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COMPARISON RESULTS FOR TWO-DIMENSIONAL HALF-LINEAR CYCLIC DIFFERENTIAL SYSTEMS

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Abstract. Two identities of the Picone type for a class of half-linear differential systems in the plane are established and the Sturmian comparison theory for such systems is developed with the help of these new formulas.

1. Introduction

The purpose of this paper is to study the existence and location of zeros of components of solutions for the system of ordinary differential equations

$$x' - p(t)\varphi_{1/\alpha}(y) = 0\,, \quad y' + q(t)\varphi_\alpha(x) = 0\,, \tag{1.1}$$

where α is a positive constant, p and q are continuous functions on an interval J and $\varphi_\gamma(u)$ denotes the odd function in $u \in \mathbb{R}$ defined for a fixed $\gamma > 0$ by $\varphi_\gamma(u) := |u|^\gamma \mathrm{sgn}\, u$.

Systems of the form (1.1) are often called half-linear because if (x, y) is a solution of (1.1), then so is $(cx, \varphi_\alpha(c)y)$ for any constant $c \in \mathbb{R}$.

It is useful to associate with (1.1) the "dual" system

$$x' + p(t)\varphi_{1/\alpha}(y) = 0\,, \quad y' - q(t)\varphi_\alpha(x) = 0\,, \tag{1.2}$$

which has the property that if (x, y) is a solution of (1.1) (resp. (1.2)), then $(-x, y)$ and $(x, -y)$ satisfy (1.2) (resp. (1.1)). This elementary, but useful relationship between systems (1.1) and (1.2) is referred to as the *duality principle* and will be used to simplify the proofs of some of our results. One of its immediate consequences is that systems (1.1) and (1.2) are equivalent as far as the existence of zeros of components of solutions are concerned.

A comprehensive study of oscillatory properties of differential systems of type (1.1) (and more general nonlinear differential systems) was initiated by J.D. Mirzov [10] who was among the first to extend Sturm's comparison theorem to a pair of

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half-linear systems of the form (1.1). Specifically, he proved that if P and Q are continuous functions on J satisfying

$$0 < p(t) \leq P(t)\,, \quad q(t) \leq Q(t)\,, \quad t \in J\,, \tag{1.3}$$

and if there exists a nontrivial solution (x, y) of system (1.1) such that its first component $x(t)$ has consecutive zeros at t_1, $t_2 \in J$, $t_1 < t_2$, then for any solution (X, Y) of the majorant system

$$X' - P(t)\varphi_{1/\alpha}(Y) = 0\,, \quad Y' + Q(t)\varphi_\alpha(X) = 0\,, \tag{1.4}$$

the first component $X(t)$ has at least one zero between t_1 and t_2. This result was generalized to a pair of Hamiltonian systems of the type

$$x' = r(t)x + p(t)\varphi_{1/\alpha}(y)\,, \quad y' = -q(t)\varphi_\alpha(x) - r(t)y\,, \tag{1.5}$$

and

$$X' = R(t)x + P(t)\varphi_{1/\alpha}(Y)\,, \quad Y' = -Q(t)\varphi_\alpha(X) - R(t)Y\,, \tag{1.6}$$

where r and R are continuous functions on J, by Á. Elbert in [3] (see also [6]). Although it is useful to have criteria for the existence of zeros of components of solutions which apply to (1.5) and (1.6) directly, it should be noted that general systems of the form

$$x' = r(t)x + p(t)\varphi_{1/\alpha}(y)\,, \quad y' = -q(t)\varphi_\alpha(x) - s(t)y\,,$$

can always be transformed by the change of dependent variables

$$u = r^*(t)^{-1}x\,, \quad v = s^*(t)y\,,$$

where $r^*(t) = \exp\big(\int_0^t r(\tau)d\tau\big)$ and $s^*(t) = \exp\big(\int_0^t s(\tau)d\tau\big)$ into the reduced (cyclic) form

$$u' = r^*(t)^{-1}\big(s^*(t)\big)^{-\frac{1}{\alpha}}p(t)\varphi_{1/\alpha}(v)\,, \quad v' = -\big(r^*(t)\big)^{\alpha}s^*(t)q(t)\varphi_\alpha(u)\,,$$

so that the oscillation criteria for (1.5) and (1.6) can be obtained from those for (1.1) and (1.4).

Comparison results given by the present authors in [6] were obtained by an application of the differential identities reminiscent of the classical Picone's formula (see [12]). The disadvantage of these identities was that because of their specific forms they could not be used to prove the variational comparison results of the Leighton type. Thus, the main purpose of this note is to show that the above pointwise comparison criterion of Mirzov can be generalized to an integral comparison theorem by replacing the Picone-type formulas from [6] by identities of another form established in the next section.

We will use the following classification of zero points of components of solutions of systems of the form (1.1).

Let (x, y) be a solution of system (1.1) satisfying the condition $x(a) = 0$ and $y(a) \neq 0$ for some $a \in J$. A value $t = b > a$ from J is called a *conjugate* (resp. *pseudoconjugate*) point to $t = a$ if $x(b) = 0$ (resp. $y(b) = 0$).

If (x,y) is a solution of (1.1) satisfying $y(a)=0$ and $x(a)\neq 0$ for some $a\in J$, then a value $t=b>a$ from J is called a *focal* (resp. *deconjugate*) point to $t=a$ if $x(b)=0$ (resp. $y(b)=0$) (see Reid's book [13]).

It is to be remarked that if $p(t)>0$ in J, then system (1.1) is equivalent with the scalar second-order half-linear differential equation

$$\big(\tilde p(t)\varphi_\alpha(x')\big)' + q(t)\varphi_\alpha(x) = 0 \tag{HL}$$

where $\tilde p(t) = p(t)^{-\alpha}$. Sturm's comparison theory for (HL) has been developed by Li and Yeh in [9] and by the present authors in [5]. Comparison results concerning pseudoconjugate points of solutions of (HL) can be found in [14]. However, application of the results for first order systems (1.1) or (1.2) to (HL) proves to be more effective way than analyzing (HL) directly as a second order equation.

For related results concerning (1.1) and more general nonlinear differential systems based on the two-dimensional version of Picone's identity, which is a special case of the formula given in [6], see [4]. The most comprehensive development of Sturmian theory for linear differential equations can be found in the monograph of Reid [13]. Comparison and oscillation results based on the Picone-type identities for pairs of systems of the form (1.1) and (1.4) (resp. (1.5) and (1.6)) in the case $\alpha=1$ were established also in Díaz and McLaughlin [1] and Kreith [7, 8]. For most recent results on oscillatory properties of half-linear system (1.1) we refer to [2].

2. Comparison theorem for conjugate and deconjugate points

First, we show that a pair of Picone type identities for solutions (x,y) and (X,Y) of systems (1.1) and (1.2), respectively, can be established via straightforward calculations.

To formulate the results we use the following notation:

$$\Phi_\gamma(U,V) := |U|^{\gamma+1} + \gamma|V|^{\gamma+1} - (\gamma+1)U\varphi_\gamma(V)\,, \quad \gamma>0,\ U,\ V\in\mathbb{R}\,.$$

An important property of $\Phi_\gamma(U,V)$ is that $\Phi_\gamma(U,V)\geq 0$ for any U and V, and $\Phi_\gamma(U,V)=0$ if and only if $U=V$.

Lemma 2.1 (Picone's identity)**.** *Let (x,y) and (X,Y) be solutions on J of systems (1.1) and (1.4), respectively.*

(i) *If $X(t)\neq 0$, $p(t)\geq 0$ and $P(t)>0$ in J, then*

$$\begin{aligned}\frac{d}{dt}\Big\{\frac{x}{\varphi_\alpha(X)}\big[\varphi_\alpha(X)y-\varphi_\alpha(x)Y\big]\Big\} &= \big[Q(t)-q(t)\big]|x|^{\alpha+1}\\ &\quad + p(t)\big[1-\big(p(t)/P(t)\big)^\alpha\big]|y|^{\frac{1}{\alpha}+1}\\ &\quad + P(t)^{-\alpha}\Phi_\alpha\Big(p(t)\varphi_{1/\alpha}(y), P(t)\frac{x}{X}\varphi_{1/\alpha}(Y)\Big)\,.\end{aligned} \tag{2.1}$$

(ii) *If $Y(t) \neq 0$, $q(t) \geq 0$ and $Q(t) > 0$ in J, then*

$$\frac{d}{dt}\Big\{\frac{y}{\varphi_{1/\alpha}(Y)}\big[\varphi_{1/\alpha}(y)X - \varphi_{1/\alpha}(Y)x\big]\Big\} = \big[P(t) - p(t)\big]|y|^{\frac{1}{\alpha}+1}$$
$$+ q(t)\big[1 - \big(q(t)/Q(t)\big)^{1/\alpha}\big]|x|^{\alpha+1}$$
$$+ Q(t)^{-1/\alpha}\Phi_{1/\alpha}\Big(q(t)\varphi_\alpha(x), Q(t)\frac{y}{Y}\varphi_\alpha(X)\Big)\,. \tag{2.2}$$

Proof. (i) Performing the differentiation on the left-hand side of (2.1), we obtain

$$\frac{d}{dt}\Big\{\frac{x}{\varphi_\alpha(X)}\big[\varphi_\alpha(X)y - \varphi_\alpha(x)Y\big]\Big\} \tag{2.3}$$
$$= x'y + xy' - \big(\alpha+1\big)\frac{\varphi_\alpha(x)}{\varphi_\alpha(X)}x'Y + \alpha\frac{|x|^{\alpha+1}}{|X|^{\alpha+1}}X'Y - \frac{|x|^{\alpha+1}}{\varphi_\alpha(X)}Y'\,.$$

Using the fact that (x, y) and (X, Y) are solutions of (1.1) and (1.4), respectively, and completing the form Φ_α by adding and subtracting $P(t)^{-\alpha}p(t)^{\alpha+1}|y|^{\frac{1}{\alpha}+1}$, the right hand-side of (2.3) becomes

$$\big[Q(t) - q(t)\big]|x|^{\alpha+1} + \Big(p(t) - \frac{p(t)^{\alpha+1}}{P(t)^\alpha}\Big)|y|^{\frac{1}{\alpha}+1}$$
$$+ P(t)^{-\alpha}\Phi_\alpha\Big(p(t)\varphi_{1/\alpha}(y), P(t)\frac{x}{X}\varphi_{1/\alpha}(Y)\Big)$$

as claimed.

(ii) Formula (2.2) can be verified directly by expanding the left-hand side of (2.2), using (1.1) and (1.4) and completing the form $\Phi_{1/\alpha}$, or it can be obtained from (2.1) by application of the duality principle. □

Let (x, y) be a solution of system (1.1). Then, it is clear that for any nonzero constant c the vector function $(cx, \varphi_\alpha(c)y)$ is also a solution of (1.1). In this note the latter function will be referred to as a *constant multiple* of the former one.

Theorem 2.1. *Let (x, y) and (X, Y) be solutions on J of systems (1.1) and (1.4), respectively.*

(i) *Suppose that $p(t) \geq 0$, $P(t) > 0$ in J, $x(t)$ has consecutive zeros a and b, $a < b$, in J and*

$$\int_a^b \big\{[Q(t) - q(t)]|x|^{\alpha+1} + p(t)\big[1 - \big(p(t)/P(t)\big)^\alpha\big]|y|^{\frac{1}{\alpha}+1}\big\}\,dt \geq 0\,. \tag{2.4}$$

Then, for any solution (X, Y) of system (1.4) the component $X(t)$ has a zero in (a, b), or otherwise $(X(t), Y(t))$ is a constant multiple of $(x(t), y(t))$ which is possible only if $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ in (a, b).

(ii) *Suppose that $q(t) \geq 0$ and $Q(t) > 0$ in J, $y(t)$ has consecutive zeros a and b, $a < b$, in J and*

$$\int_a^b \big\{q(t)\big[1 - \big(q(t)/Q(t)\big)^\alpha\big]|x|^{\alpha+1} + \big[P(t) - p(t)\big]|y|^{\frac{1}{\alpha}+1}\big\}\,dt \geq 0\,. \tag{2.5}$$

Then, for any solution (X, Y) of system (1.4) the component $Y(t)$ has a zero in (a, b), or otherwise $(X(t), Y(t))$ is a constant multiple of $(x(t), y(t))$ which is possible only if $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ in (a, b).

Proof. The proof of (i) proceeds as follows. Consider the component $X(t)$ of $(X(t), Y(t))$. Suppose that $X(t)$ has no zero in (a, b). Then the Picone identity given in (2.1) of Lemma 2.1 holds in (a, b). It is obvious that the right-hand side of (2.1) is integrable on $[a, b]$ if both $X(a) \neq 0$ and $X(b) \neq 0$. The same is true even if $X(t)$ vanishes at one or both of the endpoints a and b, since in such a case the l'Hôpital's rule guarantees the existence of the finite limits

$$\lim_{t\to a+0} \frac{x(t)}{X(t)} = \frac{p(a)\varphi_{1/\alpha}(y(a))}{P(a)\varphi_{1/\alpha}(Y(a))} \neq 0\,, \quad \lim_{t\to b-0} \frac{x(t)}{X(t)} = \frac{p(b)\varphi_{1/\alpha}(y(b))}{P(b)\varphi_{1/\alpha}(Y(b))} \neq 0\,.$$

We now integrate (2.1) from a to b. Note that if $X(a) \neq 0$, then

$$\frac{x(a)}{\varphi_\alpha(X(a))}\big(\varphi_\alpha(X(a))y(a) - \varphi_\alpha(x(a))Y(a)\big) = 0\,,$$

and that if $X(a) = 0$, then

$$\begin{aligned}\lim_{t\to a+0}\Big[&\frac{x(t)}{\varphi_\alpha(X(t))}\big(\varphi_\alpha(X(t))y(t) - \varphi_\alpha(x(t))Y(t)\big)\Big]\\ &= \lim_{t\to a+0}\Big(x(t)y(t) - |x(t)|^{\alpha+1}\frac{Y(t)}{\varphi_\alpha(X(t))}\Big) = 0\,,\end{aligned}$$

where in the last step we have used the fact that

$$\lim_{t\to a+0} \frac{Y(t)}{\varphi_\alpha(X(t))} = -\frac{1}{\alpha}\lim_{t\to a+0}\frac{Q(t)X(t)}{P(t)\varphi_{1/\alpha}(Y(t))} = 0\,.$$

Likewise it is shown that

$$\lim_{t\to b-0}\Big[\frac{x(t)}{\varphi_\alpha(X(t))}\big(\varphi_\alpha(X(t))y(t) - \varphi_\alpha(x(t))Y(t)\big)\Big] = 0\,.$$

Taking into account the above observations, we find that the integral over $[a, b]$ of the right-hand side of (2.1) is zero, and this fact implies that

$$p(t) \equiv P(t)\,, \quad q(t) \equiv Q(t)\,, \quad \varphi_{1/\alpha}(y(t)) \equiv \frac{x(t)}{X(t)}\varphi_{1/\alpha}(Y(t)) \quad \text{on} \quad [a, b]\,. \tag{2.6}$$

The first two identities of (2.6) show that systems (1.1) and (1.4) coincide, so that (x, y) and (X, Y) must be solutions of the same system (1.1). Note that the third identity forces that $X(t)$ vanishes at both endpoints; $X(a) = 0 = X(b)$. In fact, from the identity it follows that $X(a) \neq 0$ implies $y(a) = 0$. This, however, is impossible because we have in view of the first equation of (1.1)

$$y(a) = \varphi_\alpha\big(x'(a)\big)/p(a) \neq 0\,.$$

Similar argument applies at the endpoint b. As is already known, if $x(t)$ and $X(t)$ (resp. $y(t)$ and $Y(t)$) have at least one zero in common, then there exists a constant $c \neq 0$ such that $X(t) = cx(t)$ and $Y(t) = \varphi_\alpha(c)y(t)$, that is, $(X(t), Y(t))$ is a constant multiple of $(x(t), y(t))$. It is concluded therefore that $X(t)$ has a zero in (a, b), or else $(X(t), Y(t))$ is a constant multiple of $(x(t), y(t))$ on $[a, b]$.

To prove the second statement (ii) it suffices to apply essentially the same arguments as above by making use of the second Picone identity given in (2.2) of Lemma 2.1. The details are omitted. □

Remark 2.1. Clearly, if (1.3) holds, then (2.4) is satisfied and the conclusion (i) of Theorem 2.1 is true. Similarly, the satisfaction of the inequalities

$$p(t) \le P(t)\,, \quad 0 < q(t) \le Q(t)\,, \quad t \in J\,, \tag{2.7}$$

guarantees the validity of (2.5) and the conclusion (ii) of Theorem 2.1 follows.

Remark 2.2. Theorem 2.1 includes the possibility that systems (1.1) and (1.4) coincide which means that a variant of Sturm's separation theorem holds true for (1.1) in the sense that zeros of components of solutions which are not proportional interlace. An important consequence of the interlacing phenomenon is that either all nontrivial solutions are oscillatory or none of them is.

2.1. **Applications.**

(I) As a first comparison system consider

$$X' - p^* \varphi_{1/\alpha}(Y) = 0\,, \quad Y' + q^* \varphi_\alpha(X) = 0\,, \tag{B}$$

where $p^* = \max_{t\in[a,b]} p(t)$ and $q^* = \max_{t\in[a,b]} q(t)$.

Then the first component x of the solution (x, y) of system (1.1) does not oscillate more rapidly in (a, b) than the first component of the solutions of (B).

System (B) has the oscillatory solution

$$\left(p^{*\frac{1}{\alpha+1}} \sin_\alpha\left(p^{*\frac{\alpha}{\alpha+1}} q^{*\frac{1}{\alpha+1}} t\right), q^{*\frac{\alpha}{\alpha+1}} \cos_\alpha\left(p^{*\frac{\alpha}{\alpha+1}} q^{*\frac{1}{\alpha+1}} t\right)\right), \tag{2.8}$$

where $\sin_\alpha$ (resp. $\cos_\alpha$) denotes the generalized sine function (resp. generalized cosine function) defined to be the first (resp. the second) component of the solution of the system

$$X' - \varphi_{1/\alpha}(Y) = 0\,, \quad Y' + \varphi_\alpha(X) = 0\,, \tag{2.9}$$

determined by the initial condition

$$X(0) = 0\,, \quad Y(0) = \left(\frac{2}{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}}. \tag{2.10}$$

The length of the interval between consecutive zeros of the first (second) component of solution (2.8) of (B) is

$$p^{*\frac{\alpha}{\alpha+1}} q^{*\frac{1}{\alpha+1}} \pi_\alpha\,,$$

where

$$\pi_\alpha = \frac{2\alpha^{\frac{1}{\alpha+1}} \pi}{(\alpha+1) \sin \frac{\pi}{\alpha+1}}\,.$$

Therefore, if

$$p^{*\frac{\alpha}{\alpha+1}} q^{*\frac{1}{\alpha+1}} \pi_\alpha > b - a\,, \tag{2.11}$$

then for no solution of the given system (1.1) the first component can have more than one zero in the interval (a, b).

Now, as a second comparison system consider

$$X' - p_* \varphi_{1/\alpha}(Y) = 0, \quad Y' + q_* \varphi_\alpha(X) = 0\,, \tag{C}$$

where $p_* = \min_{t\in[a,b]} p(t)$ and $q_* = \min_{t\in[a,b]} q(t)$.

The first components of the solutions of (1.1) oscillate at least as rapidly as those of (C). System (C) has the oscillatory solution

$$\Big(p_*^{\frac{1}{\alpha+1}} \sin_\alpha \big(p_*^{\frac{\alpha}{\alpha+1}} q_*^{\frac{1}{\alpha+1}} t\big), q_*^{\frac{\alpha}{\alpha+1}} \cos_\alpha \big(p_*^{\frac{\alpha}{\alpha+1}} q_*^{\frac{1}{\alpha+1}} t\big)\Big)\,, \tag{2.12}$$

so that the distance between consecutive zeros of the first (second) component of (2.12) is

$$p_*^{\frac{\alpha}{\alpha+1}} q_*^{\frac{1}{\alpha+1}} \pi_\alpha\,.$$

It follows that a sufficient condition that the first (second) components of the solutions of the given system (1.1) should have at least m zeros in (a, b) is that

$$m p_*^{\frac{\alpha}{\alpha+1}} q_*^{\frac{1}{\alpha+1}} \pi_\alpha \le b - a\,. \tag{2.13}$$

In particular, a sufficient condition that system (1.1) should possess a solution the first (second) component of which has at least one zero in (a, b) is that

$$p_*^{-\frac{\alpha}{\alpha+1}} q_*^{-\frac{1}{\alpha+1}} \ge \frac{\pi_\alpha}{b-a}\,. \tag{2.14}$$

(II) In the second application of Theorem 2.1 we compare (1.1) with the Euler-type system

$$x' - p(t)\varphi_{1/\alpha}(y) = 0\,, \quad y' + \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} \frac{p(t)}{P(t)^{\alpha+1}} \varphi_\alpha(x) = 0\,, \tag{E}$$

where $\int_0^\infty p(t)\,dt = \infty$ and $P(t) = \int_0^t p(s)\,ds$. System (E) has the nonoscillatory solution

$$x(t) = P(t)^{\frac{\alpha}{\alpha+1}}\,, \quad y(t) = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha} P(t)^{-\frac{\alpha}{\alpha+1}}\,, \quad t > 0\,.$$

Thus, all nontrivial solutions of (1.1) are nonoscillatory if

$$q(t) \le \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} \frac{p(t)}{P(t)^{\alpha+1}} \tag{2.15}$$

for all sufficiently large t. Criterion (2.15) is sharp in the sense that if for some $\varepsilon > 0$

$$q(t) \ge (1+\varepsilon)\bigg(\frac{\alpha}{\alpha+1}\bigg)^{\alpha+1} p(t) P(t)^{-\alpha-1} \tag{2.16}$$

for all large t, then all nontrivial solutions of (1.1) are nonoscillatory (see Mirzov [10]).

3. Comparison theorem for pseudoconjugate and focal points

The following result generalizes and extends the comparison theorem for the scalar second-order half-linear differential equations given in [14].

Theorem 3.1 (On pseudoconjugate and focal points)**.** *Let* (x, y) *and* (X, Y) *be solutions on* J *of systems* (1.1) *and* (1.4)*, respectively.*

(i) *Suppose that* $p(t) \geq 0$, $P(t) > 0$ *in* J, $x(a) = y(b) = 0$, $a < b$, *with* $y(t) \neq 0$ *on* $[a, b)$ *and*

$$V_\alpha[x, y] := \int_a^b \left\{ \big[Q(t) - q(t)\big] |x|^{\alpha+1} + p(t)\big[1 - \big(p(t)/P(t)\big)^\alpha\big] |y|^{\frac{1}{\alpha}+1} \right\} dt \geq 0\,. \tag{3.1}$$

Then, for any solution (X, Y)*, with* $X \not\equiv 0$*, of system* (1.4) *satisfying* $X(a) = 0$ *there is a value* $c \in (a, b]$ *such that* $Y(c) = 0$*. Moreover,* $c = b$ *only if* (X, Y) *is a constant multiple of* (x, y)*.*

(ii) *Suppose that* $q(t) \geq 0$, $Q(t) > 0$ *in* J, $y(a) = x(b) = 0, a < b$, *with* $x(t) \neq 0$ *on* $[a, b)$*, and*

$$\int_a^b \left\{ q(t)\big[1 - \big(q(t)/Q(t)\big)^\alpha\big] |x|^{\alpha+1} + \big[P(t) - p(t)\big] |y|^{\frac{1}{\alpha}+1} \right\} dt \geq 0\,. \tag{3.2}$$

Then, for any solution (X, Y)*, with* $Y \not\equiv 0$*, of system* (1.4) *satisfying* $Y(a) = 0$ *there is a value* $c \in (a, b]$ *such that* $X(c) = 0$*. Moreover,* $c = b$ *only if* (X, Y) *is a constant multiple of* (x, y)*.*

Proof. (i) If $X(t_0) = 0$ at some $t_0 \in (a, b]$, then it is obvious that the component $Y(t)$ must vanish somewhere between a and t_0. Thus, we may assume that $X(t) \neq 0$ on the whole interval $(a, b]$.

Integrating (2.1) on the compact subinterval $[\sigma, \tau] \subset (a, b)$ and letting $\sigma \to a + 0$ and $\tau \to b - 0$, we obtain

$$\begin{aligned}
&\frac{x(b)}{\varphi_\alpha(X(b))} \big[\varphi_\alpha(X(b)) y(b) - \varphi_\alpha\big(x(b)\big) Y(b)\big] \\
&\qquad - \lim_{\sigma \to a+0} \Big[\frac{x(\sigma)}{\varphi_\alpha\big(X(\sigma)\big)} \big(\varphi_\alpha\big(X(\sigma)\big) y(\sigma) - \varphi_\alpha\big(x(\sigma)\big) Y(\sigma)\big)\Big] \\
&\qquad = \int_a^b \left\{ \big[Q(t) - q(t)\big] |x|^{\alpha+1} + p(t)\big[1 - \big(p(t)/P(t)\big)^\alpha\big] |y|^{\frac{1}{\alpha}+1} \right\} dt \\
&\qquad + \lim_{\sigma \to a+0} \int_\sigma^b P(t)^{-\alpha} \Phi_\alpha\big(p(t)\varphi_{1/\alpha}(y), P(t) x \varphi_{1/\alpha}(Y)/X\big)\, dt\,.
\end{aligned} \tag{3.3}$$

Since zeros of nontrivial solutions of half-linear systems (1.1) and (1.4) are simple in the sense that $x(t)$ and $y(t)$ (resp. $X(t)$ and $Y(t)$) are not both zero for any t (see [11]), $X(a) = 0$ and $y(a) = 0$ imply that $Y(a)$ and $x(a)$ must be nonzero finite values. Since $\lim_{\sigma \to a+0} x(\sigma) Y(\sigma) = 0$ and also

$$\lim_{\sigma \to a+0} \frac{x(\sigma)}{X(\sigma)} = \lim_{\sigma \to a+0} \frac{x'(\sigma)}{X'(\sigma)} = \frac{p(a)\varphi_{1/\alpha}\big(y(a)\big)}{P(a)\varphi_{1/\alpha}\big(Y(a)\big)} < \infty\,,$$

by l'Hôpital's rule, we get

$$\lim_{\sigma\to a+0}\Big[\frac{x(\sigma)}{\varphi_\alpha\big(X(\sigma)\big)}\big(\varphi_\alpha\big(X(\sigma)\big)y(\sigma)-\varphi_\alpha\big(x(\sigma)\big)Y(\sigma)\big)\Big]=0$$

and (3.3) is reduced to

$$-\frac{x(b)\varphi_\alpha\big(x(b)\big)Y(b)}{\varphi_\alpha\big(X(b)\big)}=V_\alpha[x,y]+\int_a^b P(t)^{-\alpha}\Phi_\alpha\big(p(t)\varphi_{1/\alpha}(y),P(t)x\varphi_{1/\alpha}(Y)/X\big)\,dt\,.$$

From $\Phi_\alpha\big(p(t)\varphi_{1/\alpha}(y),P(t)x\varphi_{1/\alpha}(Y)/X\big)\geq 0$ and the assumption (3.1) we get

$$-\frac{|x(b)|^{\alpha+1}Y(b)}{\varphi_\alpha\big(X(b)\big)}\geq 0\,. \tag{3.4}$$

There is no loss of generality in assuming that $y(a)$ and $Y(a)$ are both positive, that is, $x(t)>0$ and $X(t)>0$ on $(a,b]$. If $Y(t)$ does not have a zero on $(a,b]$, then $Y(b)>0$ and we obtain a contradiction with (3.4). Thus $Y(b)\leq 0$ and so there exists a value c from $(a,b]$ such that $Y(c)=0$. The case $Y(b)=0$ occurs when $V_\alpha[x,y]=0$ and

$$\Phi_\alpha\big(p(t)\varphi_{1/\alpha}(y),P(t)x\varphi_{1/\alpha}(Y)/X\big)\equiv 0\,,$$

i.e., if $X(t)\varphi_{1/\alpha}\big(y(t)\big)\equiv x(t)\varphi_{1/\alpha}\big(Y(t)\big)$ which is equivalent with the fact that (X,Y) is a constant multiple of (x,y). The proof of part (i) is complete.

Similar reasoning based on the use of identity (2.2) proves the validity of part (ii) concerning focal points. We omit the details. □

Remark 3.1. If the pointwise inequalities (1.3) hold on $[a,c]$, then (3.1) is satisfied and the conclusion (i) of Theorem 3.1 follows. Similarly, the satisfaction of the inequalities (2.7) on $[a,c]$ implies that the assertion of Theorem 3.1 (ii) holds true.

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References

[1] Diaz, J.B., McLaughlin, J.R., *Sturm comparison and separation theorems for linear, second order, self-adjointordinary differential equations and for first order systems*, Appl. Anal. **2** (1972), 355–376.

[2] Dosoudilová, M., Lomtatidze, A., Šremr, J., *Oscillatory properties of solutions to certain two-dimensional systems of non-linear ordinary differential equations*, Nonlinear Anal. **120** (2015), 57–75.

[3] Elbert, Á., *A half-linear second order differential equation*, Colloq. Math. Soc. János Bolyai **30** (1979), 153–180.

[4] Jaroš, J., *Wirtinger inequality and nonlinear differential systems*, Arch. Math. (Brno) **49** (2013), 35–41.

[5] Jaroš, J., Kusano, T., *A Picone-type identity for second order half-linear differential equations*, Acta Math. Univ. Comenian. (N.S.) **68** (1999), 137–151.

[6] Jaroš, J., Kusano, T., *Elbert-type comparison theorems for a class of nonlinear Hamiltonian systems*, Electron. J. Qual. Theory Differ. Equ. **100** (2016), 1–10.

[7] Kreith, K., *A Picone identity for first order systems*, J. Math. Anal. Appl. **31** (1970), 297–308.

[8] Kreith, K., *Oscillation theory*, Lecture Notes in Math., vol. 324, Springer, 1973.

[9] Li, H.J., Yeh, C.C., *Sturmian comparison theorem for half-linear second order differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **125A** (1995), 1193–1204.

[10] Mirzov, J.D., *On some analogs of Sturm's and Kneser's theorems for nonlinear systems*, J. Math. Anal. Appl. **53** (1976), 418–425.

[11] Mirzov, J.D., *Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations*, Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math., Mathematica, vol. 14, Masaryk University Brno, 2004.

[12] Picone, M., *Sui valori eccepzionali di un parametro da cui dipende un' equazione differenziale ordinaria del secondordine*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **II** (1910), 1–141.

[13] Reid, W.T., *Sturmian Theory for Ordinary Differential Equations*, Appl. Math. Sci., vol. 31, Springer, New York, 1980.

[14] Rostás, K., *Comparison theorems for pseudoconjugate points of half-linear ordinary differentialequations of the second order*, Acta Math. Univ. Comenian. (N.S.) **72** (2003), 223–228.

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