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RESTRICTED BOOLEAN GROUP RINGS

Dinesh Udar, R.K. Sharma, and J.B. Srivastava

Abstract. In this paper we study restricted Boolean rings and group rings. A ring R is *restricted Boolean* if every proper homomorphic image of R is boolean. Our main aim is to characterize restricted Boolean group rings. A complete characterization of non-prime restricted Boolean group rings has been obtained. Also in case of prime group rings necessary conditions have been obtained for a group ring to be restricted Boolean. A counterexample is given to show that these conditions are not sufficient.

1. Introduction

Throughout this paper R will denote an associative ring with identity $1 \neq 0$ and G a non-trivial group. A ring R is called *Boolean* if $r^2 = r$ for all $r \in R$. The Jacobson radical of a ring R is denoted by $J(R)$, and $l(J(R))$, $r(J(R))$ respectively will denote the left, right annihilator of $J(R)$ in R. We define that a ring R is *restricted Boolean* if every proper homomorphic image of R is Boolean. Clearly every Boolean ring is restricted Boolean, but the converse is not true. For example, let $R = \mathbb{Z}_4$, the ring of integers modulo 4. Then $J(R) = \langle 2 \rangle$ is the only proper ideal of R and $R/J(R) \cong \mathbb{Z}_2$. So R is a restricted Boolean ring, but it is not a Boolean ring. A ring R is called *clean* if every element of it can be written as a sum of an idempotent and a unit. A ring R is *neat* if every proper homomorphic image of R is clean. Since every Boolean ring is clean, the class of restricted Boolean rings is a proper subclass of neat rings. Commutative neat rings were studied by Warren Wm. McGovern [5].

The group ring of a group G and a ring R is denoted by RG. If H is a subgroup of G, then ωH will denote the right ideal of RG generated by $\{1 - h \mid h \in H\}$. In particular, if H is a normal subgroup of G then ωH is a two sided ideal of RG and $RG/\omega H \cong R(G/H)$. If $H = G$, then ωG is the *augmentation ideal* of RG. It is easy to see that ωG is the kernel of the *augmentation map*, $\omega\colon RG \to R$, where $\omega(\Sigma_{g\in G} r_g g) = \Sigma_{g\in G} r_g$ and $RG/\omega G \cong R$. If I is an ideal of R, then IG is an ideal of RG and $RG/IG \cong (R/I)G$. For group ring related results we refer to Connell [2] and Passman [6]; and for ring theory we refer to Lam [4].

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It is easy to see that a group ring RG is Boolean if and only if R is Boolean and G is trivial. In this paper our main focus is on the study of restricted Boolean group rings which are not Boolean. Various properties of restricted Boolean group rings have been investigated. We obtain a complete characterization of non-prime restricted Boolean group rings. It is proved that a non-prime group ring RG is restricted Boolean, but not Boolean if and only if $R \cong \mathbb{Z}_2$ and $G \cong C_2$, the finite cyclic group of order 2. Certain necessary conditions have been obtained in case of prime restricted Boolean group rings. A counterexample has been given to show that these conditions are not sufficient.

2. Main results

Lemma 2.1. *If R is a restricted Boolean ring, then any non-trivial prime ideal of R is maximal.*

Proof. Let P be a non-trivial prime ideal of R. So R/P is prime and Boolean. As a Boolean ring is commutative, R/P is a commutative prime ring, and hence it is a domain. A Boolean domain is $\mathbb{Z}_2$. So P is a maximal ideal. □

The following lemma is easy to verify.

Lemma 2.2. *A restricted Boolean ring which is semiprime, but not prime is Boolean.*

The next theorem characterizes commutative restricted Boolean rings.

Theorem 2.3. *Let R be a commutative ring. Then R is restricted Boolean if and only if any one of the following is satisfied:*

(1) *the ring R is a field, or*

(2) *the ring R is Boolean, or*

(3) *$J(R)$ is the only proper ideal of R and $R/J(R) \cong \mathbb{Z}_2$.*

Proof. First, we assume that $J(R) = 0$. Then $(xR)^2 \neq 0$, for any $x(\neq 0) \in R$. Thus, by assumption, $R/(xR)^2$ is Boolean. Since $xR/(xR)^2$ is a nilpotent ideal in the Boolean ring $R/(xR)^2$, we must have $xR = (xR)^2$. Therefore, $x \in x^2R$. Hence R is von Neumann regular. Now, if R is prime, then R is a von Neummann regular domain. Thus, R is a field. This proves (1). The converse of (1) holds by using the fact that each simple ring has the property that all its proper factors are Boolean. Now if R is not prime, then by Lemma 2.2, R is Boolean. This proves the (2).

Now suppose that $J(R) \neq 0$. Then $R/J(R)$ is Boolean. So we get that $J(R) \subseteq I$, for all non-trivial ideals I of R. Thus $l(J(R)) = r(J(R))$ is a prime ideal. Since R is not prime, there exists two non-trivial ideals I_1 and I_2 of R such that $I_1I_2 = 0$. Thus we have $J(R)^2 = 0$. Therefore, $l(J(R)) \neq 0$. By Lemma 2.1, $l(J(R))$ is maximal. Further by [1, Theorem 5 and Theorem 6], $l(J(R)) \subseteq R$. Thus $l(J(R)) = R$. Hence, $J(R)$ is the only proper ideal R and $R/J(R) \cong \mathbb{Z}_2$. This proves the (3).

The converse of (2) and (3) is straightforward. □

We now consider the restricted Boolean group rings.

Theorem 2.4. *Let G be a non-trivial group. If RG is restricted Boolean but not Boolean, then $R \cong \mathbb{Z}_2$ and G is a simple group.*

Proof. First we prove that R is a field with $R \cong \mathbb{Z}_2$. Since RG is restricted Boolean and $RG/\omega(G) \cong R$, we get that R is Boolean. Now, let $I \neq 0$ be an ideal of R, then $(R/I)G$ is Boolean, which implies that $G = \{1\}$. But this is not possible, because G is non-trivial. Thus, R does not have any non-trivial ideal I, so R is simple. And any simple Boolean ring is $\mathbb{Z}_2$. Thus, $R \cong \mathbb{Z}_2$.

Now Let H be a non-trivial normal subgroup of G. So $R(G/H)$ is Boolean, and thus G/H is trivial. So G has no non-trivial normal subgroups. Hence G is simple. □

Remark 2.5. If G is trivial, then the above Theorem need not hold. For example, if we take $R = \mathbb{Z}_4$ and $G = \{1\}$, then RG is restricted Boolean, but not Boolean and $R \ncong \mathbb{Z}_2$.

The converse of the Theorem 2.4 need not hold.

Example 2.6. Let $R = \mathbb{Z}_2$ and G be an infinite alternating group, i.e., $G = Alt_\Omega$, where Ω is an infinite set and each element of G moves only finitely many points. Clearly G is a simple locally finite group and $\Delta(G) = \{1\}$. We form the permutation module $V = \{\sum_{i\in\Omega} a_i i | a_i \in R, i \in \Omega, a_i = 0 \text{ except for finitely many } i\}$ for RG. Now V has as a R-basis the elements of Ω and G acts on V by appropriately permuting this basis. If σ and τ are two disjoint permutations in G, for example, take $\sigma = (i_1, i_2, i_3)$ and $\tau = (i_4, i_5, i_6)$, where i_1, i_2, i_3, i_4, i_5 and i_6 are distinct elements. Then it can be easily seen that $(\sigma - 1)(\tau - 1) \neq 0$ and $(\sigma - 1)(\tau - 1)$ belongs to the ideal $I = Ann_{RG}V$, but $(\sigma - 1) \notin I$. So, I is a non-trivial proper ideal of RG. We claim that RG/I is not Boolean. Because if it would had been so then $\sigma^2 - \sigma \in I$, and thus $\sigma - 1 \in I$. But as $\sigma - 1 \notin I$. Hence RG is not restricted Boolean.

We characterize, below, non-prime restricted Boolean group rings.

Theorem 2.7. *Let G be a non-trivial group. A non-prime group ring RG is restricted Boolean, but not Boolean if and only if $R \cong \mathbb{Z}_2$ and $G \cong C_2$.*

Proof. Suppose RG is restricted Boolean but not Boolean, then by Theorem 2.4, $R \cong \mathbb{Z}_2$ and G is a simple group.

First we show that ωG is the only non-zero ideal of RG. Since RG/I is Boolean for all nontrivial ideals I of RG, so $g - g^2 \in I$ for all $g \in G$. Then $(1 - g) \in I$ for all $g \in G$. So $\omega G \subseteq I$ for all nontrivial ideals I of RG. But ωG is maximal ideal because $RG/\omega G \cong \mathbb{Z}_2$. Thus ωG is the only non-zero ideal of RG.

Since RG is not a prime ring, RG has two non-zero two-sided ideals I_1 and I_2 with $I_1 I_2 = 0$. From above we have $I_1 = \omega G$ as well as $I_2 = \omega G$. This proves that $(\omega G)^2 = 0$. By Connell [2, Theorem 9], G is a finite 2-group. Since G is a simple group, $G \cong C_2$.

Conversely, let $R \cong \mathbb{Z}_2$ and $G \cong C_2$. In this case we have $J(RG) = \omega G$, and it is the only proper ideal of RG. Thus RG is restricted Boolean but not Boolean. □

From the above it can be easily seen that ωG is the only non-zero ideal of RG even if RG is prime restricted Boolean but not Boolean. Thus, restricted Boolean group rings can be characterized in terms of simple augmentation ideal as follows.

Corollary 2.8. *The group ring RG is restricted Boolean if and only if $R \cong \mathbb{Z}_2$ and ωG is the only proper ideal of RG.*

The *FC-subgroup* $\Delta(G)$ of a group G is the set of all elements of G which have finitely many conjugates in G, i.e. $\Delta(G) = \{x \in G | [G : C_G(x)] < \infty\}$.

Corollary 2.9. *Let G be an FC group (abelian or finite in particular), then RG is restricted Boolean but not Boolean if and only if $R \cong \mathbb{Z}_2$ and $G \cong C_2$.*

Proof. We prove that RG can not be prime. Let us suppose on the contrary that RG is prime, then by Connell [2, Theorem 8], $\Delta(G)$ is a torsion free abelian group. By Theorem 2.4, G is a simple group and also G is an FC group. Hence, two cases arise either $G = \Delta(G) = \{1\}$ or $G = \Delta(G) \neq \{1\}$. The first case is not possible because RG is restricted Boolean, but not Boolean. Thus, G is a non-trivial torsion free abelian group. But there is no simple torsion free abelian group possible. Thus, the second case is also not possible. Hence, our assumption is wrong and RG is non-prime. Now the result follows from Theorem 2.7. □

The following example shows that the above Corollary does not hold when G is locally finite.

Example 2.10. Let $R = \mathbb{Z}_2$ and G be a universal locally finite group, then G is a simple group, $\Delta(G) = \{1\}$ and RG is prime ([6, Theorem 9.4.9]). By Passman [6] Corollary 9.4.10, ωG is the unique proper ideal of RG. Since $RG/\omega G \cong R$, so R is the only proper homomorphic image of RG. Thus RG is restricted Boolean, but not Boolean as G is non-trivial. And a universal locally finite group need not be a 2-group ([6, Theorem 9.4.8]).

Remark 2.11. A complete characterization has been obtained if RG is non-prime restricted Boolean. But if we take RG to be prime then in view of example 2.6, example 2.10 and Corollary 2.8, a characterization of prime restricted Boolean group rings amounts to an old question due to I. Kaplansky [3] that for which groups G and which fields K the augmentation ideal ωG is the only proper two sided ideal in KG.

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Department of Mathematics,
Indian Institute of Technology,
Delhi, India
E-mail: dineshudar@yahoo.com rksharmaiitd@gmail.com jbsrivas@gmail.com