ARCHIVUM MATHEMATICUM (BRNO)
Tomus 53 (2017), 193–201

NONRECTIFIABLE OSCILLATORY SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. The second order linear differential equation

$$(p(x)y')' + q(x)y = 0\,, \quad x \in (0, x_0]$$

is considered, where p, $q \in C^1(0, x_0]$, $p(x) > 0$, $q(x) > 0$ for $x \in (0, x_0]$. Sufficient conditions are established for every nontrivial solutions to be nonrectifiable oscillatory near $x = 0$ without the Hartman–Wintner condition.

1. Introduction

We consider the second order linear differential equation

$$(p(x)y')' + q(x)y = 0\,, \quad x \in (0, x_0]\,, \tag{1.1}$$

where, p, $q \in C^1(0, x_0]$, $p(x) > 0$, $q(x) > 0$ for $x \in (0, x_0]$. A solution y of (1.1) is said to be *oscillatory near* $x = 0$ if there exists $\{z_n\}_{n=1}^{\infty}$ such that $y(z_n) = 0$ for $n \in \mathbf{N}$ and $z_n \to 0$ as $n \to \infty$. Otherwise, it is said to be *nonoscillatory near* $x = 0$.

A study of the oscillation of solutions to (1.1) has a long history. See, for example, [1, 2, 3, 4, 5, 14]. However, it seems that very little is known how oscillatory a solution of (1.1) is. In this paper, we divide oscillatory solutions into the following two classes: into rectifiable and nonrectifiable solutions, that is, those of finite and infinite length, respectively. Namely, a solution y of (1.1) is said to be *rectifiable* (resp. *nonrectifiable*) *oscillatory near* $x = 0$ if y is oscillatory near $x = 0$ and satisfies

$$\int_0^{x_0} \sqrt{1 + |y'(x)|^2}dx < \infty \quad (\text{resp. } = \infty)\,.$$

We remark that nonrectifiable oscillatory means more oscillatory than rectifiable oscillatory.

To classify oscillatory solutions of the second order linear differential equations by this geometric viewpoint began Pašić [8, 9]. Pašić [10] and J. S. W. Wong [15] presented the rectifiable and nonrectifiable oscillatory results for the Euler type equation

$$y'' + \lambda x^{-\sigma} y = 0\,, \quad x \in (0, x_0]\,,$$

2010 *Mathematics Subject Classification*: primary 34C10.
Key words and phrases: oscillatory, nonrectifiable, second order linear differential equation.
This work was supported by JSPS KAKENHI Grant Number 26400182.
Received January 30, 2017, revised June 2017. Editor R. Šimon Hilscher.
DOI: 10.5817/AM2017-4-193

where $\sigma \geq 2$ and $\lambda > 0$. Pašić [11] obtained the rectifiable and nonrectifiable oscillatory results for the Riemann–Weber version of Euler differential equation

$$y'' + x^{-\sigma}\Big(\frac{1}{4} + \frac{\lambda}{|\log x|^{\beta}}\Big)y = 0\,, \quad x \in (0, x_0]\,,$$

where $\sigma \geq 2$, $\beta > 0$ and $\lambda > 0$. Kwong, Pašić and J. S. W. Wong [7] considered the general equation

$$y'' + q(x)y = 0\,, \quad x \in (0, x_0]\,. \tag{1.2}$$

Pašić and Tanaka [13] considered more general equation (1.1) and presented the following result.

Theorem A. Assume that p, $q \in C^2(0, x_0]$,

$$\int_0^{x_0} \sqrt{\frac{q(x)}{p(x)}}dx = \infty$$

and the Hartman–Wintner condition

$$\int_0^{x_0} \frac{1}{\sqrt[4]{p(x)q(x)}}\Big|\Big(p(x)\Big(\frac{1}{\sqrt[4]{p(x)q(x)}}\Big)'\Big)'\Big|dx < \infty\,. \tag{1.3}$$

Then the following (i) and (ii) hold:

(i) every nontrivial solution of (1.1) is rectifiable oscillatory near $x = 0$ if

$$\int_0^{x_0} [p(x)]^{-\frac{3}{4}}[q(x)]^{\frac{1}{4}}dx < \infty\,;$$

(ii) every nontrivial solution of (1.1) is nonrectifiable oscillatory near $x = 0$ if

$$\int_0^{x_0} [p(x)]^{-\frac{3}{4}}[q(x)]^{\frac{1}{4}}dx = \infty\,.$$

Kwong, Pašić and J. S. W. Wong [7] gave Theorem A when $p(x) \equiv 1$. The proof of Theorem A is based on the asymptotic formula of oscillatory solutions of (1.1), which is obtained from the Hartman–Wintner condition (1.3). The purpose of this paper is to obtain nonrectifiable oscillatory results for (1.1) without the Hartman–Wintner condition (1.3). In [12], equation (1.2) is considered and condition (1.3) with $p(x) \equiv 1$ is not supposed, but it is assumed that every solution y of (1.2) satisfies either

$$\limsup_{x\to+0} x^{\frac{\theta+1}{2}}|y'(x)| < \infty \quad \text{or} \quad \limsup_{x\to+0} x^{\frac{\theta+1}{2}}|y(x)| < \infty$$

for some $\theta > 0$.

The main results of this paper are as follows.

Theorem 1.1. *Assume that every solution of* (1.1) *is oscillatory near $x = 0$ and $(p(x)q(x))' \geq 0$ for $x \in (0, x_0]$. Then every nontrivial solution of* (1.1) *is nonrectifiable oscillatory near $x = 0$.*

Theorem 1.2. *Assume that there exist* μ, $\sigma \in \mathbf{R}$ *such that* $\mu + \sigma > 2$, $(x^{\mu+\sigma-2}p(x)q(x))' \geq 0$ *for* $x \in (0, x_0]$, *and*

$$\limsup_{x\to+0} x^{-\mu}p(x) < \infty\,, \tag{1.4}$$

$$\liminf_{x\to+0} x^{\sigma}q(x) > 0\,. \tag{1.5}$$

Then every nontrivial solution of (1.1) *is nonrectifiable oscillatory near* $x = 0$.

Theorem 1.3. *Assume that there exist* a, $b \in \mathbf{R}$ *such that* $a + b > 0$, $(e^{-\frac{a+b}{x}}p(x)q(x))' \geq 0$ *for* $x \in (0, x_0]$, *and*

$$\limsup_{x\to+0} e^{\frac{a}{x}}p(x) < \infty, \tag{1.6}$$

$$\lim_{x\to+0} x^4 e^{-\frac{b}{x}}q(x) = \infty\,. \tag{1.7}$$

Then every nontrivial solution of (1.1) *is nonrectifiable oscillatory near* $x = 0$.

Remark 1.1. We obtain more general results than Theorems 1.1–1.3. See later on.

Example 1.1. We consider the Euler type equation

$$(x^{\mu}y')' + \lambda x^{-\sigma}y = 0\,, \quad x \in (0, x_0]\,, \tag{1.8}$$

where $\lambda > 0$, μ, $\sigma \in \mathbf{R}$. If $\mu + \sigma > 2$, then every solution of (1.8) is oscillatory near $x = 0$. Conversely, if $\mu + \sigma < 2$, then every nontrivial solution of (1.8) is nonoscillatory near $x = 0$. Indeed, we consider the Euler equation

$$(x^{\mu}y')' + \nu x^{\mu-2}y = 0\,, \quad x \in (0, x_0]\,, \tag{1.9}$$

which is (1.8) with $\sigma = 2 - \mu$ and $\lambda = \nu$. When $\nu > (\mu - 1)^2/4$, equation (1.9) has the oscillatory solution

$$y(x) = x^{\frac{1-\mu}{2}} \sin\Big(\frac{\sqrt{4\nu - (\mu-1)^2}}{2}\log x\Big)\,.$$

If $\mu + \sigma > 2$, $\lambda > 0$ and $\mu > 0$, then $\lambda x^{-\sigma} > \nu x^{\mu-2}$ for all sufficiently small $x > 0$, and hence the Sturm–Picone comparison theorem implies that every solution of (1.8) is oscillatory near $x = 0$. Next we assume that $\mu + \sigma < 2$. Since $x^{(1-\mu)/2}$ is a nonoscillatory solution of (1.9) with $\nu = (\mu - 1)^2/4$ and $\lambda x^{-\sigma} < \nu x^{\mu-2}$ for all sufficiently small $x > 0$, the Sturm–Picone comparison theorem implies that every nontrivial solution of (1.8) is nonoscillatory near $x = 0$.

Applying Theorem 1.1, we conclude that if $\mu + \sigma > 2$ and $\mu \geq \sigma$, then every nontrivial solution of (1.8) is nonrectifiable oscillatory near $x = 0$. Using Theorem 1.2, we find that every nontrivial solution of (1.8) is nonrectifiable oscillatory near $x = 0$, if $\mu + \sigma > 2$ and $\mu \geq 1$, which is better than $\mu + \sigma > 2$ and $\mu \geq \sigma$.

On the other hand, from Theorem A and Example 1.4 in [13], it follows that every nontrivial solution of (1.8) is rectifiable oscillatory near $x = 0$, provided $\mu+\sigma > 2$ and $3\mu+\sigma < 4$, and that every nontrivial solution of (1.8) is nonrectifiable oscillatory near $x = 0$, provided $\mu + \sigma > 2$ and $3\mu + \sigma > 4$. Therefore, for equation (1.8), Theorem A is better than Theorem 1.2. However, the Hartman–Wintner condition (1.3) is not needed in Theorems 1.1 and 1.2.

Example 1.2. We consider the equation

$$(e^{-\frac{\alpha}{x}}y')' + \lambda e^{\frac{\beta}{x}}y = 0\,, \quad x \in (0, x_0]\,, \tag{1.10}$$

where $\lambda > 0$ and α, $\beta \in \mathbf{R}$. Theorem 1.3 implies that every nontrivial solution of (1.10) is nonrectifiable oscillatory near $x = 0$ when $\alpha > 0$ and $\alpha + \beta > 0$. Indeed, setting $a = \alpha$, $b = \beta - \varepsilon$ and $\varepsilon = \min\{(\alpha+\beta)/2, \alpha\}$, we find that

$$a + b = \alpha + \beta - \varepsilon > \alpha + \beta - \frac{\alpha+\beta}{2} = \frac{\alpha+\beta}{2} > 0\,.$$

Moreover, $p(x) = e^{-\frac{\alpha}{x}}$ and $q(x) = \lambda e^{\frac{\beta}{x}}$ satisfy

$$(e^{-\frac{a+b}{x}}p(x)q(x))' = \left(\lambda e^{\frac{-2\alpha+\varepsilon}{x}}\right)' = \lambda\frac{2\alpha-\varepsilon}{x^2}e^{\frac{\varepsilon}{x}} = \lambda\frac{\alpha+(\alpha-\varepsilon)}{x^2}e^{\frac{\varepsilon}{x}} > 0\,,$$

$$\limsup_{x\to+0} e^{\frac{a}{x}}p(x) = 1\,,$$

and

$$\lim_{x\to+0} x^4 e^{-\frac{b}{x}}q(x) = \lim_{x\to+0} \lambda x^4 e^{\frac{\varepsilon}{x}} = \infty\,.$$

To prove Theorems 1.1–1.3, we use the following nonrectifiable criteria by Pašić [10, Proposition 4.2].

Proposition 1.1. *Let $y \in C^1(0, x_0]$. Assume that there exists a strictly decreasing sequence $\{a_n\}_{n=1}^{\infty}$ such that $0 < a_1 \le x_0$, $a_n \to 0$ as $n \to \infty$, $y(a_n)y(a_{n+1}) < 0$ for $n \in \mathbf{N}$, and $\sum_{n=1}^{\infty} |y(a_n)| = \infty$. Then y is nonrectifiable oscillatory near $x = 0$.*

To use Proposition 1.1, we have to estimate the amplitude of oscillatory solutions of (1.1). To this end, we employ the following energy function

$$E[y](x) = \frac{1}{f(x)}\big(p(x)y'\big)^2 + \frac{p(x)q(x)}{f(x)}y^2\,, \tag{1.11}$$

where $f \in C^1(0, x_0]$ and $f(x) > 0$ for $x \in (0, x_0]$. Assume that y is an oscillatory solution near $x = 0$ of (1.1). Then

$$\frac{d}{dx}E[y](x) = \Big(\frac{p(x)q(x)}{f(x)}\Big)' y^2 - \frac{f'(x)[p(x)]^2}{[f(x)]^2}(y')^2\,. \tag{1.12}$$

Condition (2.1) below implies that $\frac{d}{dx}E[y](x) \le 0$. By (1.11) and (1.12), we can estimate $|y(x)|$. Recently, the amplitude of oscillatory solutions to (1.1) has been studied by Kusano and Yoshida [6]. We take a decreasing sequence $\{a_n\}_{n=1}^{\infty}$ such that $0 < a_1 \le x_0$, $a_n \to 0$ as $n \to \infty$ and $y(a_n)y(a_{n+1}) < 0$, $y'(a_n) = 0$ for $n \in \mathbf{N}$. If we have $\liminf_{n\to\infty} |y(a_n)| > 0$, then $\sum_{n=1}^{\infty} |y(a_n)| = \infty$, and hence Proposition 1.1 implies that y is nonrectifiable oscillatory near $x = 0$. By this approach, we will prove Theorem 1.1 in Section 2. Moreover, by the Sturm-Picone comparison theorem with a concrete equation, we can know the distribution of zeros of y, and hence we obtain the asymptotic behavior of a_n as $n \to \infty$. In this way, we will prove Theorems 1.2 and 1.3 in Sections 3 and 4, respectively.

2. Proof of the first main result

To prove Theorem 1.1 we begin with the following more general result.

Theorem 2.1. *Assume that every solution of* (1.1) *is oscillatory near* $x = 0$ *and there exists* $f \in C^1(0, x_0]$ *such that*

$$f(x) > 0\,, \;\; f'(x) \geq 0\,, \;\; \Big(\frac{p(x)q(x)}{f(x)}\Big)' \leq 0 \quad \textit{for } x \in (0, x_0]\,, \tag{2.1}$$

$$\limsup_{x \to +0} \frac{p(x)q(x)}{f(x)} < \infty\,. \tag{2.2}$$

Then every nontrivial solution of (1.1) *is nonrectifiable oscillatory near* $x = 0$.

Proof. Let y be a nontrivial solution of (1.1). Then y is oscillatory near $x = 0$, which implies that y' is also oscillatory near $x = 0$. Let $\{a_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence $\{a_n\}_{n=1}^{\infty}$ such that $0 < a_1 \leq x_0$, $a_n \to 0$ as $n \to \infty$ and $y(a_n)y(a_{n+1}) < 0$, $y'(a_n) = 0$ for $n \in \mathbf{N}$. We use the energy function (1.11). Then $E[y](x) \geq 0$ for $x \in (0, x_0]$. We note that $E[y](x) > 0$ for $x \in (0, x_0]$. Indeed, if $E[y](\xi) = 0$ for some $\xi \in (0, x_0]$, then $y(\xi) = y'(\xi) = 0$, which means that $y(x) \equiv 0$ on $(0, x_0]$ by the uniqueness of initial value problems. This is a contradiction. From (1.12) and (2.1), it follows that

$$\frac{d}{dx} E[y](x) \leq 0\,, \quad x \in (0, x_0]\,.$$

Hence,

$$E[y](x) \geq K\,, \quad x \in (0, x_0]$$

for some $K > 0$. Since $y'(a_n) = 0$, we obtain

$$E[y](a_n) = \frac{p(a_n)q(a_n)}{f(a_n)} [y(a_n)]^2 \geq K\,, \quad n \in \mathbf{N}\,,$$

that is,

$$|y(a_n)| \geq \sqrt{\frac{K f(a_n)}{p(a_n)q(a_n)}}\,, \quad n \in \mathbf{N}\,. \tag{2.3}$$

By (2.2), there exists $c > 0$ such that

$$\frac{p(x)q(x)}{f(x)} \leq c\,, \quad x \in (0, x_0]\,.$$

Consequently,

$$|y(a_n)| \geq \sqrt{\frac{K}{c}} > 0\,, \quad n \in \mathbf{N}\,,$$

which implies that

$$\sum_{n=1}^{\infty} |y(a_n)| = \infty\,.$$

Therefore, Proposition 1.1 implies that y is nonrectifiable oscillatory near $x = 0$. □

Proof of Theorem 1.1. Setting $f(x) = p(x)q(x)$ in Theorem 2.1, we obtain Theorem 1.1 immediately. □

3. Proof of the second main result

In this section we give a proof of Theorem 1.2. To this end, we prove the following result.

Theorem 3.1. *Assume that r, $s \in C(0, x_0]$,*

$$p(x) \le r(x), \quad q(x) \ge s(x) > 0, \quad x \in (0, x_0].$$

Let w be a nontrivial solution of

$$(r(x)w')' + s(x)w = 0. \tag{3.1}$$

Assume moreover that there exist a strictly decreasing sequence $\{t_n\}_{n=1}^{\infty}$ and a function $f \in C^1(0, x_0]$ such that $w(t_n) = 0$ for $n \in \mathbf{N}$, $t_n \to 0$ as $n \to \infty$, and (2.1) *and the following condition hold:*

$$\sum_{n=1}^{\infty} \sqrt{\frac{f(t_n)}{p(t_n)q(t_n)}} = \infty. \tag{3.2}$$

Then every nontrivial solution of (1.1) *is nonrectifiable oscillatory near $x = 0$.*

Proof. The Sturm–Picone comparison theorem implies that every nontrivial solution of (1.1) is oscillatory near $x = 0$. Let y be a nontrivial solution of (1.1). Then y is oscillatory near $x = 0$, and hence there exists $\{z_n\}_{n=1}^{\infty}$ such that $z_n \to 0$ as $n \to \infty$, $y(z_n) = 0$ for $n \in \mathbf{N}$, $y(x) \ne 0$ for $x \in (z_{n+1}, z_n)$, and

$$0 < \cdots < z_{n+1} < z_n < \cdots < z_1 < x_1.$$

By Rolle's theorem, for each $n \in \mathbf{N}$, there exists $a_n \in (z_{n+1}, z_n)$ such that $y'(a_n) = 0$. We take $k \in \mathbf{N}$ so large that $t_{1+k} < z_1$. Now we set $b_n = t_{n+k}$ for $n \in \mathbf{N}$. Then $b_1 < z_1$. The Sturm–Picone comparison theorem implies that $y(x)$ has at least $n+1$ zeros in (b_{n+2}, b_1), which means that $b_{n+2} < z_{n+1}$. Since $z_{n+1} < a_n$, we have

$$a_n > z_{n+1} > b_{n+2}, \quad n \in \mathbf{N}.$$

In exactly same way as in the proof in Theorem 2.1, we conclude that (2.3) holds for some $K > 0$. From (2.1) and $a_n > b_{n+2}$, it follows that

$$|y(a_n)| \ge \sqrt{\frac{Kf(a_n)}{p(a_n)q(a_n)}} \ge \sqrt{\frac{Kf(b_{n+2})}{p(b_{n+2})q(b_{n+2})}}, \quad n \in \mathbf{N}.$$

Since $b_n = t_{n+k}$, we have

$$\begin{aligned}
\sum_{n=1}^{N} |y(a_n)| &\ge \sqrt{K} \sum_{n=1}^{N} \sqrt{\frac{f(t_{n+k+2})}{p(t_{n+k+2})q(t_{n+k+2})}} \\
&= \sqrt{K} \sum_{n=k+3}^{N+k+2} \sqrt{\frac{f(t_n)}{p(t_n)q(t_n)}} \\
&= \sqrt{K} \sum_{n=1}^{N+k+2} \sqrt{\frac{f(t_n)}{p(t_n)q(t_n)}} - \sqrt{K} \sum_{n=1}^{k+2} \sqrt{\frac{f(t_n)}{p(t_n)q(t_n)}}.
\end{aligned}$$

Letting $N \to \infty$ and using (3.2), we obtain

$$\sum_{n=1}^{\infty} |y(a_n)| = \infty .$$

Consequently, Proposition 1.1 implies that y is nonrectifiable oscillatory near $x = 0$. $\square$

From Theorem 3.1, we have the following result.

Theorem 3.2. *Assume that there exist $f \in C^1(0, x_0]$ and μ, $\sigma \in \mathbf{R}$ such that $\mu + \sigma > 2$,* (1.4) *and* (1.5) *hold, $f(x)$ satisfies* (2.1) *and*

$$\limsup_{x \to +0} x^{\mu+\sigma-2} \frac{p(x)q(x)}{f(x)} < \infty . \tag{3.3}$$

Then every nontrivial solution of (1.1) *is nonrectifiable oscillatory near $x = 0$.*

Proof. By (1.4) and (1.5), there exist $c_1 > 0$ and $c_2 > 0$ such that

$$x^{-\mu} p(x) < c_1 , \quad x^{\sigma} q(x) > c_2 , \quad x \in (0, x_0] . \tag{3.4}$$

We define $r(x)$, $s(x)$ and $w(x)$ by

$$r(x) = c_1 x^{\mu} , \quad s(x) = x^{-\sigma} \left[\frac{c_2}{2} - \frac{c_1}{16} (\sigma - \mu)(\sigma + 3\mu - 4) x^{\mu+\sigma-2} \right]$$

and

$$w(x) = x^{\frac{\sigma-\mu}{4}} \sin \left(\gamma x^{-\frac{\mu+\sigma}{2}+1} \right) ,$$

respectively, where

$$\gamma = \frac{2\sqrt{c_2}}{(\mu + \sigma - 2)\sqrt{2c_1}} .$$

Then w is an oscillatory solution near $x = 0$ of (3.1). Now we set

$$t_n = \left(\frac{n\pi}{\gamma} \right)^{-\frac{2}{\mu+\sigma-2}} , \quad n \in \mathbf{N} .$$

Then $w(t_n) = 0$ for $n \in \mathbf{N}$. By $\mu + \sigma > 2$ and (3.4), there exists $x_1 \in (0, x_0]$ such that

$$s(x) \le c_2 x^{-\sigma} < q(x) , \quad x \in (0, x_1] .$$

Moreover, from (3.4), it follows that $p(x) < r(x)$ for $x \in (0, x_0]$. By (3.3), there exists $c_3 > 0$ such that

$$x^{\mu+\sigma-2} \frac{p(x)q(x)}{f(x)} \le c_3 , \quad x \in (0, x_0] .$$

Therefore,

$$\sqrt{\frac{f(t_n)}{p(t_n)q(t_n)}} \ge \frac{1}{\sqrt{c_3}} (t_n)^{\frac{\mu+\sigma-2}{2}} = \frac{\gamma}{\sqrt{c_3}\pi n} , \quad n \ge n_1$$

for some n_1, which means that (3.2) is satisfied. Consequently, Theorem 3.1 implies that every nontrivial solution of (1.1) is nonrectifiable oscillatory near $x = 0$. $\square$

Proof of Theorem 1.2. Letting $f(x) = x^{\mu+\sigma-2} p(x)q(x)$ in Theorem 3.2, we have Theorem 1.2. $\square$

4. Proof of the third main result

In this section we prove Theorem 1.3.

Theorem 4.1. *Assume that there exist* $f \in C^1(0, x_0]$ *and* a, $b \in \mathbf{R}$ *such that* $a+b>0$, (1.6) *and* (1.7) *hold and* $f(x)$ *satisfies* (2.1),

$$\limsup_{x\to+0} e^{-\frac{a+b}{x}} \frac{p(x)q(x)}{f(x)} < \infty \,. \tag{4.1}$$

Then every nontrivial solution of (1.1) *is nonrectifiable oscillatory near* $x=0$.

Proof. By (1.6), there exists $c_1 > 0$ such that

$$e^{\frac{a}{x}} p(x) < c_1 \,, \quad x \in (0, x_0] \,. \tag{4.2}$$

By (1.7), there exists $x_1 \in (0, x_0]$ such that

$$x^4 e^{-\frac{b}{x}} q(x) > \frac{(a+b)^2}{2} c_1 \,, \quad x \in (0, x_1] \,. \tag{4.3}$$

We define $r(x)$, $s(x)$ and $w(x)$ by

$$r(x) = c_1 e^{-\frac{a}{x}} \,,$$

$$s(x) = \frac{c_1}{x^4} e^{\frac{b}{x}} \Big[\frac{(a+b)^2}{4} + \Big(\frac{(a-b)(3a+b)}{16} - ax \Big) e^{-\frac{a+b}{x}} \Big]$$

and

$$w(x) = x e^{\frac{a-b}{4x}} \sin\big(e^{\frac{a+b}{2x}}\big) \,,$$

respectively. Then w is an oscillatory solution near $x=0$ of (3.1). Now we set

$$t_n = \frac{a+b}{2\log n\pi} \,, \quad n \in \mathbf{N} \,.$$

Then $w(t_n) = 0$ for $n \in \mathbf{N}$. By (4.3), there exists $x_2 \in (0, x_1]$ such that

$$s(x) < \frac{c_1}{x^4} e^{\frac{b}{x}} \frac{(a+b)^2}{2} < q(x) \,, \quad x \in (0, x_2] \,.$$

Moreover, from (4.2), it follows that $p(x) < r(x)$ for $x \in (0, x_0]$. By (4.1), there exists $c_3 > 0$ such that

$$e^{-\frac{a+b}{x}} \frac{p(x)q(x)}{f(x)} \le c_3 \,, \quad x \in (0, x_0] \,. \tag{4.4}$$

Therefore,

$$\begin{aligned} \sqrt{\frac{f(t_n)}{p(t_n)q(t_n)}} &\ge \frac{1}{\sqrt{c_3}} \exp\Big(-\frac{a+b}{2t_n} \Big) \\ &= \frac{1}{\sqrt{c_3}} \exp(-\log n\pi) = \frac{1}{\sqrt{c_3}\pi n} \,, \quad n \ge n_1 \end{aligned}$$

for some n_1, which means that (3.2) is satisfied. Consequently, Theorem 3.1 implies that every nontrivial solution of (1.1) is nonrectifiable oscillatory near $x=0$. □

Proof of Theorem 1.3. Setting $f(x) = e^{-\frac{a+b}{x}} p(x)q(x)$ in Theorem 4.1, we obtain Theorem 1.3. □

Acknowledgement. The authors would like to thank the referee for carefully reading our manuscript and for giving positive, constructive and useful comments.

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