 & 0_{n_a-1} \end{pmatrix},
$$

where $\mathbb{I}_m$ is the $m \times m$ identity matrix and 0_m is the $m \times m$ zero matrix. Notice that the matrices C_1, C_2 satisfy the Kalman rank condition

$$
\operatorname{rank}\{C_2, C_1C_2,\dots,C_1^{n-1}C_2\} = n\,. \tag{10}
$$

Analogously, the matrices $C_i(D_a)$ satisfy (10) with $n = n_a$.

Let $\{X_{ai}\}_{ai\in D}$ be the moving frame along $\gamma(t)$ defined by $X_{ai}|_{\gamma(t)} = \pi_* F_{ai}|_{\lambda(t)}$, for some choice of a canonical Darboux frame. Theorem 3.3 implies that the following definitions are well posed.

Definition 3.5. The *canonical splitting* of $T_{\gamma(t)}M$ is

$$T_{\gamma(t)}M = \bigoplus_{\alpha} S^{\alpha}_{\gamma(t)}\,, \qquad S^{\alpha}_{\gamma(t)} := \operatorname{span}\{X_{ai}|_{\gamma(t)} \mid ai \in \alpha\}\,,$$

where the sum is over the superboxes α of D. Notice that the dimension of $S^{\alpha}_{\gamma(t)}$ is equal to the size r of the level to which the superbox α belongs.

Definition 3.6. The *canonical curvature* (along $\lambda(t)$), is the quadratic form $\mathfrak{R}_{\lambda(t)} : T_{\gamma(t)}M \times T_{\gamma(t)}M \to \mathbb{R}$ whose representative matrix, in terms of the basis $\{X_{ai}\}_{ai\in D}$, is $R_{ai,bj}(t)$. In other words

$$\mathfrak{R}_{\lambda(t)}(v,v) := \sum_{ai,bj\in D} R_{ai,bj}(t) v_{ai} v_{bj}\,, \qquad v = \sum_{ai\in D} v_{ai} X_{ai}|_{\gamma(t)} \in T_{\gamma(t)}M\,.$$

We denote the restrictions of $\mathfrak{R}_{\lambda(t)}$ on the appropriate subspaces by:

$$\mathfrak{R}^{\alpha\beta}_{\lambda(t)} : S^{\alpha}_{\gamma(t)} \times S^{\beta}_{\gamma(t)} \to \mathbb{R}.$$

For any superbox α of D, the *canonical Ricci curvature* is the partial trace:

$$\mathfrak{Ric}^{\alpha}_{\lambda(t)} := \sum_{ai\in\alpha} \mathfrak{R}^{\alpha\alpha}_{\lambda(t)}(X_{ai}, X_{ai})\,.$$

The Jacobi equation, written in terms of the components $(p(t), x(t))$ with respect to a canonical Darboux frame $\{E_{ai}, F_{ai}\}_{ai\in D}$, becomes

$$\begin{pmatrix} \dot p \\ \dot x \end{pmatrix} = \begin{pmatrix} -C_1 & -R(t) \\ C_2 & C_1^* \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}.$$

This is the sub-Riemannian generalization of the classical Jacobi equation seen as first-order equation for fields on the cotangent bundle. Its structure depends on the Young diagram of the geodesic through the matrices $C_i(D)$, while the remaining invariants are contained in the curvature matrix $R(t)$. Notice that this includes the Riemannian case, where D is the same for every geodesic, with $C_1 = 0$ and $C_2 = \mathbb{I}$.

3.2. **Homogeneity properties.** For all $c > 0$, let $H_c := H^{-1}(c/2)$ be the Hamiltonian level set. In particular H_1 is the unit cotangent bundle: the set of initial covectors associated with unit-speed geodesics. Since the Hamiltonian function is fiber-wise quadratic, we have the following property for any $c > 0$

$$e^{t\vec{H}}(c\lambda) = c e^{ct\vec{H}}(\lambda)\,, \tag{11}$$

where, for $\lambda \in T^*M$, the notation $c\lambda$ denotes the fiber-wise multiplication by c. Let $P_c\colon T^*M \to T^*M$ be the map $P_c(\lambda) = c\lambda$. Indeed $\alpha \mapsto P_{e^{\alpha}}$ is a one-parameter group of diffeomorphisms. Its generator is the *Euler vector field* $\mathfrak{e} \in \Gamma(\mathcal{V})$, and is characterized by $P_c = e^{(\ln c)\mathfrak{e}}$. We can rewrite (11) as the following commutation rule for the flows of $\vec{H}$ and $\mathfrak{e}$:

$$e^{t\vec{H}} \circ P_c = P_c \circ e^{ct\vec{H}} .$$

Observe that P_c maps H_1 diffeomorphically on H_c. Let $\lambda \in H_1$ be associated with an ample, equiregular geodesic with Young diagram D. Clearly also the geodesic associated with $\lambda^c := c\lambda \in H_c$ is ample and equiregular, with the same Young diagram. This corresponds to a reparametrization of the same curve: in fact $\lambda^c(t) = e^{t\vec{H}}(c\lambda) = c(\lambda(ct))$, hence $\gamma^c(t) = \pi(\lambda^c(t)) = \gamma(ct)$.

Theorem 3.7 (Homogeneity properties of the canonical curvature)**.** *For any superbox $\alpha \in D$, let $|\alpha|$ denote the column index of α. Denoting $\lambda^c(t) = e^{t\vec{H}}(c\lambda)$ we have, for any $c > 0$*

$$\mathfrak{R}^{\alpha\beta}_{\lambda^c(t)} = c^{|\alpha|+|\beta|}\mathfrak{R}^{\alpha\beta}_{\lambda(ct)} ,$$

Remark 3.8. In the Riemannian setting, D has only one superbox with $|\alpha| = 1$ (see Fig. 1). Then $\mathfrak{R}_\lambda := \mathfrak{R}^{\alpha\alpha}_{\lambda(0)}$ is homogeneous of degree 2 as a function of λ.

Theorem 3.7 follows directly from the next result and Definition 3.6. In the next proposition, for any $\eta \in T^*M$ and $c > 0$, we denote with $d_\eta P_c : T_\eta(T^*M) \to T_{c\eta}(T^*M)$ the differential of the map P_c, computed at η.

Proposition 3.9. *Let $\lambda \in H_1$ and $\{E_{ai}, F_{ai}\}_{ai\in D}$ be the associated canonical frame along the extremal $\lambda(t)$. Let $c > 0$ and define, for $ai \in D$*

$$E^c_{ai}(t) := \frac{1}{c^i}(d_{\lambda(ct)}P_c)E_{ai}(ct) , \qquad F^c_{ai}(t) := c^{i-1}(d_{\lambda(ct)}P_c)F_{ai}(ct) .$$

The moving frame $\{E^c_{ai}(t), F^c_{ai}(t)\}_{ai\in D} \in T_{\lambda^c(t)}(T^*M)$ is a canonical frame associated with the initial covector $\lambda^c = c\lambda \in H_c$, with curvature matrix

$$R^{\lambda^c}_{ai,bj}(t) = c^{i+j}R^\lambda_{ai,bj}(ct) . \tag{12}$$

Proof. We check all the relations of Theorem 3.3. Indeed P_α sends fibers to fibers, hence (i) is trivially satisfied. For what concerns (ii), let θ be the Liouville one-form, and $\sigma = d\theta$. Indeed $P_c^*\theta = c\theta$. Hence $P_c^*\sigma = c\sigma$. It follows that $\{E^c_{ai}(t), F^c_{ai}(t)\}_{ai\in D}$ is a Darboux frame at $\lambda^c(t)$:

$$\sigma_{\lambda^c(t)}(E^c_{ai}(t), F^c_{bj}(t)) = \tfrac{1}{c}(P_c^*\sigma)_{\lambda(t)}(E_{ai}(t), F_{bj}(t)) = \delta_{ab}\delta_{ij} ,$$

and similarly for the others Darboux relations.

For what concerns (iii) (the structural equations), let $\xi(t)$ be any vector field along $\lambda(t)$, and $(d_{\lambda(t)}P_c)\xi(ct)$ be the corresponding vector field along $\lambda^c(t)$. Then

$$\begin{aligned}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} e_*^{-\varepsilon\vec{H}} \circ (d_{\lambda(t)}P_c)\xi(c(t+\varepsilon)) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (e^{-\varepsilon\vec{H}} \circ P_c)_*\xi(c(t+\varepsilon)) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (P_c \circ e^{-c\varepsilon\vec{H}})_*\xi(c(t+\varepsilon)) \\ &= c\left.\frac{d}{d\tau}\right|_{\tau=0} (P_c \circ e^{-\tau\vec{H}})_*\xi(ct+\tau) \\ &= c(d_{\lambda(ct)}P_c)\dot{\xi}(ct)\,.\end{aligned}$$

Applying the above identity to compute the derivatives of the new frame, and using (8), one finds that $\{E^c_{ai}(t), F^c_{ai}(t)\}_{ai\in D}$ satisfies the structural equations, with curvature matrix given by (12). For example

$$\begin{aligned}\dot{F}^c_{ai}(t) &= c^{i-1}c(d_{\lambda(ct)}P_c)\dot{F}_{ai}(ct) \\ &= c^i(d_{\lambda(ct)}P_c)\big[R^\lambda_{ai,bj}(ct)E_{bj}(ct) - F_{a(i+1)}(ct)\big] \\ &= c^i\big[c^jR^\lambda_{ai,bj}(ct)E^c_{bj}(t) - c^{-i}F^c_{a(i+1)}(t)\big] \\ &= c^{i+j}R^\lambda_{ai,bj}(ct)E^c_{bj}(t) - F^c_{a(i+1)}(t)\,,\end{aligned}$$

where we suppressed a summation over $bj \in D$. □

Proposition 3.3 defines not only a curvature, but also a (non-linear) connection, in the sense of Ehresmann, that we now introduce.

4. Ehresmann curvature and curvature operator

For any smooth vector bundle N over M, let $\Gamma(N)$ denote the smooth sections of N. Recall that $\mathcal{V} := \ker \pi_* \subset T(T^*M)$ is the *vertical distribution*. An *Ehresmann connection* on T^*M is a smooth distribution $\mathcal{H} \subset T(T^*M)$ such that

$$T(T^*M) = \mathcal{H} \oplus \mathcal{V}\,.$$

We call $\mathcal{H}$ the *horizontal distribution*[2]. An Ehresmann connection $\mathcal{H}$ is *linear* if $\mathcal{H}_{c\lambda} = (d_\lambda P_c)\mathcal{H}_\lambda$ for every $\lambda \in T^*M$ and $c > 0$.

For any $X \in \Gamma(TM)$ there exists a unique *horizontal lift* ∇_X in $\Gamma(\mathcal{H})$ such that $\pi_*\nabla_X = X$.

Remark 4.1. A function $h \in C^\infty(T^*M)$ is fiber-wise linear if it can be written as $h(\lambda) = \langle\lambda, Y\rangle$, for some $Y \in \Gamma(TM)$. Such an Y is clearly unique, and for this reason we denote $h_Y := \lambda \mapsto \langle\lambda, Y\rangle$ the fiber-wise linear function associated with $Y \in \Gamma(TM)$. A connection ∇ is linear if, for every $X \in \Gamma(TM)$, the derivation ∇_X maps fiber-wise linear functions to fiber-wise linear functions. In this case, we recover the classical notion of covariant derivative by defining $\nabla_X Y = Z$ if $\nabla_X h_Y = h_Z$, where $Y, Z \in \Gamma(TM)$.

[2]Note that this is a distribution on T^*M, i.e. a sub-bundle of $T(T^*M)$ and should not be confused with the sub-Riemannian distribution $\mathcal{D}$, that is a subbundle of TM.

We recall the definition of curvature of an Ehresmann connection [9].

Definition 4.2. The *Ehresmann curvature* of the connection ∇ is the $C^\infty(M)$-linear map $R^\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(\mathcal{V})$ defined by

$$R^\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\,, \qquad X, Y \in \Gamma(TM)\,.$$

R^∇ is skew-symmetric, namely $R^\nabla(X,Y) = -R^\nabla(Y,X)$. Notice that $R^\nabla = 0$ if and only if $\mathcal{H}$ is involutive.

4.1. **Canonical connection.** Let $\gamma(t)$ be a fixed ample and equiregular geodesic with Young diagram D, projection of the extremal $\lambda(t)$, with initial covector λ. Let $\{E_{ai}(t), F_{ai}(t)\}$ be a canonical frame along $\lambda(t)$. For $t = 0$, this defines a subspace at $\lambda \in T^*M$, namely

$$\mathcal{H}_\lambda := \operatorname{span}\{F_{ai}|_\lambda\}_{ai \in D}\,, \qquad \lambda \in T^*M\,. \tag{13}$$

Indeed this definition makes sense on the subset of covectors $N \subset T^*M$ associated with ample and equiregular geodesics. In the Riemannian case, every non-trivial geodesic is ample and equiregular, with the same Young diagram. Hence $N = T^*M \setminus H^{-1}(0)$. A posteriori one can show that this connection is linear and can be extended smoothly on the whole T^*M. In the sub-Riemannian case, $N \subset T^*M \setminus H^{-1}(0)$.

In general, using the results of [3, Section 5.2] and [15, Section 5], one can prove that N is open and dense in T^*M. Moreover, the elements of the frame depend rationally (in charts) on the point λ, hence $\mathcal{H}$ is smooth on N.

For simplicity, we assume that it is possible to extend $\mathcal{H}$ to a smooth distribution on the whole T^*M. This is indeed possible in some cases of interest: on corank 1 structures with symmetries [10] and on contact sub-Riemannian structures [4] (see also [14] for fat structures). In the general case, we replace T^*M with N.

Definition 4.3. The *canonical Ehresmann connection* associated with the sub-Riemannian structure is the horizontal distribution $\mathcal{H} \subset T(T^*M)$ defined by (13).

As a consequence of Proposition 3.9, $\mathcal{H}$ is non-linear, in general. However, if the structure is Riemannian, one has $\mathcal{H}_{c\lambda} = (d_\lambda P_c)\mathcal{H}_\lambda$ and the connection is linear.

Proposition 4.4. *Let H be the sub-Riemannian Hamitonian and $\mathcal{H}$ the canonical connection. Then $\nabla_X H = 0$ for every $X \in \Gamma(TM)$. Equivalently, $\vec{H} \in \mathcal{H}$.*

Remark 4.5. The above condition is the compatibility of the canonical connection with the sub-Riemannian metric. In the Riemannian setting, $\mathcal{H}$ is linear and this condition can be rewritten, in the sense of covariant derivative, as $\nabla g = 0$.

Proof. The equivalence of the two statements follows from the definition of Hamiltonian vector field and the fact that $\mathcal{H}$ is Lagrangian, by construction. Indeed

$$\nabla_X H = dH(\nabla_X) = \sigma(\vec{H}, \nabla_X).$$

Then we prove that $\vec{H} \in \mathcal{H}$. □

Lemma 4.6. *Let $\mathfrak{e}$ be the Euler vector field. Then $\dot{\mathfrak{e}} = -\vec{H}$.*

Proof of Lemma 4.6. Let $P_s = e^{(\ln s)\mathfrak{e}}$ be the dilation along the fibers. We have the following commutation rule for the flows of $\vec{H}$ and $\mathfrak{e}$

$$P_{-s} \circ e^{-t\vec{H}} \circ P_s = e^{-ts\vec{H}} .$$

Computing the derivative w.r.t. t and s at $(t,s) = (0,1)$ we obtain $[\vec{H}, \mathfrak{e}] = -\mathfrak{e}$, that implies the statement. □

Lemma 4.7. *Since $\mathfrak{e}$ is vertical, then $\mathfrak{e} = v(t)^* E(t)$ for some smooth $v(t) \in \mathbb{R}^n$. Accordingly with the decomposition of Remark 3.4, we set*

$$v(t) = (v_1(t), \dots, v_k(t))^* , \quad \textit{with} \quad v_a(t) = (v_{a1}(t), \dots, v_{an_a}(t))^* .$$

Then $v(t)$ is constant and we have

$$\mathfrak{e} = \sum_{\substack{ai \in D \\ n_a = 1}} v_{ai} E_{ai} .$$

Proof of Lemma 4.7. As a consequence of Lemma 4.6, $\ddot{\mathfrak{e}} = 0$. Using the structural equations (9), we obtain

$$C_1^* C_2 v - C_2 C_1 v - 2C_2 \dot{v} = 0 , \tag{14}$$

$$\ddot{v} + 2C_1 \dot{v} + C_1^2 v - RC_2 v = 0 . \tag{15}$$

We show that for any row index of the Young diagram $a = 1, \dots, k$

$$v_a = \begin{cases} (0, \dots, 0)^* & n_a > 1 , \\ \text{constant} & n_a = 1 . \end{cases}$$

Let us focus on (14). For each $a = 1, \dots, k$, we take its a-th block. By the block structure of C_1 and C_2, this is

$$C_1^* C_2 v_a - C_2 C_1 v_a - 2C_2 \dot{v}_a = 0 , \qquad \forall\, a = 1, \dots, k , \tag{16}$$

where here $C_1 = C_1(D_a)$ and $C_2 = C_2(D_a)$. If $n_a = 1$, then $C_1 = 0$ and $C_2 = 1$. In this case (16) implies $v_a(t) = v_a$ is constant. Now let $n_a > 1$. In this case, the particular form of C_1, C_2 for (16) yields

$$C_1^* C_2 v_a = 0 , \qquad \text{and} \qquad C_2 C_1 v_a + 2C_2 \dot{v}_a = 0 , \qquad (n_a > 1) .$$

Indeed the kernel of C_1^* is orthogonal to the image of C_2. Hence $C_1^* C_2 v_a = 0$ implies $C_2 v_a = 0$. In particular (16) is equivalent to

$$C_2 v_a = 0 , \qquad C_2 C_1 v_a = 0 , \qquad (n_a > 1) . \tag{17}$$

More explicitly, $v_a = (0, 0, v_{a3}, \dots, v_{an_a})$. For the case $n_a = 2$ this is sufficient to completely determine v_a. In all the other cases, let us turn to (15). The latter does not split immediately, as the curvature matrix R is not block-diagonal. However, let us consider a copy of (15) multiplied by $C_2 C_1^i$. For each a such that $n_a > 2$ we consider its a-th block, obtaining the following:

$$C_2 C_1^i \ddot{v}_a + 2C_2 C_1^{i+1} \dot{v}_a + C_2 C_1^{i+2} v_a - [C_2 C_1^i R C_2 v]_a = 0\,, \qquad (n_a > 2)\,.$$

We claim that $[C_2 C_1^i R C_2 v]_a = 0$ if $n_a > 2$ and $i < n_a - 2$.

By setting the matrix $[R_{ab}]_{ij} := R_{ai,bj}$, with $ai, bj \in D$ (this is a block of R, corresponding to the rows a, b of the Young diagram D), we compute

$$\begin{aligned}
[C_2 C_1^i R C_2 v]_a &= \sum_{b,c,d=1}^{k} [C_2 C_1^i]_{ab} R_{bc} [C_2]_{cd} v_d = \sum_{b=1}^{k} (C_2 C_1^i) R_{ab} (C_2 v_b) \\
&= \sum_{n_b=1} (C_2 C_1^i) R_{ab} (C_2 v_b) = \sum_{n_b=1} R_{a(i+1),b1} v_{b1}\,,
\end{aligned}$$

where we used the block structure of the C_i's and (17). The last sum involves only $R_{a(i+1),b1}$ with $n_b = 1$ and $n_a > 2$. If $i < n_a - 2$, then $R_{a(i+1),b1}$ is *not* in the last $2n_b = 2$ elements of Table 1, and vanishes by the normal conditions (see Appendix A). Thus we have:

$$C_2 C_1^i \ddot{v}_a + 2C_2 C_1^{i+1} \dot{v}_a + C_2 C_1^{i+2} v_a = 0\,, \qquad (n_a > 2\,, \quad i < n_a - 2)\,. \tag{18}$$

In particular using (17), and taking $i = 0, \dots, n_a - 3$ we see that (18) is equivalent to $C_2 C_1^{i+2} v_a = 0$ for all $i = 0, \dots, n_a - 3$. Combining all the cases

$$v_a \in \ker\{C_2, C_2 C_1, C_2 C_1^2, \dots, C_2 C_1^{n_a - 1}\}\,, \qquad (n_a > 1)\,.$$

This yields $v_a = 0$, by Kalman rank condition (10). □

Lemma 4.7 implies our statement since

$$\vec{H} = -\dot{\mathfrak{e}} = -\sum_{\substack{ai \in D \\ n_a = 1}} v_{ai} \dot{E}_{ai} = \sum_{\substack{ai \in D \\ n_a = 1}} v_{ai} F_{ai} \in \mathcal{H}\,,$$

where we used the structural equations (8) for the E_{ai}'s with $n_a = 1$.

4.2. **Relation with the canonical curvature.** We now discuss the relation between the curvature of the canonical Ehresmann connection and the sub-Riemannian curvature operator. In what follows we denote by $\mathfrak{R}_\lambda := \mathfrak{R}_{\lambda(0)}$, where $\lambda(t)$ is the extremal with initial datum λ. Then $\mathfrak{R}$ extends to a well defined map

$$\begin{aligned}
\mathfrak{R} \colon \Gamma(T^*M) \times \Gamma(TM) \times \Gamma(TM) &\to C^\infty(M)\,, \\
(\lambda, X, Y) &\mapsto \mathfrak{R}_\lambda(X, Y)\,.
\end{aligned} \tag{19}$$

We stress that here the first argument is a section $\lambda \in \Gamma(T^*M)$.

Although $\mathfrak{R}$ is $C^\infty(M)$-linear in the last two arguments by construction, it is in general non-linear in the first argument, so it does not define a $(1, 2)$ tensor. Nevertheless, for any fixed section $\lambda \in \Gamma(T^*M)$, the restriction $\mathfrak{R}_\lambda \colon \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ is a $(0, 2)$ symmetric tensor.

Theorem 4.8. *Let $R^\nabla : \Gamma(TM)\times\Gamma(TM) \to \Gamma(\mathcal{V})$ be the curvature of the canonical Ehresmann connection, and let $\mathfrak{R}\colon \Gamma(T^*M) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ be the canonical curvature map* (19)*. Then*

$$\mathfrak{R}_\lambda(X,Y) = \sigma_\lambda(R^\nabla(\mathsf{T},X),\nabla_Y)\,, \qquad \forall\,\lambda \in \Gamma(T^*M)\,, \quad X,Y \in \Gamma(TM)\,, \tag{20}$$

where $\mathsf{T} = \pi_*\vec{H}|_\lambda \in \Gamma(TM)$.

Proof. We evaluate the right hand side of (20) at the point x, for any fixed section $\lambda = \lambda(x) \in \Gamma(T^*M)$. By linearity, it is sufficient to take $X = X_{ai}$ and $Y = Y_{bj}$, projections of a canonical frame $F_{ai}|_\lambda, F_{bj}|_\lambda$ at $t = 0$. Indeed, by definition, $\nabla_{X_{ai}}|_\lambda = F_{ai}|_\lambda$. Then

$$\begin{aligned}\sigma_\lambda(R^\nabla(\mathsf{T},X_{ai}),\nabla_{X_{bj}}) &= \sigma_\lambda([\nabla_\mathsf{T},F_{ai}],F_{bj}) = \sigma_\lambda([\vec{H},F_{ai}],F_{bj})\\ &= \sigma_\lambda(\dot{F}_{ai},F_{bj}) = R^\lambda_{ai,bj}(0)\,.\end{aligned}$$

Here we used the structural equations and that $\vec{H} \in \mathcal{H}$, thus $\nabla_\mathsf{T} = \vec{H}$. By definition of canonical curvature map, we obtain the statement. □

Remark 4.9. For $\lambda \in \Gamma(T^*M)$, the corresponding tangent field $\mathsf{T} \in \Gamma(\mathcal{D}) \subsetneq \Gamma(TM)$. Therefore, $\mathfrak{R}$ recovers only part of the whole Ehresmann connection.

Remark 4.10 (On the Riemannian case)**.** As we proved in (7), we have

$$\mathfrak{R}_\lambda(X,Y) = R^\nabla(\mathsf{T},X,Y,\mathsf{T})\,,$$

where $\mathsf{T} = \pi_*\vec{H}|_\lambda$ is the tangent vector associated with the covector λ. For completeness, let us recover the same formula by the r.h.s. of (20). Indeed, for any vertical vector $V \in \mathcal{V}_\lambda$ and $W \in T_\lambda(T^*M)$, we have $\sigma_\lambda(V,W) = V(h_{\pi_*W})|_\lambda$ as one can check from a direct computation. Thus the r.h.s. of (20) is

$$\begin{aligned}\sigma_\lambda([\nabla_\mathsf{T},\nabla_X]-\nabla_{[\mathsf{T},X]},\nabla_Y) &= \left(\nabla_\mathsf{T}\nabla_X(h_Y) - \nabla_X\nabla_\mathsf{T}(h_Y) - \nabla_{[\mathsf{T},X]}(h_Y)\right)|_\lambda\\ &= h_{\nabla_\mathsf{T}\nabla_XY-\nabla_X\nabla_\mathsf{T}Y-\nabla_{[\mathsf{T},X]}Y}(\lambda)\\ &= \langle\lambda, \nabla_\mathsf{T}\nabla_XY - \nabla_X\nabla_\mathsf{T}Y - \nabla_{[\mathsf{T},X]}Y\rangle\\ &= g(\nabla_\mathsf{T}\nabla_XY - \nabla_X\nabla_\mathsf{T}Y - \nabla_{[\mathsf{T},X]}Y,\mathsf{T})\\ &= R^\nabla(\mathsf{T},X,Y,\mathsf{T})\,.\end{aligned}$$

Appendix A. Normal condition for the canonical frame

Here we rewrite the *normal* condition for the matrix $R_{ai,bj}$ mentioned in Theorem 3.3 (and defined in [15]) according to our notation.

Definition A.1. The matrix $R_{ai,bj}$ is *normal* if it satisfies:

(i) global symmetry: for all ai, $bj \in D$

$$R_{ai,bj} = R_{bj,ai}\,.$$

(ii) partial skew-symmetry: for all ai, $bi \in D$ with $n_a = n_b$ and $i < n_a$

$$R_{ai,b(i+1)} = -R_{bi,a(i+1)}\,.$$

(iii) vanishing conditions: the only possibly non vanishing entries $R_{ai,bj}$ satisfy

(iii.a) $n_a = n_b$ and $|i - j| \leq 1$,
(iii.b) $n_a > n_b$ and (i, j) belong to the last $2n_b$ elements of Table 1.

Tab. 1: Vanishing conditions.

i	1	1	2	$\cdots$	ℓ	ℓ	$\ell+1$	$\cdots$	n_b	n_b+1	$\cdots$	n_a-1	n_a
j	1	2	2	$\cdots$	ℓ	$\ell+1$	$\ell+1$	$\cdots$	n_b	n_b	$\cdots$	n_b	n_b

The sequence is obtained as follows: starting from $(i, j) = (1, 1)$ (the first boxes of the rows a and b), each next even pair is obtained from the previous one by increasing j by one (keeping i fixed). Each next odd pair is obtained from the previous one by increasing i by one (keeping j fixed). This stops when j reaches its maximum, that is $(i, j) = (n_b, n_b)$. Then, each next pair is obtained from the previous one by increasing i by one (keeping j fixed), up to $(i, j) = (n_a, n_b)$. The total number of pairs appearing in the table is $n_b + n_a - 1$.

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References

[1] Agrachev, A.A., *Some open problems*, Geometric control theory and sub-Riemannian geometry, Springer INdAM Ser., 5, Springer, Cham, 2014, pp. 1–13.

[2] Agrachev, A.A., Barilari, D., Boscain, U., *Introduction to Riemannian and sub-Riemannian geometry (Lecture Notes)*, 2015, `http://webusers.imj-prg.fr/~davide.barilari/notes.php`.

[3] Agrachev, A.A., Barilari, D., Rizzi, L., *Curvature: a variational approach*, Memoirs of the AMS (in press).

[4] Agrachev, A.A., Barilari, D., Rizzi, L., *Sub-riemannian curvature in contact geometry*, J. Geom. Anal. (2016), 1–43, DOI10.1007/s12220-016-9684-0.

[5] Agrachev, A.A., Zelenko, I., *Geometry of Jacobi curves. I*, J. Dynam. Control Systems **8** (1) (2002), 93–140. DOI: 10.1023/A:1013904801414

[6] Agrachev, A.A., Zelenko, I., *Geometry of Jacobi curves. II*, J. Dynam. Control Systems **8** (2) (2002), 167–215, DOI10.1023/A:1015317426164.

[7] Barilari, D., Rizzi, L., *Comparison theorems for conjugate points in sub-Riemannian geometry*, ESAIM Control Optim. Calc. Var. **22** (2) (2016), 439–472.

[8] Jean, F., *Control of nonholonomic systems: from sub-Riemannian geometry to motion planning*, SpringerBriefs in Mathematics, Springer, Cham, 2014.

[9] Kobayashi, S., Nomizu, K., *Foundations of differential geometry. Vol. I*, Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996, Reprint of the 1963 original, A Wiley-Interscience Publication.

[10] Li, C., Zelenko, I., *Jacobi equations and comparison theorems for corank 1 sub-Riemannian structures with symmetries*, J. Geom. Phys. **61** (4) (2011), 781–807.

[11] Montgomery, R., *Abnormal minimizers*, SIAM J. Control Optim. **32** (6) (1994), 1605–1620.

[12] Montgomery, R., *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs ed., vol. 91, AMS, Providence, RI, 2002.

[13] Rifford, L., *Sub-Riemannian geometry and optimal transport*, SpringerBriefs in Mathematics, Springer, Cham, 2014.

[14] Rizzi, L., Silveira, P., *Sub-Riemannian Ricci curvatures and universal diameter bounds for 3-Sasakian manifolds*, ArXiv e-prints, Sept. 2015, J. Inst. Math. Jussieu (in press).

[15] Zelenko, I., Li, C., *Differential geometry of curves in Lagrange Grassmannians with given Young diagram*, Differential Geom. Appl. **27** (6) (2009), 723–742.

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