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SOME FIXED POINT THEOREMS IN GENERATING SPACES OF QUASI-METRIC FAMILY

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Abstract. The aim of this paper is to introduce the concepts of compatible mappings and compatible mappings of type (R) in non-Archimedean Menger probabilistic normed spaces and to study the existence problems of common fixed points for compatible mappings of type (R), also, we give an applications by using the main theorems.

1. Introduction and preliminaries

Recently, a number of fixed point theorems for single-valued and multi-valued mappings in probabilistic metric spaces have been proved by many authors ([1, 2, 3, 4, 10, 11, 17]). Since every metric space is a probabilistic metric space, we can use many results in probabilistic metric spaces to prove some fixed point theorems in metric spaces.

In this paper, first, we prove some common fixed point theorems in generating space of quasi-metric and probabilistic metric spaces. Secondly, we give some convergence theorems for sequences of self-mappings on a metric space. Finally, we extend Caristi's fixed point theorem and Ekeland's variational principle in metric spaces to probabilistic metric spaces.

For notations and properties of probabilistic metric spaces, refer to [2, 3, 13, 14]. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^+$ the set of non-negative real numbers. A mapping $F\colon \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is a nondecreasing and let continuous function with $\inf F = 0$ and $\sup F = 1$. We will denote $\mathbf{D}$ by the set of all distribution functions.

Definition 1.1. A probabilistic metric space (briefly, a PM-space) is a pair (X, F), where X is a nonempty set and F is a mapping from $X \times X$ to $\mathbf{D}$. For $(x, y) \in X \times X$, the distribution function $F(x, y)$ is denoted by $F_{x,y}$. The functions $F_{x,y}$ are assumed to satisfy the following conditions:

(PM-1) $F_{x,y}(t) = 1$ for every $t > 0$ if and only if $x = y$,

(PM-2) $F_{x,y}(0) = 0$ for all x, $y \in X$,

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(PM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all for all $x, y \in X$ and $t > 0$,

(PM-4) If $F_{x,y}(u) = 1$ and $F_{y,z}(v) = 1$, then $F_{x,z}(u + v) = 1$ for every $x, y, z \in X$ and $u, v > 0$.

Definition 1.2. A t-norm is a function $\Delta\colon [0, 1] \to [0, 1]$ which is associative, commutative, nondecreasing in each coordinate and $\Delta(a, 1) = a$ for every $a \in [0, 1]$.

Definition 1.3. A Menger PM-space is a triple (X, F, Δ), where (X, F) is a PM-space and Δ is a t-norm with the following condition:

(PM-5) $F_{x,y}(u + v) \geq \Delta(F_{x,z}(u), F_{z,y}(v))$ for all $x, y, z \in X$ and $u, v > 0$.

Definition 1.4. A non-Archimedean Menger PM-space (an N.A. Menger PM-space) is a triple (X, F, Δ), where Δ is a t-norm and the space (X, F) satisfies the conditions (PM-1)$\sim$ (PM-3) and (PM-6):

(PM-6) $F_{x,y}\big(\max\{t_1, t_2\}\big) \geq \Delta\big(F_{x,z}(t_1), F_{z,y}(t_2)\big)$ for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}^+$.

The concept of neighborhoods in PM-spaces was introduced by Schweizer and Sklar [13]. If $x \in X$, $\epsilon > 0$ and $\lambda \in (0, 1)$, then the (ϵ, λ)-neighborhood of x, denoted by $U_x(\epsilon, \lambda)$, is defined by

$$U_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}\,.$$

If (X, F, Δ) is a Menger PM-space with the continuous t-norm Δ, then the family $\{U_x(\epsilon, \lambda) : x \in X, \epsilon, \lambda \in (0, 1)\}$ of neighborhoods induces a Hausdorff topology on X, which is denoted by the (ϵ, λ)-topology τ.

Definition 1.5. A PM-space (X, F) is said to be of type $(C)_g$ if there exists an element $g \in \Gamma$ such that

$$g\big(F_{x,y}(t)\big) \leq g\big(F_{x,z}(t)\big) + g\big(F_{z,y}(t)\big) \quad \text{for all} \quad x, y, z \in X \quad \text{and} \quad t \geq 0\,,$$

where $\Gamma = \{g|g : [0, 1] \to [0, \infty]$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < \infty\}$.

Definition 1.6. An N.A. Menger PM-space (X, F, Δ) is said to be of type $(D)_g$ if there exists an element $g \in \Gamma$ such that

$$g\big(\Delta(s, t)\big) \leq g(s) + g(t) \quad \text{for all} \quad s, t \in [0, 1]\,.$$

Remark 1.7. Let (X, F, Δ) be an N.A. Menger PM-space. Then we have

(1) If (X, F, Δ) is of type $(D)_g$, then it is of type $(C)_g$.

(2) If $\Delta \geq \Delta_m$, where $\Delta_m(s, t) = \max\{s + t - 1, 0\}$ for $s, t \in [0, 1]$, then (X, F, Δ) is of type $(D)_g$ for $g \in \Gamma$ defined by $g(t) = 1 - t$.

Remark 1.8. (i) If a PM-space (X, F) is of type $(C)_g$, then it is metrizable, if the metric d on X is defined by

$$d(x, y) = \int_0^1 g\big(F_{x,y}(t)\big)\, dt \quad \text{for all} \quad x, y \in X\,. \tag{1.1}$$

(ii) If an N.A. Menger PM-space (X, F, Δ) is of type $(D)_g$ then it is metrizable, where the metric d on X is defined by (1.1). On the other hand, the (ϵ, λ)-topology τ coincides with the topology induced by the metric d defined by (1.1).
(iii) If (X, F, Δ) is an N.A. Menger PM-space with the t-norm such that $\Delta \geq \Delta_m$, where $\Delta_m(s,t) = \max\{s+t-1, 0\}$ for $s, t \in [0,1]$, then (ii) is also true.

2. Fixed point theorems in metric spaces

In this section, we give several fixed point theorems for compatible mappings of type (R) in a metric space (X, d). The following definitions and properties of compatible mappings and compatible mappings of type (R) are given in [9, 10, 12].

Definition 2.1. Let S and T be mappings from a complete metric space (X, d) into itself. The mappings S and T are said to be compatible if $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Definition 2.2. Let S and T be mappings from a complete metric space (X, d) into itself. The mappings S and T are said to be compatible of type (R) if $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$ and $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Proposition 2.3. *Let S and T be mappings from a complete metric space (X, d) into itself. If a pair (S, T) is compatible of type (R) on X and $Tz = Sz$ for $z \in X$, then $STz = TSz = S^2z = T^2z$.*

Proposition 2.4. *Let S and T be mappings from a complete metric space (X, d) into itself. If a pair (S, T) is compatible of type (R) on X and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$, then we have*

(i) *if S is continuous, then $\lim_{n\to\infty} d(TSx_n, Sz) = 0$,*

(ii) *if T is continuous, then $\lim_{n\to\infty} d(STx_n, Tz) = 0$ and*

(iii) *if T and S are continuous, then $STz = TSz$ and $Tz = Sz$.*

Let Φ be the family of all mappings $\phi\colon (\mathbb{R}^+)^5 \to \mathbb{R}^+$ such that ϕ is upper semi-continuous, nondecreasing in each coordinate variable, and for any $t > 0$,

$$\phi(t, t, 0, \alpha t, t) \leq \beta t \quad \text{and} \quad \phi(t, t, 0, 0, \alpha t) \leq \beta t\,,$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$, and

$$\gamma(t) = \phi(t, t, a_1 t, a_2 t, a_3 t) < t\,,$$

where $\gamma\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping and $a_1 + a_2 + a_3 = 4$.

Lemma 2.5 ([16]). *For any $t > 0$, $\gamma(t) < 1$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the n-iteration composition of γ.*

Let A, B, S, T be mappings from a metric space (X, d) into itself such that

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X)\,, \tag{2.1}$$

there exists $\phi \in \Phi$ such that

$$d(Ax, By) \leq \phi\big(d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx), d(Sx, Ty)\big) \tag{2.2}$$

for all $x, y \in X$. Then, by (2.1), since $A(X) \subset T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \tag{2.3}$$

for $n = 0, 1, 2, \ldots$.

Lemma 2.6. *The sequence $\{y_n\}$ defined by* (2.3) *is a Cauchy sequence in X.*

Theorem 2.7. *Let A, B, S, and T be mappings from a complete space (X, d) into itself satisfying the following conditions:*

(1) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*

(2) *$d(Ax, By) \leq \phi(d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx), d(Sx, Ty))$ for all $x, y \in X$,*

(3) *one of A, B, S and T is continuous,*

(4) *the pairs (A, S) and (B, T) are compatible of type (R).*

Then A, B, S and T have a unique common fixed point in X.

Proof. By Lemma 2.6, the sequence $\{y_n\}$ defined by (2.3) is a Cauchy sequence in X and so, since (X, d) is complete, it converges to a point $z \in X$. On the other hand, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to the point z.

Now, suppose T is continuous, since B and T are compatible of type (R), by Proposition 2.4, $BTx_{2n+1}, TTx_{2n+1} \to Tz$ as $n \to \infty$. Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (2.2), we have

$$\begin{aligned} d(Ax_{2n}, BTx_{2n+1}) \leq \phi\big(&d(Ax_{2n}, Sx_{2n}), d(BTx_{2n+1}, TTx_{2n+1})\,,\\ &d(Ax_{2n}, TTx_{2n+1}), d(BTx_{2n+1}, Sx_{2n}), d(Sx_{2n}, TTx_{2n+1})\big)\,. \end{aligned} \tag{2.4}$$

Taking $n \to \infty$ in (2.4), since $\phi \in \Phi$, we have

$$d(z, Tz) \leq \phi\big(0, 0, d(z, Tz), d(z, Tz), d(z, Tz)\big) < \gamma\big(d(z, Tz)\big) < d(z, Tz)\,,$$

which is a contradiction. Thus, we have $Tz = z$. Similarly, if we replace x by x_{2n} and y by z in (2.2), respectively, and take $n \to \infty$, then we have $Bz = z$. Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $Bz = Su = z$. By using (2.2) again, we have

$$\begin{aligned} d(Au, z) = d(Au, Bz) &\leq \phi\big(d(Au, Su), d(Bz, Tz), d(Au, Tz), d(Bz, Su), d(Su, Tz)\big)\\ &= \phi\big(d(Au, z), 0, d(Au, z), 0, 0)\big) < \gamma\big(d(Au, z)\big) < d(Au, z)\,, \end{aligned}$$

which is a contradiction. Thus, we have $Au = z$. Since A and S are compatible of type (R) and $Au = Su = z$, by Proposition 2.4, $d(ASu, SAu) = 0$ and hence $Az = ASu = SAu = Sz$. By (2.2) again, we have

$$d(Az, z) = d(Az = Bz) \leq \phi\big(d(Az, Sz), d(Bz, Tz), d(Az, Tz), d(Bz, Sz), d(Sz, Tz)\big)$$
$$= \phi\big(d(Az, z), 0, d(Az, z), 0, 0\big) < \gamma\big(d(Az, z)\big) < d(Az, z)\,,$$

which implies that $Az = z$. Therefore, $Az = Bz = Tz = z$, i.e., z is a common fixed point of the given mappings A, B, S and T.

Finally, in order to prove the uniqueness of z, suppose w be another common fixed point of A, B, S and T. Then we have

$$d(z, w) = d(Az, Bw) \leq \phi\big(d(Az, Sz), d(Bw, Tw), d(Az, Tw), d(Bw, Sz), d(Sz, Tw)\big)$$
$$= \phi\big(0, 0, d(z, w), d(z, w), d(z, w)\big) < \gamma\big(d(z, w)\big) < d(z, w)$$

which implies that $z = w$. Similarly, we can prove the theorem when A or B or S is continuous. Therefore, the proof is achieved. □

3. Fixed point theorems for single-valued mapping in generating spaces of quasi-metric family

In this section, we give some fixed point theorems for single-valued mapping in generating spaces of quasi-metric family. First, we give the definition of a generating spaces of quasi-metric family, and some properties and examples of generating spaces of quasi-metric family, (see [6, 7]).

Definition 3.1. Let X be a non-empty set, $\{d_\alpha\colon \alpha \in (0,1]\}$ be a family of mappings and d_α be mapping of $X \times X$ into $\mathbb{R}^+$. $(X, d_\alpha\colon \alpha \in (0,1])$ is called a generating space of quasi-metric family if it satisfies the following conditions:

(QM-1) $d_\alpha(x, y) = 0$ for all $\alpha \in (0,1]$ if and only if $x = y$,

(QM-2) $d_\alpha(x, y) = d_\alpha(y, x)$ for all $x, y \in X$ and $\alpha \in (0,1]$,

(QM-3) For any $\alpha \in (0,1]$, there exists an $\mu \in (0, \alpha]$ such that
$$d_\alpha(x, y) \leq d_\mu(x, z) + d_\mu(z, y), \quad x, y, z \in X\,,$$

(QM-4) For any x, $y \in X$, $d_\alpha(x, y)$ is non-increasing and left continuous in α.

Later $\{d_\alpha : \alpha \in (0,1]\}$ is called a family of quasi-metrics.

Example 3.2. (1) Let (X, d) be a metric space. Letting $d_\alpha(x, y) = d(x, y)$ for all $\alpha \in (0,1]$ and x, $y \in X$, then (X, d) is a generating space of quasi-metric family.
(2) Every fuzzy metric space and every probabilistic metric space are both examples of generating spaces of quasi-metric family.

Remark 3.3. In [6], J.X. Fan proved that $\{d_\alpha : \alpha \in (0,1]\}$ is a generating space of quasi-metric family, then there exists a topology τ_{d_α} on X such that (X, τ_{d_α}) is a Hausdorff space and $\mathcal{U}(x) = \{U_x(\epsilon, \alpha) : \epsilon > 0, \alpha \in (0,1]\}$, $x \in X$, is a basis of neighborhoods of the point x for the topology τ_{d_α}, where
$$U_x(\epsilon, \alpha) = \{y \in X : d_\alpha(x, y) < \epsilon\}\,.$$

Throughout this paper, we assume that $\kappa\colon (0,1] \to [0,\infty)$ is a non-increasing function satisfying the following condition:

$$M = \sup_{\alpha\in(0,1]} \kappa(\alpha) < \infty\,. \tag{3.1}$$

Theorem 3.4. *Let $(X, d_\alpha : \alpha \in (0,1])$ be complete generating space of quasi-metric family. Let f be a self-mapping on X and $\phi\colon X \to [0,\infty)$ be a lower semi-continuous function satisfying the following condition:*

$$d_\alpha\big(p, f(p)\big) \le \kappa(\alpha)\big\{\phi(p) - \phi(f(p))\big\}\,, \quad p \in X\,. \tag{3.2}$$

Then f has a fixed point in X.

Proof. If there exists some point $p_0 \in X$ such that $\phi(p_0) - \phi(f(p_0)) < 0$. By condition (3.2), it is obvious that $p_0 = f(p_0)$, i.e., p_0 is a fixed point of f. But this is a contradiction. Hence we have $\phi(p) - \phi(f(p)) \ge 0$. Now we define a relation "$\le$" on X as follows:

(3.3)

$$p \le q \Longleftrightarrow d_\alpha(p,q) \le \kappa(\alpha)\{\phi(p) - \phi(q)\}\,, \quad \text{for any } \; p, q \in X \;\text{ and }\; \alpha \in (0,1].$$

for any $p, q \in X$ and $\alpha \in (0,1]$. From (3.3), it follows that for any p, $q \in X$, if $p \le q$, then we have

$$\phi(q) \le \phi(p)\,. \tag{3.4}$$

First we prove that "$\le$" is a partial ordering on X.
The reflexivity and antisymmetry of "$\le$" are obvious. Now we prove the transitivity as follows: If p, q, $r \in X$ and $p \le q$, $q \le r$, by (3.4) we have

$$\phi(r) \le \phi(q) \le \phi(p)\,. \tag{3.5}$$

Since $\{d_\alpha : \alpha \in (0,1]\}$ is a family of quasi-metric, then they are non-increasing in α. Hence for any given $\alpha \in (0,1]$, there exists a number $\mu \in (0,1]$ such that

$$d_\alpha(p,r) \le d_\mu(p,q) + d_\mu(q,r)\,.$$

By (3.3), we have

$$d_\alpha(p,r) \le \kappa(\mu)\{\phi(p) - \phi(q) + \phi(q) - \phi(r)\} = \kappa(\mu)\{\phi(p) - \phi(r)\}\,. \tag{3.6}$$

Noting that the function κ is non-decreasing, so we have

$$d_\alpha(p,r) \le \kappa(\alpha)\big\{\phi(p) - \phi(r)\big\}\,. \tag{3.7}$$

By the arbitrariness of $\alpha \in (0,1]$, we know that $p \le r$ and so "$\le$" is a partial ordering on X.

Next we prove that there exists a maximal element in X. To this end, let $\{p_\mu\}_{\mu\in J}$ be any totally ordered subset of $(X, \le)$, where J is an index set. We define

$$p_\mu \le p_\nu \Longleftrightarrow \mu \le \nu\,.$$

Hence $(J, \le)$ is a directed set and $\{\phi(p_\mu)\}_{\mu\in J} \subset [0,\infty)$ is a monotone decreasing net. Let $\phi(p_\mu) \to \gamma \ge 0$. Hence for any given $\epsilon > 0$, $\lambda > 0$, $\epsilon > \lambda$, there exists an $\mu_0 \in J$ such that $\mu_0 \ge \mu$ we have

$$\gamma \le \phi(p) < \gamma + \lambda\,.$$

Hence for any $\alpha \in (0,1]$ and μ, ν with $\mu_0 \le \mu \le \nu$, we have $\phi(p_\mu) - \phi(p_\nu) \le \lambda$, we have

$$d_\alpha(p_\mu, p_\nu) \le \kappa(\alpha)\big\{\phi(p_\mu) - \phi(p_\nu)\big\} \le M\lambda < \epsilon\,.$$

This implies that $\{p_\mu\}$ is Cauchy net in X. By completeness of X, we assume that $p_\mu \to p \in X$. In view of the lower semi-continuity of ϕ, it follows that

$$\phi(p) \le \varliminf_{\mu} \phi(p_\mu) = \lim_{\mu} \phi(p_\mu) = \gamma \le \phi(p_\mu\,, \quad \mu \in J\,. \tag{3.8}$$

Now we prove that p is an upper bound of $\{p_\mu\}_{\mu\in J}$. In fact, for any μ, $\nu \in J$ with $\mu \le \nu$, from (3.8) we have

$$d_\alpha(p_\mu, p_\nu) \le \kappa(\alpha)\{\phi(p_\mu) - \gamma\}$$

for all $\alpha \in (0,1]$. Taking the limit for ν, we obtain

$$d_\alpha(p_\mu, p) \le \kappa(\alpha)\big\{\phi(p_\mu) - \phi(p)\big\}\,,$$

which means that $p_\mu \le p$ for all $\mu \in J$, i.e., p is an upper bound of $\{p_\mu\}_{\mu\in J}$. Applying Zorn's Lemma, $(X, \le)$ has a maximal element, say $p_* \in X$.

Finally, we prove that p_* is a fixed point of f. In fact, it follows that

$$\phi(p_*) - \phi\big(f(p_*)\big) \ge 0\,.$$

And from (3.2) we have

$$d_\alpha(p_*, f(p_*)) \le \kappa(\alpha)\big\{\phi(p_*) - \phi(f(p_*))\big\}$$

which shows that $p_* \le f(p_*)$. Since p_* is a maximal element in $(X, \le)$, we have $p_* = f(p_*)$. Therefore, the proof is achieved. □

Theorem 3.5. *Let $(X, d_\alpha : \alpha \in (0,1])$ be complete generating space of quasi-metric family. Let $\phi\colon X \to \mathbb{R}$ be a lower semi-continuous functional bounded from below and $\phi(x) = +\infty$. Suppose that for any given $\epsilon > 0$, there exists an $x_0 \in X$ such that*

$$\phi(x_0) \le \inf\{\phi(x) : x \in X\} + \epsilon\,.$$

Then there exists an $\tilde{x} \in X$ such that

1. $\phi(\tilde{x}) \le \phi(x_0)$,
2. $d_\alpha(\tilde{x}, x_0) \le 1$ *for all* $\alpha \in (0,1]$,
3. $\phi(x) > \phi(\tilde{x}) - \epsilon \cdot d_\alpha(x, \tilde{x})$ *for all* $x \in X$, $\alpha \in (0,1]$, $x \ne \tilde{x}$.

Proof. Let

$$S = \Big\{x \in X : d_\alpha(x, x_0) \le \frac{1}{\epsilon}\big(\phi(x_0) - \phi(x)\big) \text{ for all } \alpha \in (0,1]\Big\}\,. \tag{3.9}$$

It is obvious that $S \ne \emptyset$ (since $x_0 \in S$). Now we prove that S is a closed set. In fact, let $\{x_n\}$ be a sequence in X and $x_n \to \tilde{x}$. By the continuity of d_α, then we have

$$\begin{aligned} d_\alpha(x_0, \tilde{x}) &= \lim_{n\to\infty} d_\alpha(x_0, x_n) \\ &\le \lim_{n\to\infty} \sup \frac{1}{\epsilon}\big(\phi(x_0) - \phi(x_n)\big) \\ &\le \frac{1}{\epsilon}\big(\phi(x_0) - \phi(\tilde{x})\big)\,. \end{aligned} \tag{3.10}$$

Imitating the proof of Theorem 3.4, we can prove that the inequality (3.10) holds for all $\alpha \in (0,1]$. This implies that $\tilde{x} \in S$. Consequently, S is closed set of X. Now we introduce a partial order "$\leq$" in S as follows:

$$x \leq y \Longleftrightarrow d_\alpha(x,y) \leq \frac{1}{\epsilon}\big(\phi(x) - \phi(y)\big)\,, \quad \alpha \in (0,1]\,. \tag{3.11}$$

Noting that if x, $y \in S$ with $x \leq y$, then from (3.11) we have

$$\phi(y) \leq \phi(x)\,. \tag{3.12}$$

Let $\{x_\mu\}_{\mu\in J}$ be a totally ordered set of S and J be an indexed set. We stipulate that $\mu \leq \nu$ if and only if $x_\mu \leq x_\nu$. Then $(J, \leq)$ is a directed set. It follows from (3.12) that $\{\phi_\mu\}_{\mu\in J}$ is a monotone decreasing net bounded from below. Let $\phi_\mu \to \gamma > -\infty$. Finally, from (3.11), we have

$$d_\alpha(x_\mu, x_\nu) \leq \frac{1}{\epsilon}\big(\phi(x_\mu) - \phi(x_\nu)\big)\,, \qquad \nu \geq \mu, \alpha \in (0,1]\,. \tag{3.13}$$

It is easy to see that $\{x_\mu\}_{\mu\in J}$ is a Cauchy net of S. By the closedness of S we can assume that $x_\mu \to \tilde{x} \in S$. Hence for any given $\mu \in J$, since d_α, $\alpha \in (0,1]$ is continuous, it follows that

$$\begin{aligned} d_\alpha(x_\mu, \tilde{x}) = \lim_\nu d_\alpha(x_\mu, x_\nu) &\leq \underline{\lim}_\nu \frac{1}{\epsilon}\big(\phi(x_\mu) - \phi(x_\nu)\big) \\ &\leq \frac{1}{\epsilon}\big(\phi(x_\mu) - \phi(\tilde{x})\big)\,. \end{aligned}$$

Imitating the proof of Theorem 3.4, we can prove that, for any $\alpha \in (0,1]$, the above inequality holds. Hence we have

$$d_\alpha(x_\mu, \tilde{x}) \leq \frac{1}{\epsilon}\big(\phi(x_\mu) - \phi(\tilde{x})\big)\,, \qquad \mu \in J, \alpha \in (0,1]\,,$$

which means that $x_\mu \leq \tilde{x}$ for all $\mu \in J$, i.e., $\tilde{x}$ is an upper bound of $\{x_\mu\}_{\mu\in J}$. By Zorn's Lemma, $(S, \leq)$ has a maximal element, say $\tilde{x}$. Hence we have

$$d_\alpha(x_0, \tilde{x}) \leq \frac{1}{\epsilon}\big(\phi(x_0) - \phi(\tilde{x})\big)\,, \quad \alpha \in (0,1]\,. \tag{3.14}$$

This implies that $\phi(\tilde{x}) \leq \phi(x_0)$. The conclusion (1) is proved.

Next, by the condition (3.9), $\phi(x_0) - \phi(\tilde{x}) \leq \epsilon$. Using (3.14), we have

$$d_\alpha(x_0, \tilde{x}) \leq 1\,, \quad \alpha \in (0,1]\,.$$

Thus the conclusion (2) is proved.

Now we verify that the conclusion (3) is true too. Suppose that this is not the case. Then there exists some $x \in X, x \neq \tilde{x}$, such that

$$d_\alpha(\tilde{x}, x) \leq \frac{1}{\epsilon}\big(\phi(\tilde{x}) - \phi(x)\big)\,, \quad \alpha \in (0,1]\,. \tag{3.15}$$

Next we prove that the following inequality (3.16) can be deduced from (3.15):

$$d_\alpha(x_0, x) \leq \frac{1}{\epsilon}\big(\phi(x_0) - \phi(x)\big)\,, \quad \alpha \in (0,1]\,. \tag{3.16}$$

In fact, when $\frac{1}{\epsilon}(\phi(x_0) - \phi(x)) \leq 1$, it obvious that (3.16) is true. When $1 < \frac{1}{\epsilon}(\phi(x_0) - \phi(x)) = \frac{1}{\epsilon}(\phi(x_0) - \phi(\tilde{x}) + \phi(\tilde{x}) - \phi(x))$, by conclusion (1), we have $\phi(x_0) \geq \phi(\tilde{x})$. It follows from (3.15) that $\phi(\tilde{x}) \geq \phi(x)$. Hence for any $\alpha \in (0,1]$, there exists a number $\mu \in (0,1]$ such that

$$d_\mu(x_0, \tilde{x}) < \frac{1}{\epsilon}\big(\phi(x_0) - \phi(\tilde{x})\big)\,, \quad d_\mu(\tilde{x}, x) < \frac{1}{\epsilon}\big(\phi(\tilde{x}) - \phi(x)\big)\,,$$

From (3.14) and (3.15), we have

$$\begin{aligned} d_\alpha(x_0, x) \leq d_\mu(x_0, \tilde{x}) + d_\mu(\tilde{x}, x) &\leq \frac{1}{\epsilon}\big(\phi(x_0) - \phi(\tilde{x})\big) + \frac{1}{\epsilon}\big(\phi(\tilde{x}) - \phi(x)\big) \\ &= \frac{1}{\epsilon}\big(\phi(x_0) - \phi(x)\big)\,. \end{aligned}$$

This implies that (3.16) is true, and so $x \in S$. By virtue of (3.15) and (3.11), it follows that $\tilde{x} \leq x$, which is a contradiction (since x is a maximal element of S). Therefore the conclusion (3) is true. This completes the proof. □

4. Caristi type fixed point theorems the Ekeland's variational principle

In this section, we extend the Caristi's fixed point theorem and the Ekeland's variational principle in PM-spaces. Also, we prove some common fixed point theorems in PM-spaces by using the results in Section 2 and Section 3.

Theorem 4.1. *Let $(X, \mathcal{F})$ be a PM-space of $(C)_g$-type and $(X, d_\alpha : \alpha \in (0,1])$ be complete generating space of quasi-metric family, where d_α is defined by*

$$d_\alpha(x,y) = \int_0^1 g\big(F_{x,y}(t)\big)\,dt\,, \qquad x, y \in X \;\text{ and }\; \alpha \in (0,1]\,. \tag{4.1}$$

If $\phi\colon X \to \mathbb{R}$ is a lower semi-continuous and bounded below function and a mapping $T\colon X \to X$ satisfies the following condition:

$$g\big(F_{x,Tx}(t)\big) \leq \phi(x) - \phi(Tx)\,, \quad x \in X\,,\; t \geq 0\,, \tag{4.2}$$

then T has a fixed point in X.

Proof. From (4.2) and for all $\alpha \in (0,1]$, we have

$$\begin{aligned} d_\alpha(x, Tx) = \int_0^1 g\big(F_{x,Tx}(t)\big)\,dt &\leq \int_0^1 \big(\phi(x) - \phi(Tx)\big)\,dt \\ &= \phi(x) - \phi(Tx) \end{aligned}$$

and thus, by Theorem 3.4, T has a fixed point in X. □

Corollary 4.2. *Let $(X, \mathcal{F})$ be a PM-space of $(C)_g$-type and $(X, d_\alpha : \alpha \in (0,1])$ be complete generating space of quasi-metric family, where d_α is defined by (4.1), and a function $\phi(x,t)\colon E \times \mathbb{R}^+ \to \mathbb{R}^+$ be integrable in t. If a function $\psi(x) = \displaystyle\int_0^1 \phi(x,t)\,dt$*

is a lower semi-continuous and bounded below and a mapping $T\colon X \to X$ satisfies the condition:

$$g\big(F_{x,y}(t)\big) \le \phi(x,t) - \phi(Tx,t)\,, \quad x \in X\,,\ t \ge 0\,, \tag{4.3}$$

then T has a fixed point in X.

Proof. From (4.3) and for all $\alpha \in (0,1]$, we have

$$\begin{aligned} d_\alpha(x,Tx) &= \int_0^1 g\big(F_{x,Tx}(t)\big)\,dt \le \int_0^1 \big(\phi(x,t) - \phi(Tx,t)\big)\,dt \\ &= \int_0^1 \phi(x,t)\,dt - \int_0^1 \phi(Tx,t)\,dt = \psi(x) - \psi(Tx)\,. \end{aligned}$$

Therefore, by Theorem 4.1, T has a fixed point in X. $\square$

Theorem 4.3. *Let $(X,\mathcal{F})$ be a PM-space of $(C)_g$-type and $(X, d_\alpha : \alpha \in (0,1])$ be complete generating space of quasi-metric family, where d_α is defined by (4.1). If a function $\phi\colon X \to \mathbb{R}$ is proper, lower semi-continuous and bounded below, and T is a multi-valued mapping from X into 2^X such that for each $x \in X$, there exists a point $fx \in Tx$ such that $f\colon X \to X$ is a function satisfying the following condition:*

$$g\big(F_{x,fx}(t)\big) \le \phi(x) - \phi(fx)\,, \quad x \in X, t \ge 0\,, \tag{4.4}$$

then f and T have a common fixed point in X.

Proof. Since ϕ is proper, there exists a point $u \in X$ such that $\phi(u) < \infty$ and so $A = \{x \in X : g(F_{u,x}(t)) \le \phi(u) - \phi(x)\}$ is non-empty closed set in X. By the condition (4.4), for any $x \in A$, we have

$$\phi(fx) + g\big(F_{x,fx}(t)\big) \le \phi(x) \le \phi(u) - g\big(F_{u,x}(t)\big)\,.$$

Thus we have

$$\begin{aligned} g\big(F_{u,fx}(t)\big) &\le g\big(F_{u,x}(t)\big) + g\big(F_{x,fx}(t)\big) \\ &\le \phi(u) - \phi(x) + \phi(x) - \phi(fx) = \phi(u) - \phi(fx)\,, \end{aligned}$$

which implies that f is a mapping from A into A. Therefore, by Theorem 4.1, the function $f\colon A \to A$ has a fixed point in A, say x_0, and so $x_0 = fx_0 \in Tx_0$, that is, the point x_0 is a common fixed point of f and T. So, the proof is achieved. $\square$

Theorem 4.4. *Let $(X,\mathcal{F})$ be a PM-space of $(C)_g$-type and $(X, d_\alpha : \alpha \in (0,1])$ be complete generating space of quasi-metric family, where d_α is defined by (4.1). If a function $\phi\colon X \to \mathbb{R}$ is proper, lower semi-continuous and bounded below and, for each $\epsilon > 0$, there exists a point $u \in X$ such that $\phi(u) \le \inf\{\phi(x) : x \in X\} + \epsilon$, then there exists a point $v \in X$ such that*

(1) $\phi(v) \le \phi(u)$,

(2) $g(F_{u,v}(t)) \le 1$,

(3) $\phi(v) - \phi(u) \le \epsilon g(F_{u,x}(t))$ *for all $x \in X$ and $t \ge 0$.*

Proof. Let $\epsilon > 0$ and let a point $u \in X$ such that $\phi(u) \leq \inf\{\phi(x) : x \in X\} + \epsilon$. Letting $A = \{x \in X : \phi(x) \leq \phi(u) - \epsilon g(F_{u,x}(t))\}$, then A is non-empty closed subset in X and so, since $(X, d_\alpha : \alpha \in (0,1])$ is complete, A is complete. For each $x \in A$, let $Sx = \{y \in X : \phi(y) \leq \phi(x) - \epsilon g(F_{x,y}(t)), x \neq y\}$ and define

$$Tx = \begin{cases} x\,, & \text{if} \quad Sx \quad \text{is empty;} \\ Sx\,, & \text{if} \quad Sx \quad \text{is nonempty.} \end{cases}$$

Then T is a multi-valued mapping from A into 2^A. In fact, since $Tx = x \in A$ if $Sx = \emptyset$ and $Tx = Sx$ if $Sx \neq \emptyset$, we have, for each $y \in Tx = Sx$,

$$\phi(y) \leq \phi(x) - \epsilon g\big(F_{x,y}(t)\big)$$

and

$$\begin{aligned} \epsilon g\big(F_{u,y}(t)\big) &\leq \epsilon g\big(F_{u,x}(t)\big) + \epsilon g\big(F_{x,y}(t)\big) \\ &\leq \phi(u) - \phi(x) + \phi(x) - \phi(y) = \phi(u) - \phi(y)\,, \end{aligned}$$

which implies $y \in A$ and so we have $Tx = Sx \in A$. Assume that T has no fixed point in A. Then for each $x \in A$ and $y = Tx = Sx$, we have

$$\epsilon g\big(F_{x,y}(t)\big) \leq \phi(x) - \phi(y)\,,$$

and

$$g\big(F_{x,y}(t)\big) \leq \frac{1}{\epsilon}\big(\phi(x) - \phi(y)\big)\,.$$

Thus, by Theorem 4.3, T has a fixed point v in A, which is a contradiction. Therefore $Sv = \emptyset$, i.e., for each $x \in X$, $x \neq v$, $\phi(x) > \phi(v) - \epsilon g(F_{v,x}(t))$. Since $v \in A$, $\phi(v) \leq \phi(u) - \epsilon g\big(F_{u,v}(t)\big)$ and so $\phi(v) \leq \phi(u)$. On the other hand, we have

$$\begin{aligned} \epsilon g(F_{u,v}(t)) &\leq \phi(u) - \phi(v) \\ &\leq \phi(u) - \inf\{\phi(x) : x \in X\} \leq \epsilon \end{aligned}$$

and so, $g(F_{u,v}(t)) \leq 1$. This completes the proof. □

Next, by using Theorem 4.4, we can prove some common fixed point theorems in PM-spaces. Now, we introduce some definitions and properties of compatible mappings of type (R) in PM-spaces.

Definition 4.5. Let $(X, \mathcal{F}, \Delta)$ be a N.A. Menger PM-space of type $(D)_g$ and A, S be mappings from X into itself. A and S are said to be compatible if

$$\lim_{n\to\infty} g\big(F_{ASx_n, SAx_n}(t)\big) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 4.6. Let $(X, \mathcal{F}, \Delta)$ be a N.A. Menger PM-space of type $(D)_g$ and A, S be mappings from X into itself. A and S are said to be compatible if

$$\lim_{n\to\infty} g\big(F_{ASx_n, SAx_n}(t)\big) = 0 \quad \lim_{n\to\infty} g\big(F_{AAx_n, SSx_n}(t)\big) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Theorem 4.7. *Let $(X, \mathcal{F}, \Delta)$ be a τ-complete N.A. Menger PM-space with t-norm Δ such that $\Delta(s,t) \geq \Delta_m(s,t) = \max\{s+t-1, 0\}$, $s, t \in [0,1]$. Let A, B, S and T be mappings from X into itself such that*

(1) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*

(2) *one of A, B, S and T is τ-continuous,*

(3) *(A, S) and (B, T) are compatible of type (R),*

(4) *there exists $\phi\colon (\mathbb{R}^+)^5 \to \mathbb{R}^+$ such that ϕ is upper continuous, non-decreasing in each coordinate variable, and for any $t > 0$*

$$\phi(t, t, 0, \alpha t, 0) \leq \beta t \quad \textit{and} \quad \phi(t, t, 0, 0, \alpha t) \leq \beta t$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$, $\gamma(t) = \phi(t, t, a_1 t, a_2 t, a_3 t) < t$, where $\gamma\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping and $a_1 + a_2 + a_3 = 4$ and the following holds:

$$\int_0^1 F_{Ax,By}(t)\,dt \geq 1 - \phi\Big[1 - \int_0^1 F_{Ax,Sx}(t)\,dt, 1 - \int_0^1 F_{By,Ty}(t)\,dt,$$
$$1 - \int_0^1 F_{Ax,Ty}(t), 1 - \int_0^1 F_{Sx,Ty}(t)\,dt, 1 - \int_0^1 F_{By,Sx}(t)\,dt\Big]$$

for all x, $y \in X$ and $t \geq 0$. Then A, B, S and T have a unique common fixed point in X.

Proof. Since $(X, \mathcal{F}, \Delta)$ is an N.A. Menger PM-space with the t-norm Δ such that $\Delta \geq \Delta_m(s,t) = \max\{s+t-1, 0\}$, s, $t \in [0,1]$, by Remark 1.8, it is metrizable by the metric d defined by (1.1). Thus, if we define $g(t) = 1 - t$, from the condition (4), we have

$$d(Ax, By) \leq \phi\big(d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx), d(Sx, Ty)\big)$$

for all x, $y \in X$. Therefore, by Theorem 2.7, A, B, S and T have a unique common fixed point in X. This completes the proof. $\square$

5. Caristi's coincidence theorem for set-valued functions in quasi-metric space

By using the partial ordering method, a more general type of Caristi's Coincidence Theorem for Set-valued functions in Quasi-metric space is given in this section.

Lemma 5.1. *Let $(X, d_\alpha : \alpha \in (0,1])$ be complete generating space of quasi-metric family, $\phi\colon X \to X$ be a lower semi-continuous function bounded from below and $\kappa\colon (0,1) \to (0,\infty)$ be a non-increasing function. Let*

$$m = \inf_{\alpha\in(0,1]} \kappa(\alpha) \quad \textit{and} \quad M = \sup_{\alpha\in(0,1]} \kappa(\alpha) < \infty \tag{5.1}$$

and define a relation "$\leq$" on X by

$$x \leq y \Longleftrightarrow d_\alpha(x,y) \leq \kappa(\alpha)\big(\phi(x) - \phi(y)\big)$$

for all $\alpha \in (0,1]$. Then we have the following:

(i) *If $x \leq y$, then $\phi(y) \leq \phi(x)$.*

(ii) *"$\leq$" is a partial ordering on X.*

(iii) *there exists at least one maximal element in the partial ordering set $(X, \leq)$.*

Proof. (i) Let $x \leq y$ and assume that $\phi(y) \leq \phi(x)$. Then there exists a number $\mu \in (0,1]$ such that $d_\mu(x,y) = a > 0$ for $a \in \mathbb{R}^+$ and so we have

$$a = d_\mu(x,y) \leq \kappa(\alpha)\big(\phi(x) - \phi(y)\big) \tag{5.2}$$

for all $\alpha \in (0,1]$. Letting $\alpha \to 0^+$ in (5.2), we have $a \leq 0$, which is contradicts $a > 0$. Therefore the assertion (i) follows.

(ii) The reflexivity and the anti-symmetry of "$\leq$" are obvious. Now, we shall prove the transitivity of "$\leq$". If $x \leq y$ and $y \leq z$ for x, y, $z \in X$, by (i), we have

$$\phi(z) \leq \phi(y) \leq \phi(x)\,. \tag{5.3}$$

(1) If $d_\alpha(x,y) \leq m(\phi(x) - \phi(z))$, where m is the constant defined by (5.1), since we have $m(\phi(x) - \phi(z)) = 1$ and κ is non-increasing, it follows that

$$\kappa(\alpha)\big(\phi(x) - \phi(z)\big) = 1$$

for all $\alpha \in (0,1]$ and so we have

$$d_\alpha(x,z) \leq \kappa(\alpha)\big(\phi(x) - \phi(y)\big) \tag{5.4}$$

for all $\alpha \in (0,1]$.

(2) If $d_\alpha(x,z) > m(\phi(x) - \phi(z))$, then we can take $\mu \in (0,1]$ such that

$$d_\mu(x,y) > m\big(\phi(x) - \phi(y)\big) \quad \text{and} \quad d_\mu(y,z) > m\big(\phi(y) - \phi(z)\big)$$

and so for all $\alpha \in (0,1]$, we have

$$d_\alpha(x,z) \leq d_\mu(x,y) + d_\mu(y,z) \leq \kappa(\alpha)\big(\phi(x) - \phi(y) + \phi(y) - \phi(z)\big)\,. \tag{5.5}$$

Thus, letting $r \to 1^-$ in (5.5), we have

$$d_\alpha(x,z) \leq M\big(\phi(x) - \phi(z)\big) \tag{5.6}$$

for all $\alpha \in (0,1]$, which implies that $x \leq z$.

(3) Let $\{x_\mu | \mu \in J\}$ be any totally ordering subset os $(X, \leq)$, where J is an index set. We define

$$x_\mu \leq x_\nu \Longleftrightarrow \mu \leq \nu\,.$$

Then $(J, \leq)$ is a direct set and $\{\phi(x_\mu\}_{\mu \in J}$ is a monotonically decreasing net in $\mathbb{R}$. Letting $\phi(x_\mu) \to s$, by the boundedness from below of ϕ, it follows that s is a finite number. Hence for all $\lambda > 0$ and $\epsilon > M\lambda$, there exists an $\mu_0 \in J$ such that $s \leq \phi(x) < s + \lambda$ for $\mu \geq \mu_0$. Thus, for any μ, $\nu \in J$ with $\mu_0 \leq \mu \leq \nu$, we have

$$0 \leq \phi(x_\mu) - \phi(x_\nu) \leq \lambda$$

and

$$d_\alpha(x_\mu, x_\nu) \le \kappa(\alpha)\big(\phi(x_\mu) - \phi(x_\nu)\big) \le \kappa(\alpha)\lambda \le \epsilon$$

for all $\alpha \in (0, 1]$ and $\epsilon > 0$, which implies that $\{x_\mu\}_{\mu\in J}$ is a Cauchy net in X. Since $(X, d_\alpha : \alpha \in (0, 1])$ is complete, the net $\{x_\mu\}_{\mu\in J}$ in X converges to an element $x^* \in X$. From the continuity of ϕ, it follows that

$$\phi(x^*) \le \liminf_\mu \phi(x_\mu) = \lim_\mu = \phi(x_\mu) = s \le \phi(x_\mu) \tag{5.7}$$

for all $\mu \in J$.

Next, we shall prove that x^* is an upper bound of $\{x_\mu\}_{\mu\in J}$. In fact, for any $\mu \in J$, if $d_\alpha(x_\mu, x^*)$ is continuous, then we have

$$\begin{aligned} d_\alpha(x_\mu, x^*) &= \lim_\nu d_\alpha(x_\mu, x_\nu) \\ &\le \lim_\nu \kappa(\alpha)\big(\phi(x_\mu) - \phi(x_\nu)\big) && (5.8) \\ &\le \kappa(\alpha)\big(\phi(x_\mu) - \phi(x^*)\big) && (5.9) \end{aligned}$$

for all $\alpha \in (0, 1]$. Which implies that $x_\mu \le x^*$ for all $\mu \in J$, i.e., x^* is an upper bound of $\{x_\mu\}_{\mu\in J}$. Applying Zorn's Lemma, $(X, \le)$ has a maximal element. Therefore the proof is achieved. □

Theorem 5.2. *Let $(X, d_\alpha : \alpha \in (0, 1])$ be complete generating space of quasi-metric family. Let D be a non-empty subset of X, $f : D \to X$ be a surjective mapping and $\phi : X \to \mathbb{R}$ be a lower semi-continuous function bounded from below, Let $\{S_\mu\}_{\mu\in J}$ be a family of set-valued mappings $S_\mu : D \to 2^X \setminus \{\emptyset\}$. Suppose that for if each $x \in D$, $f(x) \notin \bigcap_{\mu\in J} S_\mu(x)$, then there exists an $\mu_0 \in J$ and a $y \in S_{\mu_0}(x) \setminus \{f(x)\}$ such that*

$$d_\alpha\big(f(x), y\big) \le \kappa(\alpha)\big(\phi\big(f(x)\big) - \phi(y)\big)$$

for all $x \in X$, $\alpha \in (0, 1]$. Then there exists a coincidence point $u \in X$ of f and $\{S_\mu\}_{\mu\in J}$, that is, there exists a u in X such that $f(u) \in \bigcap_{\mu\in J} S_\mu(u)$.

Proof. We define a partial ordering "$\le$" on X by

$$x \le y \Longleftrightarrow d_\alpha(x, y) \le \kappa(\alpha)\big(\phi(x) - \phi(y)\big)$$

for all $\alpha \in (0, 1]$. By Lemma 5.1, $(X, \le)$ has a maximal element $z \in X$. Since $f : D \to X$ is surjective, there exists a $u \in D$ such that $f(u) = z$. If we have $f(u) \notin \bigcap_{\mu\in J} S_\mu(u)$, by the assumption, there exists an $\mu_0 \in J$ and a $y \in S_{\mu_0}(u) \setminus \{f(u)\}$ such that

$$d_\alpha(f(u), y) \le \kappa(\alpha)\big(\phi\big(f(u)\big) - \phi(y)\big)$$

for all $\alpha \in (0, 1]$ and so we have $f(u) \le y$. But, since $f(u) = z$ is a maximal element in X, we have

$$z = f(u) = y \in S_{\mu_0}(u) \setminus \{f(u)\},$$

which is a contradiction. Therefore, we have $f(u) \notin \bigcap_{\mu \in J} S_\mu(u)$, and so the proof is achieved. □

Corollary 5.3. *Let D, X, f, ϕ be as in Theorem 5.2. Let $S\colon D \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that if for each $x \in D$, $f(x) \notin S(x)$, then there exists a $y \in S(x)$ such that*

$$d_\alpha\big(f(x), y\big) \leq \kappa(\alpha)\big(\phi\big(f(x)\big) - \phi(y)\big)$$

for all $x \in X$, $\alpha \in (0,1]$. Then S has a fixed point in X.

Corollary 5.4. *Let $(X, d_\alpha : \alpha \in (0,1])$ be complete generating space of quasi-metric family. Let $\phi\colon X \to \mathbb{R}^+$ be a lower semi-continuous function, $S\colon X \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping and $f\colon X \to X$ be a surjective mapping. Suppose that for each $x \in X$, there exists a $y \in S(x)$ such that*

$$d_\alpha\big(f(x), y\big) \leq \kappa(\alpha)\big(\phi\big(f(x)\big) - \phi(y)\big)$$

for all $x \in X$, $\alpha \in (0,1]$. Then there exists a $u \in X$ such that $f(u) \in S(u)$.

Theorem 5.5. *Let $(X, d_\alpha : \alpha \in (0,1])$ be complete generating space of quasi-metric family. Let $\phi\colon X \to \mathbb{R}$ be a lower semi-continuous function bounded from below. Suppose that for any $\epsilon > 0$, there exists an $x_0 \in X$ such that*

$$\phi(x_0) \leq \inf\{\phi(x) : x \in X\} + \epsilon\,. \tag{5.10}$$

If $\kappa\colon (0,1) \to (0,\infty)$ is a non-increasing function satisfying the condition (5.1), then there exists a $u \in X$ such that

(1) *$d_\alpha(x_0, u) \leq \kappa(\alpha)(\phi(x_0) - \phi(u))$ for all $r \in (0,1)$ and $\alpha \in (0,1]$,*
(2) *$d_\alpha(x_0, u) \leq \epsilon\kappa(\alpha)$ for all $r \in (0,1)$ and $\alpha \in (0,1]$,*
(3) *for any $x \in X$, $x \neq u$, there exist an $\alpha_0 \in (0,1]$ such that*

$$d_{\alpha_0}(u, x) > \kappa(\alpha_0)\big(\phi(u) - \phi(x)\big)\,.$$

Proof. (1) Let

$$X_0 = \{x \in X : d_\alpha(x_0, u) \leq \kappa(\alpha)(\phi(x_0) - \phi(u)),\ r \in (0,1), \alpha \in (0,1]\}\,.$$

Since $x_0 \in X$, $X_0 \neq \emptyset$. Now, we shall prove that X_0 is a closed set in X. In fact, let $\{x_n\}$ be a sequence in X and $x_n \to x^*$ as $n \to \infty$. If $d_\alpha(x_0, x^*)$ is continuous at $\alpha \in (0,1]$, we have

$$\begin{aligned} d_\alpha(x_0, x^*) &= \lim_{n\to\infty} d_\alpha(x_0, x_n) \\ &\leq \kappa(\alpha) \lim_{n\to\infty} \big(\phi(x_0) - \phi(x_n)\big) \\ &\leq \kappa(\alpha)\big(\phi(x_0) - \phi(x^*)\big) \end{aligned} \tag{5.11}$$

for all $\alpha \in (0,1]$. As in the proof of Lemma 5.1, we can prove the relation (5.11) holds for all $\alpha \in (0,1]$, which implies that $x^* \in X_0$, that is, X_0 is a closed subset in X and $(X_0, d_\alpha : \alpha \in (0,1])$ be complete generating space of quasi-metric family.

Now, we define a partial ordering "$\leq$" by

$$x \leq y \Longleftrightarrow d_\alpha(x,y) \leq \kappa(\alpha)\big(\phi(x) - \phi(y)\big) \tag{5.12}$$

for all $\alpha \in (0,1]$. Then $(X_0, \leq)$ is a partial ordering set and hence, by Lemma 5.1, $(X_0, \leq)$ has a maximal element $u \in X_0$. Thus we have

$$d_\alpha(x_0, u) \leq \kappa(\alpha)\big(\phi(x_0) - \phi(u)\big) \tag{5.13}$$

for all $\alpha \in (0,1]$, which implies that the relation (1) follows.

(2) By the condition (5.10), we have

$$0 \leq \phi(x_0) - \phi(u) \leq \epsilon\,. \tag{5.14}$$

Thus, by (1), we have

$$d_\alpha(x_0, u) \leq \epsilon\kappa(\alpha)$$

for all $\alpha \in (0,1]$, which implies that the relation (2) follows.

(3) Assume that the assertion (3) is False. Then there exists a non-increasing function $\kappa\colon (0,1] \to (0,\infty)$ such that for each $x \in X$ there exists a $y \in X$, $y \neq x$, such that

$$d_\alpha(x, y) \leq \kappa(\alpha)\big(\phi(x) - \phi(y)\big) \tag{5.15}$$

for all $\alpha \in (0,1]$. Define $f\colon X_0 \to X_0$ by $f(x) = y$. Then the function $f\colon X_0 \to X_0$ satisfies the following condition:

$$d_\alpha\big(x, f(x)\big) \leq \kappa(\alpha)\big(\phi(x) - \phi\big(f(x)\big)\big) \tag{5.16}$$

for all $\alpha \in (0,1]$. Hence, by Corollary 5.3, f has a fixed point in X_0. But f cann't have a fixed point in X_0, which is a contradiction. Therefore, the assertion (3) follows. Therefore the proof is achieved. $\square$

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