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ON THE CONSTRUCTION AND THE REALIZATION OF WILD MONOIDS

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Abstract. We develop elementary methods of computing the monoid $\boldsymbol{\mathcal{V}}(\boldsymbol{R})$ for a directly-finite regular ring $\boldsymbol{R}$. We construct a class of directly finite non-cancellative refinement monoids and realize them by regular algebras over an arbitrary field.

1. Introduction

The commutative monoid $\boldsymbol{\mathcal{V}}(\boldsymbol{R})$, assigned to a unital associative ring $\boldsymbol{R}$, consists of all isomorphism classes of finitely generated projective right $\boldsymbol{R}$-modules, with the operation induced from direct sums. Alternatively, the monoid $\boldsymbol{\mathcal{V}}(\boldsymbol{R})$ is defined as *Murray-von Neumann equivalence* classes of idempotent $\omega \times \omega$-matrices with finitely many nonzero entries over $\boldsymbol{R}$.

For a von Neumann regular ring $\boldsymbol{R}$, the monoid $\boldsymbol{\mathcal{V}}(\boldsymbol{R})$ faithfully reflects the structure of the ring. Not surprisingly, many of direct sum decomposition problems of von Neumann regular rings have reformulation in terms of the corresponding refinement monoids. Let us mention the separativity problem whether there are non-isomorphic finitely generated projective right $\boldsymbol{R}$-modules $\boldsymbol{\mathcal{M}}$, $\boldsymbol{\mathcal{N}}$ such that $\boldsymbol{\mathcal{M}} \oplus \boldsymbol{\mathcal{M}} \simeq \boldsymbol{\mathcal{M}} \oplus \boldsymbol{\mathcal{N}} \simeq \boldsymbol{\mathcal{N}} \oplus \boldsymbol{\mathcal{N}}$ as a prominent example (cf. [9, Problem 1]).

If $\boldsymbol{R}$ is a von Neumann regular ring or a C^*-algebra with real rank zero, then the monoid $\boldsymbol{\mathcal{V}}(\boldsymbol{R})$ satisfies the *Riesz refinement property. The realization problem* [10] asks which refienement monoids are realized as $\boldsymbol{\mathcal{V}}(\boldsymbol{R})$ of von Neumann regular rings. As shows an example of F. Wehrung [12, Corollary 2.12], not all of them. But the size of the Wehrung's counter-example is $\geq \aleph_2$, which leaves the realization problem open for refinement monoids of smaller cardinalities. The countable case is particularly important for the direct sum decomposition problems of von Neumann rings are usually reduced to realization problems of certain countable refinement monoids.

There are classes of refinement monoids for which the realization problem has a positive solution. The monoids $\boldsymbol{\mathcal{M}}(E)$ associated to row-finite directed graphs

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(cf. [6]) are realized functorially in [3]. The method used in [3] is extended in [2], where finitely generated primitive monoids are realized. We refer to [1] for a survey on this result.

The refinement monoids obtained by these canonical constructions have a common feature; they are direct limits of finitely generated refinement monoids. Such refinement monoids are called *tame* in [4]. The remaining ones are *wild*. Two examples of wild monoids $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ are studied in detail in [4] and realized in [5].

The refinement monoid $\mathfrak{M}$ is non-cancellative but admits faithful state, consequently, it cannot be realized as $\mathcal{V}(\boldsymbol{R})$ for any von Neumann regular algebra over an uncountable field [2, Proposition 4.1]. Surprisingly, $\mathfrak{M}$ is realized by an exchange algebra over any field with involution [5, Theorem 4.10] as well as a regular algebra over a countable field [5, Theorem 5.5]. Note that the first such example goes back to [7].

The monoid $\overline{\mathfrak{M}}$ is a factor of $\mathfrak{M}$ by an o-ideal and it is isomorphic to $\mathcal{V}(\boldsymbol{S})$ for a regular algebra $\boldsymbol{S}$ invented by Bergman and Goodearl [9, Example 5.10]. It is in some sense a canonical example of a wild monoid. The modification of this construction was used by Moncasi who constructed a directly finite regular Hermite ring such that $K_0(\mathbf{R})$ is not a Riesz group [11], in particular, the monoid $\mathcal{V}(\boldsymbol{R})$ does not satisfy the Riesz interpolation property. Modifications of the Bergman-Goodearl construction play a crucial role also in this paper.

The paper consists of three parts. Firstly, we develop quite elementary but useful methods of computing the monoid $\mathcal{V}(\boldsymbol{R})$ for a regular ring $\boldsymbol{R}$. We define a *partial H-map* from a hereditary subset H of a monoid and we understand when the partial H-map map uniquely extends to a monoid isomorphism. This idea leads to Lemma 3.5 that allows us to compute the monoid $\mathcal{V}(\boldsymbol{R})$ of a regular ring $\boldsymbol{R}$ knowing the structure of the partial monoid of its finitely generated right ideals. We refine Lemma 3.5 in Corollary 3.9, which is designed to compute $\mathcal{V}(\boldsymbol{R})$ of directly finite regular rings $\boldsymbol{R}$; in this case it suffices to describe the ordered set of traces of idempotents of the ring $\boldsymbol{R}$.

In the second part of the paper, consisting of Sections 4 and 5, we construct a class of directly finite non-cancellative refinement monoids. In Sections 4 we aim to construct a class of refinement monoids rich enough to provide interesting examples with potential of further applications. In Section 5 we restrict ourselves to particular refinement monoids $\mathfrak{B}_{2n}$, $n \in \mathbb{N}$, obtained by the previous construction. We prove that the monoids $\mathfrak{B}_{2n}$ for $n \geq 2$, do not satisfy the Riesz interpolation property.

The remaining Sections 6–8 are devoted to construction of regular rings $\boldsymbol{R}_{2n}$ and the proof that $\mathcal{V}(\mathbf{R}_{2n}) \simeq \mathfrak{B}_{2n}$, for all positive integers n. The auxiliary Section 6 is elementary linear algebra. In Section 7 we recall the Goodearl's modification [9, Example 5.10] of the Bergman's example, denoted by $\boldsymbol{R}_2$, and we prove that $\mathfrak{B}_2 \simeq \mathcal{V}(\mathbf{R}_2)$. In the final Section 8, we generalize the constructions of Bergman and Goodearl. This results in rings $\boldsymbol{R}_{2n}$ such that $\mathcal{V}(\mathbf{R}_{2n}) \simeq \mathfrak{B}_{2n}$.

2. Basic concepts

We denote by $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$, the set of all, all positive, and all non-negative integers, respectively.

2.1. **Refinement monoids.** Within the paper, all the monoids are supposed to be commutative. Let $\mathcal{M}$ be a monoid. We will use $\leq_{\mathcal{M}}$ to denote the *algebraic preorder* of $\mathcal{M}$ defined by $x \leq_{\mathcal{M}} y$ if and only if there is $u \in \mathcal{M}$ such that $x + u = y$, for all x, $y \in \mathcal{M}$. We denote by $\equiv_{\mathcal{M}}$ the equivalence relation induced by by the algebraic preorder $\leq_{\mathcal{M}}$, defined by $x \equiv_{\mathcal{M}} y$ provided that $x \leq_{\mathcal{M}} y$ and $y \leq_{\mathcal{M}} x$, for all x, $y \in \mathcal{M}$.

A subset H of a monoid $\mathcal{M}$ is called *hereditary* provided that $y \in H$ and $x \leq_{\mathcal{M}} y$ implies that $x \in H$, for all x, $y \in \mathcal{M}$. Given a subset X of the monoid $\mathcal{M}$, we set

$$\downarrow(X)_{\mathcal{M}} := \{x \in \mathcal{M} \mid \exists y \in X : x \leq_{\mathcal{M}} y\}\,.$$

Thus $\downarrow(X)_{\mathcal{M}}$ is the least hereditary subset of $\mathcal{M}$ containing X. A hereditary submonoid of the monoid $\mathcal{M}$ will be called an *o-ideal* of $\mathcal{M}$. We will denote by $\mathcal{O}(X)_{\mathcal{M}}$ the least o-ideal of $\mathcal{M}$ containing the set X, i.e.,

$$\mathcal{O}(X)_{\mathcal{M}} := \{x \in \mathcal{M} \mid \exists\, y_1, \dots, y_n \in X : x \leq_{\mathcal{M}} y_1 + \cdots + y_n\}\,.$$

When $X = \{x\}$ is a singleton set, we will write shortly $\downarrow(x)_{\mathcal{M}}$ and $\mathcal{O}(x)_{\mathcal{M}}$. An element $\boldsymbol{u} \in \mathcal{M}$ is an *order unit* of $\mathcal{M}$ provided that $\mathcal{O}(\boldsymbol{u})_{\mathcal{M}} = \mathcal{M}$; equivalently, there is a positive integer λ such that $x \leq_{\mathcal{M}} \lambda \boldsymbol{u}$, for each $x \in \mathcal{M}$.

A monoid $\mathcal{M}$ is *conical* provided that $x + y = 0 \implies x = y = 0$, for all x, $y \in \mathcal{M}$. A monoid $\mathcal{M}$ satisfies the *Riesz refinement property* provided that whenever $x_1 + x_2 = y_1 + y_2$ in $\mathcal{M}$, there are elements $z_{ij} \in \mathcal{M}$, i, $j = 1, 2$, such that

$$x_i = z_{i1} + z_{i2} \quad \text{and} \quad y_j = z_{1j} + z_{2j} \quad \text{for all} \quad i, j = 1, 2\,. \tag{2.1}$$

A *refinement monoid* is a conical monoid satisfying the Riesz refinement property. A monoid $\mathcal{M}$ satisfies the *interpolation property* provided that for all x_i, $y_j \in \mathcal{M}$, $i, j = 1, 2$, with $x_i \leq_{\mathcal{M}} y_j$, for all i, $j \in \{1, 2\}$, there is $z \in \mathcal{M}$ such that $x_i \leq_{\mathcal{M}} z \leq_{\mathcal{M}} y_j$, for all i, $j \in 1, 2$. A cancellative conical monoid is a refinement monoid if and only if it satisfies the interpolation property [8, Proposition 2.1]. In general, there are refinement monoids that do not satisfy the interpolation property (cf. [11] and Section 4).

2.2. **Rings and modules.** A ring $\boldsymbol{R}$ is (*von Neumann*) *regular*[1] provided that for every $a \in \boldsymbol{R}$ there is $b \in \boldsymbol{R}$ such that $aba = a$. There are many characterizations of regular rings. Probably the most prominent one is that a ring $\boldsymbol{R}$ is regular if and only if each right (resp. left) finitely generated ideal of $\boldsymbol{R}$ is generated by an idempotent [9, Theorem 1.1].

Given a ring $\boldsymbol{R}$, we denote by proj$-\boldsymbol{R}$ the class of all finitely generated projective right $\boldsymbol{R}$-modules. Given $\boldsymbol{R}$-modules $\boldsymbol{A}$ and $\boldsymbol{B}$, the notation $\boldsymbol{A} \leq \boldsymbol{B}$ means that $\boldsymbol{A}$ is a submodule of $\boldsymbol{B}$ and $\boldsymbol{A} \lesssim \boldsymbol{B}$ denotes that the module $\boldsymbol{A}$ is isomorphic to a

[1] It is common to shorten the title by dropping *von Neumann* and call the von Neumann regular rings just *regular* (cf. [9]). In the paper we will follow this custom.

submodule of $\boldsymbol{B}$. We will use the notation $\boldsymbol{A} \leq^{\oplus} \boldsymbol{B}$, resp. $\boldsymbol{A} \lesssim^{\oplus} \boldsymbol{B}$, to denote that $\boldsymbol{A}$ is a direct summand of $\boldsymbol{B}$, resp. that $\boldsymbol{A}$ is isomorphic to a direct summand of $\boldsymbol{B}$.

An element e of a ring $\boldsymbol{R}$ is an *idempotent* if $e = ee$. We denote by $\operatorname{Idem}(\boldsymbol{R})$ the set of all idempotents in the ring $\boldsymbol{R}$. Idempotents e and f are *orthogonal* provided that $ef = fe = 0$.

Given a ring $\boldsymbol{R}$ and right $\boldsymbol{R}$-modules $\boldsymbol{A}$ and $\boldsymbol{B}$, we denote by $\hom_{\boldsymbol{R}}(\boldsymbol{A}, \boldsymbol{B})$ the set of all $\boldsymbol{R}$-linear maps $\boldsymbol{A} \to \boldsymbol{B}$. We denote by 0 the zero monoid, the zero module, the zero vector space, depending on the context.

3. Partial H-maps and their applications

Let $\mathcal{M}$, $\mathcal{N}$ be monoids and H a hereditary subset of $\mathcal{M}$. A *partial H-map* is a one-to-one map $\alpha\colon H \to \mathcal{N}$ such that for all $z \in H$ and all u, $v \in \mathcal{N}$, the equality $\alpha(z) = u + v$ holds true if and only if there are (necessarily unique) x, $y \in H$ such that $\alpha(x) = u$, $\alpha(y) = v$ and $x + y = z$.

By induction we readily prove that if $\alpha\colon H \to \mathcal{N}$ is a partial H-map, then for all $x \in \mathcal{M}$, all $n \in \mathbb{N}$ and all $u_1, \dots, u_n \in \mathcal{N}$: $\alpha(x) = u_1 + \cdots + u_n$ if and only if $x = x_1 + \cdots + x_n$ for (necessarily unique) $x_i \in H$, $i \in \{1, 2, \dots, n\}$, such that $u_i = \alpha(x_i)$ for all $i = \{1, 2, \dots, n\}$.

Lemma 3.1. *Let $\mathcal{M}$, $\mathcal{N}$ be monoids and let H be a hereditary subset of $\mathcal{M}$. If $\alpha\colon H \to \mathcal{N}$ is a partial H-map then for all x, y, $z \in H$:*

$$z = x + y \iff \alpha(z) = \alpha(x) + \alpha(y)\,.$$

Proof. If $z = x + y$, then $\alpha(z) = \alpha(x) + \alpha(y)$ readily by the definition of a partial H-map. Conversely, the equality $\alpha(z) = \alpha(x) + \alpha(y)$ implies that there are x', $y' \in H$ such that $z = x' + y'$, $\alpha(x) = \alpha(x')$ and $\alpha(y) = \alpha(y')$. Since a partial H-map is by definition one-to-one, we conclude that $x = x'$ and $y = y'$. □

Keeping the setting of Lemma 3.1, we get by induction that for every $n \in \mathbb{N}$ and all x, $y_1, \dots, y_n \in H$:

$$x = \sum_{i=1}^{n} y_i \iff \alpha(x) = \sum_{i=1}^{n} \alpha(y_i)\,. \tag{3.1}$$

Lemma 3.2. *Let $\mathcal{M}$, $\mathcal{N}$ be refinement monoids and H a hereditary subset of $\mathcal{M}$. Then every partial H-map $\alpha\colon H \to \mathcal{N}$ extends to a unique isomorphism $\beta\colon \mathcal{O}(H)_{\mathcal{M}} \to \mathcal{O}(\alpha(H))_{\mathcal{N}}$.*

Proof. By the definition, for every $x \in \mathcal{O}(H)_{\mathcal{M}}$ there are $n \in \mathbb{N}$ and $y_1, \dots, y_n \in H$ with $x \leq_{\mathcal{M}} y_1 + \cdots + y_n$. Since $\mathcal{M}$ is a refinement monoid, there are $x_i \leq_{\mathcal{M}} y_i$, $i = 1, \dots, n$, such that $x = x_1 + \cdots + x_n$. We define a map $\beta\colon \mathcal{O}(H)_{\mathcal{M}} \to \mathcal{N}$ by $x \mapsto \alpha(x_1) + \cdots + \alpha(x_n)$.

Claim 1. *The map β is a well-defined monoid homomorphism.*

Proof of Claim 1. Let $x_1 + \cdots + x_m = y_1 + \cdots + y_n$ for some m, $n \in \mathbb{N}$ and $x_1, \dots, x_m$, $y_1, \dots, y_n \in H$. Since $\mathcal{M}$ is a refinement monoid, there are $z_{ij} \in H$ such that $x_i = \sum_{j=1}^{n} z_{ij}$ for all $i \leq m$ and $y_j = \sum_{i=1}^{m} z_{ij}$ for all $j \leq n$. By (3.1) we have that $\alpha(x_i) = \sum_{j=1}^{n} \alpha(z_{ij})$ for all $i \leq m$ and $\alpha(y_j) = \sum_{i=1}^{m} \alpha(z_{ij})$ for all $j \leq n$. It follows that

$$\sum_{i=1}^{m} \alpha(x_i) = \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha(z_{ij}) = \sum_{j=1}^{n}\sum_{i=1}^{m} \alpha(z_{ij}) = \sum_{j=1}^{n} \alpha(y_j)\,.$$

Thus the map $\beta\colon \mathcal{O}(H)_{\mathbf{M}} \to N$ is well-defined. It is straightforward that it is a monoid homomorphism. $\square_{\text{Claim}}$

Claim 2. *The homomorphism β is one-to-one.*

Proof of Claim 2. Suppose that $\beta(x) = \beta(y)$ for some x, $y \in \mathcal{O}(H)_{\mathbf{M}}$. By the definition, there are m, $n \in \mathbb{N}$ and $x'_1, \dots, x'_m$, $y'_1, \dots, y'_n \in H$ such that $x \leq_{\mathbf{M}} x'_1 + \cdots + x'_m$ and $y \leq_{\mathbf{M}} y'_1 + \cdots + y'_n$. Since $\mathbf{M}$ is a refinement monoid, there are $x_i \leq_{\mathbf{M}} x'_i$, $i = 1, \dots, m$, and $y_j \leq_{\mathbf{M}} y'_j$, $j = 1, \dots, n$, in H such that $x = x_1 + \cdots + x_m$ and $y = y_1 + \cdots + y_n$. Since $\beta(x) = \beta(y)$, we get that $\sum_{i=1}^{m} \alpha(x_i) = \sum_{j=1}^{n} \alpha(y_j)$. Since $\mathbf{N}$ is a refinement monoid, there are $w_{i,j} \in \mathbf{N}$ such that $\alpha(x_i) = \sum_{j=1}^{n} w_{i,j}$, for all $i = 1, \dots, m$, and $\alpha(y_j) = \sum_{i=1}^{m} w_{i,j}$, for all $j = 1, \dots, n$. Since α is a partial H-map, there are elements $z_{i,j} \in H$ such that

$$w_{i,j} = \alpha(z_{i,j})\,, \quad \forall \quad i \in \{1, 2, \dots, m\},\ j \in \{1, 2, \dots, n\}\,, \tag{3.2}$$

and

$$x_i = \sum_{j=1}^{n} z_{i,j}\,, \quad \forall \quad i \in \{1, 2, \dots, m\}\,.$$

Applying that α is a partial H-map again, we infer that there are elements $z'_{i,j} \in H$ such that

$$w_{i,j} = \alpha(z'_{i,j})\,, \quad \forall \quad i \in \{1, 2, \dots, m\},\ j \in \{1, 2, \dots, n\}\,, \tag{3.3}$$

and

$$y_j = \sum_{i=1}^{m} z'_{i,j}\,, \quad \forall \quad j \in \{1, 2, \dots, n\}.$$

Since the map $\alpha\colon H \to \mathbf{N}$ is by definition one-to-one, we get from (3.2) and (3.3) that $z_{i,j} = z'_{i,j}$ for all $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$. It follows that

$$x = \sum_{i=1}^{m} x_i = \sum_{i=1}^{m}\sum_{j=1}^{n} z_{i,j} = \sum_{j=1}^{n}\sum_{i=1}^{m} z'_{i,j} = \sum_{j=1}^{n} y_j = y\,.$$

This proves that β is one-to-one. $\square_{\text{Claim}}$

Claim 3. *The equality $\beta(\mathcal{O}(H)_{\mathbf{M}}) = \mathcal{O}(\alpha(H))_{\mathbf{N}}$ holds true.*

Proof of Claim 3. As we have shown above, each $x \in \mathcal{O}(H)_{\mathbf{M}}$ is a sum of elements from H. It follows that $\beta(\mathcal{O}(H)_{\mathbf{M}}) \subseteq \mathcal{O}(\alpha(H))_{\mathbf{N}}$. It is straightforward to see from the definition of a partial H-map, that the image $\alpha(H)$ is a hereditary subset of $\mathbf{N}$. Since $\mathbf{N}$ is a refinement monoid, each element of $\mathcal{O}(\alpha(H))_{\mathbf{N}}$ is a sum of elements of $\downarrow(\alpha(H))_{\mathbf{N}}$. Therefore $\mathcal{O}(\alpha(H))_{\mathbf{N}}$ is a submonoid of $\mathbf{N}$ generated by $\alpha(H)$. From $H \subseteq \mathcal{O}(H)_{\mathbf{M}}$ we infer that $\alpha(H) \subseteq \beta(\mathcal{O}(H)_{\mathbf{M}})$. Since $\beta(\mathcal{O}(H)_{\mathbf{M}})$ is a submonoid of $\mathbf{N}$, we conclude that $\mathcal{O}(\alpha(H))_{\mathbf{N}} \subseteq \beta(\mathcal{O}(H)_{\mathbf{M}})$. $\square_{\text{Claim}}$

The three claims prove the lemma. □

Corollary 3.3. *Let $\mathcal{M}$, $\mathcal{N}$ be refinement monoids, $\boldsymbol{u} \in \mathcal{M}$, and let $\alpha\colon \downarrow(\boldsymbol{u})_{\mathcal{M}} \to \mathcal{N}$ be a partial $\downarrow(\boldsymbol{u})_{\mathcal{M}}$-map. If $\boldsymbol{u}$ is an order unit in $\mathcal{M}$ and $\alpha(\boldsymbol{u})$ is an order unit in $\mathcal{N}$, then α extends to a unique isomorphism $\beta\colon \mathcal{M} \to \mathcal{N}$.*

Let $\boldsymbol{R}$ be a ring. Given a finitely generated right $\boldsymbol{R}$-module $\boldsymbol{A}$, we denote by $[\boldsymbol{A}]$ the isomorphism class of the module $\boldsymbol{A}$, and by $\mathcal{V}(\boldsymbol{R})$ the monoid of all isomorphism classes of finitely generated projective right $\boldsymbol{R}$-modules with the operation of addition defined by

$$[\boldsymbol{A}] + [\boldsymbol{B}] = [\boldsymbol{A} \oplus \boldsymbol{B}]$$

for all $\boldsymbol{A}$, $\boldsymbol{B} \in \text{proj}-\boldsymbol{R}$. As above, we will use $\leq_{\mathcal{V}(\boldsymbol{R})}$ to denote the algebraic preorder on $\mathcal{V}(\boldsymbol{R})$ and $\equiv_{\mathcal{V}(\boldsymbol{R})}$ to denote the corresponding equivalence relation. If the ring $\boldsymbol{R}$ is regular, then $\mathcal{V}(\boldsymbol{R})$ is a refinement monoid due to [9, Theorem 2.8].

Lemma 3.4. *Let $\boldsymbol{R}$ be a ring and $\boldsymbol{A}$, $\boldsymbol{B}$ finitely generated right R-modules. Then $[\boldsymbol{A}] + [\boldsymbol{B}] \leq_{\mathcal{V}(\boldsymbol{R})} [\boldsymbol{R}]$ if and only if there are orthogonal idempotents e, $f \in \boldsymbol{R}$ such that $[e\boldsymbol{R}] = [\boldsymbol{A}]$ and $[f\boldsymbol{R}] = [\boldsymbol{B}]$.*

Proof. ($\Leftarrow$) Let e, f be orthogonal idempotents such that $e\boldsymbol{R} \simeq \boldsymbol{A}$ and $f\boldsymbol{R} \simeq \boldsymbol{B}$. Since the idempotents e and f are orthogonal, $\boldsymbol{R} = e\boldsymbol{R} \oplus f\boldsymbol{R} \oplus (1 - e - f)\boldsymbol{R}$. Therefore $\boldsymbol{A} \oplus \boldsymbol{B} \lesssim^{\oplus} \boldsymbol{R}$, hence $[\boldsymbol{A}] + [\boldsymbol{B}] \leq_{\mathcal{V}(\boldsymbol{R})} [\boldsymbol{R}]$. ($\Rightarrow$) By the assumption $[\boldsymbol{A}] + [\boldsymbol{B}] \leq_{\mathcal{V}(\boldsymbol{R})} [\boldsymbol{R}]$, hence $\boldsymbol{A} \oplus \boldsymbol{B} \lesssim^{\oplus} \boldsymbol{R}$. It follows that $\boldsymbol{R} = \boldsymbol{A}' \oplus \boldsymbol{B}' \oplus \boldsymbol{C}$ for some $\boldsymbol{A}' \simeq \boldsymbol{A}$ and $\boldsymbol{B}' \simeq \boldsymbol{B}$. The projection $\boldsymbol{R} \to \boldsymbol{A}'$ with the kernel $\boldsymbol{B}' \oplus \boldsymbol{C}$ corresponds to a left multiplication by an idempotent, say e. Similarly, the projection $\boldsymbol{R} \to \boldsymbol{B}'$ with the kernel $\boldsymbol{A}' \oplus \boldsymbol{C}$ corresponds to a left multiplication by an idempotent, say f. As the composition of these projections, in whatever order, is the zero endomorphism, the idempotents e and f are orthogonal. Clearly $e\boldsymbol{R} = \boldsymbol{A}' \simeq \boldsymbol{A}$ and $f\boldsymbol{R} = \boldsymbol{B}' \simeq \boldsymbol{B}$, hence $[e\boldsymbol{R}] = [\boldsymbol{A}]$ and $[f\boldsymbol{R}] = [\boldsymbol{B}]$. □

Lemma 3.5. *Let $\boldsymbol{R}$ be a regular ring, $\mathcal{N}$ a refinement monoid, and $\gamma\colon \mathrm{Idem}(\boldsymbol{R}) \to \mathcal{N}$ a map satisfying:*

(1) *$\gamma(e) = \gamma(f) \iff [e\boldsymbol{R}] = [f\boldsymbol{R}]$, for all e, $f \in \mathrm{Idem}(\boldsymbol{R})$.*
(2) *The equality $x + y = \gamma(g)$ holds true for some x, $y \in \mathcal{N}$ and $g \in \mathrm{Idem}(\boldsymbol{R})$ if and only if there are orthogonal idempotents e, $f \in \boldsymbol{R}$ such that $\gamma(e) = x$, $\gamma(f) = y$, and $e + f = g$.*
(3) *$\gamma(1)$ is an order unit in $\mathcal{N}$.*

Then the map $\alpha\colon \{[e\boldsymbol{R}] \mid e \in \mathrm{Idem}(\boldsymbol{R})\} \to \mathcal{N}$ given by the correspondence $[e\boldsymbol{R}] \mapsto \gamma(e)$ extends to a (unique) isomorphism $\beta\colon \mathcal{V}(\boldsymbol{R}) \to \mathcal{N}$.

Proof. Put $\mathcal{M} := \{[e\boldsymbol{R}] \mid e \in \mathrm{Idem}(\boldsymbol{R})\}$. It follows from (1) that the $\alpha\colon \mathcal{M} \to \mathcal{N}$ given by $[e\boldsymbol{R}] \mapsto \gamma(e)$ is a well-defined one-to-one map. In view of Lemma 3.4 property (2) implies that α is a partial $\downarrow([\boldsymbol{R}])_{\mathcal{V}([\boldsymbol{R}])}$-map. Property (3) says that $\alpha([R])$ is an order unit in $\mathcal{N}$ and since $[R]$ is clearly an order unit in $\mathcal{V}([\boldsymbol{R}])$, the map α extends to a (unique) isomorphism $\beta\colon \mathcal{V}(\boldsymbol{R}) \to \mathcal{N}$ due to Corollary 3.3. □

We set
$$\operatorname{tr}_{\boldsymbol{R}}(b) := \{abc \mid a, c \in \boldsymbol{R}\} = \bigcup_{a \in \boldsymbol{R}} ab\boldsymbol{R} = \bigcup_{c \in \boldsymbol{R}} \boldsymbol{R}bc\,,$$
for every $b \in \boldsymbol{R}$.

Lemma 3.6. *Let e and f be idempotents of a ring $\boldsymbol{R}$. Then*
$$[e\boldsymbol{R}] \leq [f\boldsymbol{R}] \iff \operatorname{tr}_{\boldsymbol{R}}(e) \subseteq \operatorname{tr}_{\boldsymbol{R}}(f)\,. \tag{3.4}$$

Proof. ($\Rightarrow$) Suppose that $[e\boldsymbol{R}] \leq [f\boldsymbol{R}]$. Then $e\boldsymbol{R} \lesssim^{\oplus} f\boldsymbol{R}$, by the definition. It follows that there is a surjective homomorphism $\varphi\colon f\boldsymbol{R} \to e\boldsymbol{R}$. Since f is an idempotent, φ extends to a homomorphism $\Phi\colon \boldsymbol{R} \to e\boldsymbol{R}$. The homomorphism Φ corresponds to a left multiplication by an element $a = \Phi(1) \in \boldsymbol{R}$. It follows that $e\boldsymbol{R} = af\boldsymbol{R}$, and consequently $\operatorname{tr}_{\boldsymbol{R}}(e) \subseteq \operatorname{tr}_{\boldsymbol{R}}(f)$.

($\Leftarrow$) If $\operatorname{tr}_{\boldsymbol{R}}(e) \subseteq \operatorname{tr}_{\boldsymbol{R}}(f)$, then $e \in af\boldsymbol{R}$ for some $a \in \boldsymbol{R}$. Since e is an idempotent, the left multiplication by ea determines a surjective map $f\boldsymbol{R} \to e\boldsymbol{R}$. Since $e\boldsymbol{R}$ is a projective right $\boldsymbol{R}$-module, we infer that $e\boldsymbol{R} \lesssim^{\oplus} f\boldsymbol{R}$. Therefore $[e\boldsymbol{R}] \leq [f\boldsymbol{R}]$. ☐

A right $\boldsymbol{R}$-module $\boldsymbol{A}$ is *directly finite* provided that $\boldsymbol{A} \simeq \boldsymbol{A} \oplus \boldsymbol{B}$ implies that $\boldsymbol{B} = 0$ for all right $\boldsymbol{R}$-modules $\boldsymbol{B}$, i.e, the module $\boldsymbol{A}$ it is not isomorphic to any of its proper direct summands [9, page 49]. A ring $\boldsymbol{R}$ is *directly finite* if it is directly finite as a right $\boldsymbol{R}$-module. Note that this notion is left-right symmetric as a ring $\boldsymbol{R}$ is directly finite if and only if $ab = 1$ implies $ba = 1$ for all a, $b \in \boldsymbol{R}$ (cf. [9, Lemma 5.1]).

Lemma 3.7. *If a ring $\boldsymbol{R}$ is directly finite then*
$$[e\boldsymbol{R}] \equiv_{\mathcal{V}(\boldsymbol{R})} [f\boldsymbol{R}] \implies [e\boldsymbol{R}] = [f\boldsymbol{R}]\,,$$
for all $e, f \in \operatorname{Idem}(\boldsymbol{R})$.

Proof. Let e, $f \in \operatorname{Idem}(\boldsymbol{R})$ and suppose that $[e\boldsymbol{R}] \equiv_{\mathcal{V}(\boldsymbol{R})} [f\boldsymbol{R}]$. Then there are $\boldsymbol{A}$, $\boldsymbol{B} \in \operatorname{proj}-\boldsymbol{R}$ such that $[f\boldsymbol{R}] = [e\boldsymbol{R}] + [\boldsymbol{A}]$ and $[e\boldsymbol{R}] = [f\boldsymbol{R}] + [\boldsymbol{B}]$, i.e., $f\boldsymbol{R} \simeq e\boldsymbol{R} \oplus \boldsymbol{A}$ and $e\boldsymbol{R} \simeq f\boldsymbol{R} \oplus \boldsymbol{B}$. It follows that
$$e\boldsymbol{R} = f\boldsymbol{R} \oplus \boldsymbol{B} \simeq e\boldsymbol{R} \oplus \boldsymbol{A} \oplus \boldsymbol{B}\,,$$
hence
$$\boldsymbol{R} = (1-e)\boldsymbol{R} \oplus e\boldsymbol{R} \simeq (1-e)\boldsymbol{R} \oplus e\boldsymbol{R} \oplus \boldsymbol{A} \oplus \boldsymbol{B} = \boldsymbol{R} \oplus \boldsymbol{A} \oplus \boldsymbol{B}\,.$$
Since the ring $\boldsymbol{R}$ is directly finite, we conclude that $\boldsymbol{A} = \boldsymbol{B} = 0$, hence $e\boldsymbol{R} \simeq f\boldsymbol{R}$, whence $[e\boldsymbol{R}] = [f\boldsymbol{R}]$ ☐

Applying Lemma 3.6 we get that

Corollary 3.8. *Let $\boldsymbol{R}$ be a directly finite ring. Then*
$$[e\boldsymbol{R}] = [f\boldsymbol{R}] \iff \operatorname{tr}_{\boldsymbol{R}}(e) = \operatorname{tr}_{\boldsymbol{R}}(f)\,,$$
for all e, $f \in \operatorname{Idem}(\boldsymbol{R})$.

Combining Lemma 3.5 and Corollary 3.8 we conclude with

Corollary 3.9. *Let $\boldsymbol{R}$ be a directly finite regular ring, let $\mathbf{N}$ be a refinement monoid, and let $\gamma\colon \operatorname{Idem}(\boldsymbol{R}) \to \mathbf{N}$ be a map satisfying:*

(1) *$\gamma(e) = \gamma(f) \iff \mathrm{tr}_{\boldsymbol{R}}(e) = \mathrm{tr}_{\boldsymbol{R}}(f)$, for all e, $f \in \mathrm{Idem}(\boldsymbol{R})$.*
(2) *The equality $x + y = \gamma(g)$ holds true for some x, $y \in \mathfrak{N}$ and $g \in \mathrm{Idem}(\boldsymbol{R})$ if and only if there are orthogonal idempotents e, $f \in \boldsymbol{R}$ such that $\gamma(e) = x$, $\gamma(f) = y$, and $e + f = g$.*
(3) *$\gamma(1)$ is an order unit in $\mathfrak{N}$.*

Then the map $\alpha\colon \{[e\boldsymbol{R}] \mid e \in \mathrm{Idem}(\boldsymbol{R})\} \to \mathfrak{N}$ given by the correspondence $[e\boldsymbol{R}] \mapsto \gamma(e)$ extends to a (unique) isomorphism $\beta\colon \boldsymbol{\mathcal{V}}(\boldsymbol{R}) \to \mathfrak{N}$.

4. Non-cancellative refinement monoids

In this section we recall a construction of refinement monoids that are, under some simple conditions, non cancellative directly finite. It leads to examples that will be realized as $\boldsymbol{\mathcal{V}}(\boldsymbol{R})$ of regular rings, $\boldsymbol{R}$, in the rest of the paper. We seek both simplicity and generality hoping for further applications of the construction.

Definition 4.1. Let $\mathfrak{M}$, $\mathfrak{G}$ be monoids and $\iota\colon \mathfrak{M} \to \mathfrak{G}$ a monoid homomorphism. Given $H \subseteq \mathfrak{M}$ a hereditary subset (w.r.t. the algebraic preorder on $\mathfrak{M}$) and a submonoid $\mathfrak{F}$ of $\mathfrak{G}$, we define a relation $\Theta_H^{\mathfrak{F}}$ on the monoid $\mathfrak{M}$ by

$$x \equiv y\,(\Theta_H^{\mathfrak{F}}) \overset{\mathrm{df}}{\iff} \begin{cases} \iota(x) + p = \iota(y) + q \text{ for some } p, q \in \mathfrak{F} & : x, y \notin H, \\ x = y & : \text{otherwise}, \end{cases} \tag{4.1}$$

for all x, $y \in \mathfrak{M}$.

Lemma 4.2. *Let $\iota\colon \mathfrak{M} \to \mathfrak{G}$ be a monoid homomorphism, H a hereditary subset of $\mathfrak{M}$, and $\mathfrak{F}$ a submonoid of $\mathfrak{G}$. Then the relation $\Theta_H^{\mathfrak{F}}$ defined by* (4.1) *is a congruence of $\mathfrak{M}$.*

Proof. We shall prove separately that $\Theta_H^{\mathfrak{F}}$ is an equivalence relation on $\mathfrak{M}$ and that $\Theta_H^{\mathfrak{F}}$ is compatible with the operation of addition.

Claim 1. *$\Theta_H^{\mathfrak{F}}$ is an equivalence relation.*

Proof of Claim 1. The relation $\Theta_H^{\mathfrak{F}}$ is clearly symmetric and reflexive. Suppose that

$$x \equiv y\,(\Theta_H^{\mathfrak{F}}) \text{ and } y \equiv z\,(\Theta_H^{\mathfrak{F}}) \tag{4.2}$$

for some x, y, $z \in \mathfrak{M}$. Observe from definition (4.1) that $x \equiv y\,(\Theta_H^{\mathfrak{F}})$ implies that either both x and y belong to H, in which case they are equal, or none of them belong to H. Therefore, in order to verify transitivity of $\Theta_H^{\mathfrak{F}}$, there are two cases to discuss:

Case 1: None of the elements x, y, z belong to H. In this case there are p, q, r, $s \in \mathfrak{F}$ such that

$$\iota(x) + p = \iota(y) + q \text{ and } \iota(y) + r = \iota(z) + s\,.$$

It follows that $\iota(x) + (p + r) = \iota(z) + (q + s)$, and since $\mathfrak{F}$ is a submonoid of $\mathfrak{G}$, we conclude that $x \equiv z\,(\Theta_H^{\mathfrak{F}})$.

Case 2: All the elements x, y, z belong to H. In this case it follows readily from (4.2) that $x = y = z$, and thus trivially $x \equiv z\,(\Theta_H^{\mathfrak{F}})$.

We conclude that $\Theta_H^{\mathfrak{F}}$ is an equivalence relation on $\mathfrak{M}$. $\square_{\text{Claim}}$

Claim 2. $\Theta_H^{\mathfrak{F}}$ *is compatible with addition.*

Proof of Claim 2. Let $x_i \equiv y_i\,(\Theta_H^{\mathfrak{F}})$ for some x_i, $y_i \in \mathfrak{M}$, $i = 1, 2$. If all the elements x_i, y_i, $i = 1, 2$, belong to H, definition (4.1) gives that $x_i = y_i$, for all $i = 1, 2$. It follows that $x_1 + x_2 = y_1 + y_2$,

Suppose that not all the elements x_i, y_i, $i = 1, 2$, belong to H. By symmetry we can without loss of generality assume that $x_1 \notin H$. From $x_1 \equiv y_1\,(\Theta_H^{\mathfrak{F}})$ we infer that $y_1 \notin H$ as well. Since H is a hereditary subset of $\mathfrak{M}$, we get that $x_1 + x_2$, $y_1 + y_2 \notin H$. By definition (4.1), there are p_i, $q_i \in \mathfrak{F}$, $i = 1, 2$ (p_2, q_2 possibly zero when x_2, $y_2 \in H$) such that

$$\iota(x_i) + p_i = \iota(y_i) + q_i\,,$$

for all $i = 1, 2$. It follows that

$$\iota(x_1 + x_2) + (p_1 + p_2) = \iota(y_1 + y_2) + (q_1 + q_2)\,.$$

Since $\mathfrak{F}$ is closed under addition and none of the elements $x_1 + x_2$, $y_1 + y_2$ belongs to H, we conclude from (4.1) that $x_1 + x_2 \equiv y_1 + y_2\,(\Theta_H^{\mathfrak{F}})$. $\square_{\text{Claim}}$

This concludes the proof. $\square$

Let $\mathfrak{M}$ be a monoid and Θ a congruence of $\mathfrak{M}$. Given an element $x \in \mathfrak{M}$, we denote by $[x]_\Theta$ the Θ-block of x, i.e., $[x]_\Theta := \{y \in \mathfrak{M} \mid x \equiv y\,(\Theta)\}$. We denote by $\mathfrak{M}/\Theta$ the quotient monoid of $\mathfrak{M}$ by the congruence Θ.

Lemma 4.3. *Let* $\iota\colon \mathfrak{M} \to \mathfrak{G}$ *be a monoid homomorphism,* H *a proper hereditary subset of* $\mathfrak{M}$*, and* $\mathfrak{F}$ *a submonoid of* $\mathfrak{G}$*. Suppose that there are* $x \neq y$ *in* H *and* p*,* $q \in \mathfrak{F}$ *such that*

$$\iota(x) + p = \iota(y) + q\,. \tag{4.3}$$

Then the quotient monoid $\mathfrak{M}/\Theta_H^{\mathfrak{F}}$ *is not cancellative.*

Proof. Since H is a proper subset of $\mathfrak{M}$, there is $z \in \mathfrak{M} \setminus H$. From (4.3) we get that

$$\iota(z + x) + p = \iota(z) + \iota(x) + q = \iota(z) + \iota(y) + q = \iota(z + y) + q\,. \tag{4.4}$$

From (4.4) we infer that

$$z + x \equiv z + y\,(\Theta_H^{\mathfrak{F}})\,,$$

hence

$$[z]_{\Theta_H^{\mathfrak{F}}} + [x]_{\Theta_H^{\mathfrak{F}}} = [z + x]_{\Theta_H^{\mathfrak{F}}} = [z + y]_{\Theta_H^{\mathfrak{F}}} = [z]_{\Theta_H^{\mathfrak{F}}} + [y]_{\Theta_H^{\mathfrak{F}}}\,.$$

On the other hand since $x \neq y$ in H, we get from Definition 4.1 that

$$[x]_{\Theta_H^{\mathfrak{F}}} = \{x\} \neq \{y\} = [y]_{\Theta_H^{\mathfrak{F}}}\,.$$

Therefore $\mathfrak{M}/\Theta_H^{\mathfrak{F}}$ is not cancellative. $\square$

In the next lemma we show that under the assumptions that $H = \mathfrak{O}$ is an o-ideal and both $\mathfrak{O}$ and $\mathfrak{G}$ are cancellative, we can cancel elements from the given hereditary subset.

Lemma 4.4. *Let $\iota\colon \mathfrak{M} \to \mathfrak{G}$ be a monoid homomorphism, $\mathfrak{O}$ an order ideal of $\mathfrak{M}$, and $\mathfrak{F}$ a submonoid of $\mathfrak{G}$. Suppose that both $\mathfrak{O}$ and $\mathfrak{G}$ are cancellative. Let x, $y \in \mathfrak{M}$ and $o \in \mathfrak{O}$ satisfy*

$$[x]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} + [o]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} = [y]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} + [o]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} \,. \tag{4.5}$$

Then $[x]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} = [y]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}}$.

Proof. Equation (4.5) is equivalent to

$$x + o \equiv y + o \,(\Theta_{\mathfrak{O}}^{\mathfrak{F}}) \,.$$

First suppose that $x + o \in \mathfrak{O}$. Then also $y + o \in \mathfrak{O}$, and consequently x, $y \in \mathfrak{O}$ for $\mathfrak{O}$ is an o-ideal. By Definition 4.1 we have that $x + o = y + o \in \mathfrak{O}$. Since $\mathfrak{O}$ is cancellative, we get that $x = y$.

Assume that $x + o \notin \mathfrak{O}$. Since $o \in \mathfrak{O}$ and $\mathfrak{O}$ is an o-ideal, we infer that $x \notin \mathfrak{O}$. Similarly we get that $y \notin \mathfrak{O}$. According to Definition 4.1 there are elements p, $q \in \mathfrak{F}$ such that

$$\iota(x) + \iota(o) + p = \iota(x + o) + p = \iota(y + o) + q = \iota(y) + \iota(o) + q \,.$$

Since $\mathfrak{F}$ is cancellative, we get that

$$\iota(x) + p = \iota(y) + q \,,$$

hence $[x]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} = [y]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}}$, due to Definition 4.1. □

Let $\mathfrak{G}$ be a group and $\mathfrak{F}$ a submonoid of $\mathfrak{G}$. We set

$$\mathfrak{F}^{\natural} := \{p - q \mid p, q \in \mathfrak{F}\} \,.$$

Clearly, $\mathfrak{F}^{\natural}$ is the subgroup of $\mathfrak{G}$ generated by the monoid $\mathfrak{F}$.

Lemma 4.5. *Let $\iota\colon \mathfrak{M} \to \mathfrak{G}$ be a monoid homomorphism, H a hereditary subset of $\mathfrak{M}$. Suppose that $\mathfrak{G}$ is a group and let $\mathfrak{F}$ be a submonoid of $\mathfrak{G}$. Then $\Theta_H^{\mathfrak{F}} = \Theta_H^{\mathfrak{F}^{\natural}}$.*

Proof. It is clear that $\Theta_H^{\mathfrak{F}} \subseteq \Theta_H^{\mathfrak{F}^{\natural}}$. We prove the opposite inclusion. Let x and y be elements of $\mathfrak{M}$ such that $x \equiv y \,(\Theta_H^{\mathfrak{F}^{\natural}})$. By Definition 4.1, we have that $x = y$ unless both x, y belong to $\mathfrak{M} \setminus H$. In this case there are p, $q \in \mathfrak{F}^{\natural}$ such that

$$\iota(x) + p = \iota(y) + q \,. \tag{4.6}$$

Then there are p_i, $q_i \in \mathfrak{F}$, $i = 1, 2$, such that $p = p_1 - p_2$ and $q = q_2 - q_1$. Substituting to (4.6) we get that

$$\iota(x) + p_1 + q_1 = \iota(y) + q_2 + p_2 \,.$$

Therefore $x \equiv y \,(\Theta_H^{\mathfrak{F}})$. □

Under the assumptions of Lemma 4.5 we may restrict ourselves to the case when $\mathfrak{F}$ is a subgroup of the group $\mathfrak{G}$. Notice also that when $\iota\colon \mathfrak{M} \to \mathfrak{G}$ is the inclusion map and $\mathfrak{F}$ is a group, we have that

$$x \equiv y \,(\Theta_H^{\mathfrak{F}}) \iff \begin{cases} x = y + q \text{ for some } q \in \mathfrak{F} & : x, y \notin H \,, \\ x = y & : \text{otherwise,} \end{cases} \tag{4.7}$$

for all x, $y \in \mathfrak{M}$.

A monoid $\mathfrak{M}$ is said to be *directly finite* provided that $x + y = x$ implies that $y = 0$ for all x, $y \in \mathfrak{M}$. We can see readily from the definitions, that the monoid $\mathcal{V}(\boldsymbol{R})$ is directly finite if and only if all finitely generated projective right $\boldsymbol{R}$-modules are directly finite. Following [9, p. 50], this is equivalent to all matrix rings $\mathbb{M}_n(\boldsymbol{R})$ being directly finite. As far as we know it is still an open question whether the monoid $\mathcal{V}(\boldsymbol{R})$ of a directly finite regular ring must be directly finite (cf. [9, Problem 1 on p. 344]). A sufficient conditions for direct finiteness of the quotient monoids $\mathfrak{M}/\Theta_H^{\mathfrak{F}}$ is given by the following lemma:

Lemma 4.6. *Let $\iota\colon \mathfrak{M} \to \mathfrak{G}$ be a monoid homomorphism, H a hereditary subset of $\mathfrak{M}$. Suppose that $\mathfrak{G}$ is a group and let $\mathfrak{F}$ be a subgroup of $\mathfrak{G}$ such that $\iota^{-1}(\mathfrak{F}) = \mathbf{0}$. Then the quotient $\mathfrak{M}/\Theta_H^{\mathfrak{F}}$ is directly finite whenever the monoid $\mathfrak{M}$ is directly finite.*

Proof. Suppose that elements x, $y \in \mathfrak{M}$ satisfy

$$[x]_{\Theta_H^{\mathfrak{F}}} + [y]_{\Theta_H^{\mathfrak{F}}} = [x]_{\Theta_H^{\mathfrak{F}}}\,.$$

If $x \in H$, then $x + y = x$ by the definition of $\Theta_H^{\mathfrak{F}}$ and since $\mathfrak{M}$ is directly finite, we conclude that $y = 0$. Suppose that $x \notin H$. According to (4.7) there is $q \in \mathfrak{F}$ such that

$$\iota(x) + \iota(y) = \iota(x) + q\,. \tag{4.8}$$

Since $\mathfrak{G}$ is a group, we get from (4.8) that $\iota(y) = q$, and so $y \in \iota^{-1}(\mathfrak{F}) = \mathbf{0}$. Therefore $y = 0$. □

In the proof of forthcoming Lemma 4.8 we will repeatedly make use of the following:

Lemma 4.7. *Let $\mathfrak{M}$ be a monoid and Θ a congruence of $\mathfrak{M}$. Let x_i, $y_i \in \mathfrak{M}$, $i = 1, 2$, be such that*

$$[x_1]_\Theta + [x_2]_\Theta = [y_1]_\Theta + [y_2]_\Theta \tag{4.9}$$

and suppose that here are x_i', y_i', $i = 1, 2$, in $\mathfrak{M}$ with $x_i \equiv x_i'\,(\Theta)$ and $y_i \equiv y_i'\,(\Theta)$ for all $i = 1, 2$ and

$$x_1' + x_2' = y_1' + y_2'\,. \tag{4.10}$$

If z_{ij}, $i, j = 1, 2$, is a refinement of (4.10), then $[z_{ij}]_\Theta$, i, $j = 1, 2$, is a refinement of (4.9).

Proof. Since Θ is a congruence of $\mathfrak{M}$, the equality $x_i' = z_{i1} + z_{i2}$ implies that $[x_i]_\Theta = [x_i']_\Theta = [z_{i1}]_\Theta + [z_{i2}]_\Theta$ and $y_j' = z_{1j} + z_{2j}$ implies that $[y_j]_\Theta = [y_j']_\Theta = [z_{1j}]_\Theta + [z_{2j}]_\Theta$, for all i, $j \in 1, 2$. Therefore, if z_{ij}, $i, j = 1, 2$, is a refinement of (4.10), then $[z_{ij}]_\Theta$, $i, j = 1, 2$, is a refinement of (4.9). □

Lemma 4.8. *Let $\mathfrak{M}$, $\mathfrak{G}$ be monoids, $\mathfrak{O}$ and order ideal of $\mathfrak{M}$, and $\mathfrak{F}$ a submonoid of $\mathfrak{G}$. Let $\iota\colon \mathfrak{M} \to \mathfrak{G}$ be a one-to-one monoid homomorphism such that for every x, $y \in \mathfrak{M} \setminus \mathfrak{O}$ and every p, $q \in \mathfrak{F}$ there is $r \in \mathfrak{F}$ such that both $\iota(x) + p + r \in \iota(\mathfrak{M} \setminus \mathfrak{O})$ and $\iota(y) + q + r \in \iota(\mathfrak{M} \setminus \mathfrak{O})$. If $\mathfrak{M}$ is a refinement monoid, then the quotient $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{F}}$ is a refinement monoid as well.*

Proof. We are to verify that the quotient monoid $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{F}}$ is conical and that it satisfies the Riesz refinement property.

Claim 1. *The quotient $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{F}}$ is conical.*

Proof of Claim 1. Let

$$[x]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} + [y]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} = [0]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}}\,,$$

for some x, $y \in \mathfrak{M}$. This is equivalent to $x + y \equiv 0\,(\Theta_{\mathfrak{O}}^{\mathfrak{F}})$. Since $0 \in \mathfrak{O}$, we get from (4.1) that $x + y = 0$. Since the monoid $\mathfrak{M}$ is conical, we conclude that $x = y = 0$. $\square_{\text{Claim}}$

Claim 2. *The quotient $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{F}}$ satisfies the Riesz refinement property.*

Proof of Claim 2. Suppose that x_i, $y_i \in \mathfrak{M}$, $i = 1, 2$, satisfy

$$[x_1]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} + [x_2]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} = [y_1]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}} + [y_2]_{\Theta_{\mathfrak{O}}^{\mathfrak{F}}}\,, \tag{4.11}$$

and so equivalently

$$x_1 + x_2 \equiv y_1 + y_2\,(\Theta_{\mathfrak{O}}^{\mathfrak{F}})\,. \tag{4.12}$$

We are going to discuss the following two complementary case:

Case 1: Suppose that $x_1 + x_2 \in \mathfrak{O}$. With regard to definition (4.1), we get from (4.12) that $y_1 + y_2 \in \mathfrak{O}$ as well and that

$$x_1 + x_2 = y_1 + y_2. \tag{4.13}$$

Since (4.13) has a refinement, (4.11) has a refinement as well due to Lemma 4.7

Case 2: If $x_1 + x_2 \notin \mathfrak{O}$, then $y_1 + y_2 \notin \mathfrak{O}$ as well, due to (4.12) and (4.1). Since $\mathfrak{O}$ is an o-ideal of $\mathfrak{M}$, in particular, it is closed under addition, at least one of the elements x_1, x_2, as well as at least one of the elements y_1, y_2 does not belong to $\mathfrak{O}$. By symmetry, we can assume without loss of generality that both x_2 and y_2 are not in $\mathfrak{O}$. Since (4.12) holds true, there are $p, q \in \mathfrak{F}$ such that

$$\iota(x_1 + x_2) + p = \iota(y_1 + y_2) + q\,, \tag{4.14}$$

due to definition (4.1). According to the assumptions, there is an element $r \in \mathfrak{F}$ such that $\iota(x_2) + p + r \in \iota(\mathfrak{M} \setminus \mathfrak{O})$ and $\iota(y_2) + q + r \in \iota(\mathfrak{M} \setminus \mathfrak{O})$. Let $x_2', y_2' \in \mathfrak{M} \setminus \mathfrak{O}$ be the elements satisfying $\iota(x_2') = \iota(x_2) + p + r$ and $\iota(y_2') = \iota(y_2) + q + r$. It follows from (4.14) that

$$\begin{aligned} \iota(x_1 + x_2') &= \iota(x_1) + \iota(x_2) + p + r = \iota(x_1 + x_2) + p + r \\ &= \iota(y_1 + y_2) + q + r = \iota(y_1) + \iota(y_2) + q + r = \iota(y_1 + y_2')\,. \end{aligned} \tag{4.15}$$

From (4.15) and the injectivity of ι we conclude that

$$x_1 + x_2' = y_1 + y_2'\,. \tag{4.16}$$

Since $\mathfrak{M}$ is a refinement monoid, equation (4.16) has a refinement that induces a refinement of (4.11) due to Lemma 4.7. $\square_{\text{Claim}}$

The properties verified by Claims 1 and 2 mean that $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{F}}$ is a refinement monoid. $\square$

We state a corollary of Lemma 4.8 describing some cases when the formulation of the assumptions can be reasonably simplified. It is going to be applied in the next section.

Corollary 4.9. *Let $\mathfrak{M}$, $\mathfrak{G}$ be monoids, $\mathfrak{O}$ and order ideal of $\mathfrak{M}$, and $\mathfrak{F}$ a submonoid of $\mathfrak{G}$. Let $\iota\colon \mathfrak{M} \to \mathfrak{G}$ be a one-to-one monoid homomorphism such that*

$$\iota(\mathfrak{M} \setminus \mathfrak{O}) + \mathfrak{F} \subseteq \iota(\mathfrak{M} \setminus \mathfrak{O})\,. \tag{4.17}$$

If $\mathfrak{M}$ is a refinement monoid, then the quotient $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{F}}$ is a refinement monoid. If $\mathfrak{G}$ is a group then $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{F}^\natural}$ is a refinement monoid as well.

Proof. The fact that $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{F}}$ is a refinement monoid follows readily from Lemma 4.8 as the assumptions of the lemma follow from (4.17). The quotient $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{F}^\natural}$ is a refinement monoid due to Lemma 4.5. □

5. The monoids $\mathcal{A}_{2n}$, $\mathfrak{B}_{2n}$, and $\mathfrak{C}_{2n}$

Let $\mathfrak{O}$ be an o-ideal in a monoid $\mathfrak{M}$. We denote by $\Theta_{\mathfrak{O}}^{\mathfrak{M}}$ the relation on $\mathfrak{M}$ defined by $x \equiv y\,(\Theta_{\mathfrak{O}}^{\mathfrak{M}})$ provided that there are $o, p \in \mathfrak{O}$ such that $x + o = y + p$. Note that this definition is consistent with the notation of the previous section assuming that we are given the identity map $\iota\colon \mathfrak{M} \to \mathfrak{M}$.

Lemma 5.1. *Let $\mathfrak{M}$ be a conical cancellative monoid. Let $\mathfrak{O}$ be an o-ideal of $\mathfrak{M}$ such that*

$$o \le x \text{ for all } o \in \mathfrak{O} \text{ and all } x \in \mathfrak{M} \setminus \mathfrak{O}. \tag{5.1}$$

Then $\mathfrak{M}$ is a refinement monoid if and only if both $\mathfrak{O}$ and $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{M}}$ are refinement monoids.

Proof. ($\Rightarrow$) Suppose that $\mathfrak{M}$ is a refinement monoid. An o-ideal of a refinement monoid is clearly a refinement monoid, in particular $\mathfrak{O}$ is a refinement monoid.

Suppose that

$$[x]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} + [y]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} = [x+y]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} = [0]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}}\,,$$

for some x, $y \in \mathfrak{M}$. Note that it follows readily from the definition of the congruence $\Theta_{\mathfrak{O}}^{\mathfrak{M}}$ that $[0]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} = \mathfrak{O}$. Therefore, $x + y \in \mathfrak{O}$, hence both x, y belong to $\mathfrak{O}$, for $\mathfrak{O}$ is an o-ideal. We conclude that $[x]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} = [y]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} = [0]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}}$, and so the quotient monoid $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{M}}$ is conical.

We are going to prove that $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{M}}$ satisfies the Riesz refinement property. Let

$$[x_1]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} + [x_2]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} = [y_1]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} + [y_2]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} \tag{5.2}$$

in $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{M}}$. Then, by the definition, there are o, $p \in \mathfrak{O}$ such that $x_1 + x_2 + o = y_1 + y_2 + p$. We set $x_2' := x_2 + o$ and $y_2' := y_2 + p$. Then

$$x_1 + x_2' = y_1 + y_2'\,, \tag{5.3}$$

and since $\mathfrak{M}$ satisfies the Riesz refinement property, the equation (5.3) has a refinement. Clearly $x_2' \equiv x_2 + o\,(\Theta_{\mathfrak{O}}^{\mathfrak{M}})$ and $y_2' \equiv y_2 + p\,(\Theta_{\mathfrak{O}}^{\mathfrak{M}})$, and so this refinement leads to a refinement of (5.2) in the quotient monoid $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{M}}$.

($\Leftarrow$) Suppose that both $\mathfrak{O}$ and $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{M}}$ are refinement monoids. Note that a monoid having a conical o-ideal is conical, in particular the monoid $\mathfrak{M}$ is conical. It remains to prove that $\mathfrak{M}$ satisfies the Riesz refinement property. Given elements $o \in \mathfrak{O}$ and $x \in \mathfrak{M} \setminus \mathfrak{O}$, we denote by $x - o$ the unique element of $\mathfrak{M}$ satisfying $x = o + (x - o)$. Such an element exists due to (5.1) and it is unique since $\mathfrak{M}$ is cancellative.

Suppose that

$$x_1 + x_2 = y_1 + y_2 \tag{5.4}$$

for some x_i, $y_j \in \mathfrak{M}$, i, $j = 1, 2$. We aim to prove that the equation (5.4) has a refinement. Up to symmetry, there are three cases to discuss.

Case 1: All x_i, y_j, i, $j \in 1, 2$, are from $\mathfrak{O}$. Since $\mathfrak{O}$ satisfies the Riesz refinement property, we find a refinement of (5.4) within $\mathfrak{O}$.

Case 2: Some but not all the elements appearing in (5.4) are in $\mathfrak{O}$. Observe that in this case at most one of x_i, $i = 1, 2$, as well as at most one of y_j, $j = 1, 2$, are from $\mathfrak{M} \setminus \mathfrak{O}$. Therefore, we can without loss of generality assume that $x_1, y_1 \in \mathfrak{M} \setminus \mathfrak{O}$ while $y_2 \in \mathfrak{O}$. We put

$$z_{11} := x_1 - y_2, \quad z_{12} := y_2, \quad z_{21} := x_2, \text{ and } z_{22} := 0\,.$$

Clearly

$$\begin{aligned} x_1 &= z_{11} + z_{12} = (x_1 - y_2) + y_2\,, \\ x_2 &= z_{21} + z_{22} = x_2 + 0\,, \quad \text{and} \\ y_2 &= z_{12} + z_{22} = y_2 + 0\,. \end{aligned}$$

Thus we only need to verify that $y_1 = z_{11} + z_{21}$. This follows from

$$z_{11} + z_{21} + y_2 = (x_1 - y_2) + x_2 + y_2 = x_1 + x_2 = y_1 + y_2$$

and the cancellativity of $\mathfrak{M}$.

Case 3: All the elements x_i, y_j, i, $j = 1, 2$, are in $\mathfrak{M} \setminus \mathfrak{O}$. Since $\mathfrak{M}/\Theta_{\mathfrak{O}}^{\mathfrak{M}}$ is a refinement monoid, there are z_{ij}, i, $j = 1, 2$, such that

$$\begin{aligned} [x_i]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} &= [z_{i1}]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} + [z_{i2}]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}}\,, && \text{for all} \quad i = 1, 2\,, \quad \text{and} \\ [y_j]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} &= [z_{1j}]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} + [z_{2j}]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}}\,, && \text{for all} \quad j = 1, 2\,. \end{aligned}$$

This particularly means that there are $o_i, p_i \in \mathfrak{O}$, $i = 1, 2$, satisfying

$$x_i + o_i = z_{i1} + z_{i2} + p_i\,, \quad \text{for both } i = 1, 2\,.$$

Observe that since $x_i, y_j \in \mathfrak{M} \setminus \mathfrak{O}$, for all $i, j = 1, 2$, either $z_{11}, z_{22} \in \mathfrak{M} \setminus \mathfrak{O}$ or $z_{12}, z_{21} \in \mathfrak{M} \setminus \mathfrak{O}$. We can without loss of generality assume that the first one holds true. Set

$$u_{ii} := z_{ii} + p_i - o_i\,, \quad \text{for all} \quad i = 1, 2 \quad \text{and} \quad u_{ij} := z_{ij} \quad \text{for all} \quad i \neq j \text{ in } \{1, 2\}\,,$$

and observe that

$$\begin{aligned} x_i &= u_{i1} + u_{i2}\,, \quad \text{for all} \quad i = 1, 2\,, \\ [y_j]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} &= [u_{1j}]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}} + [u_{2j}]_{\Theta_{\mathfrak{O}}^{\mathfrak{M}}}\,, \quad \text{for all} \quad j = 1, 2\,, \end{aligned} \tag{5.5}$$

and both u_{11}, u_{22} belong to $\mathcal{M} \setminus \mathcal{O}$. It follows from (5.5) that

$$y_j + q_j = u_{1j} + u_{2j} + r_j\,, \quad j = 1, 2, \tag{5.6}$$

for some q_j, $r_j \in \mathcal{O}$, $j = 1, 2$. Therefore

$$y_1 + y_2 + q_1 + q_2 = \Big(\sum_{i=1}^{2}\sum_{j=1}^{2} u_{ij}\Big) + r_1 + r_2 = x_1 + x_2 + r_2 + r_2\,. \tag{5.7}$$

Since $\mathcal{M}$ is cancellative, we conclude from (5.4) and (5.7) that

$$q_1 + q_2 = r_1 + r_2\,.$$

Since $\mathcal{O}$ satisfies the Riesz refinement property, there are $s_{ij} \in \mathcal{O}$, i, $j = 1, 2$ such that

$$q_j = s_{j1} + s_{j2} \quad \text{and} \quad r_j = s_{1j} + s_{2j} \quad \text{for all} \quad j = 1, 2\,. \tag{5.8}$$

Substituting from (5.8) to (5.6), we get that

$$y_j + s_{j1} + s_{j2} = u_{1j} + u_{2j} + s_{1j} + s_{2j}\,, \quad \text{for all} \quad j = 1, 2\,. \tag{5.9}$$

Since the monoid $\mathcal{M}$ is cancellative, we conclude from (5.9) that

$$\begin{aligned} y_1 + s_{12} &= u_{11} + u_{21} + s_{21} \quad \text{and} \\ y_2 + s_{21} &= u_{12} + u_{22} + s_{12}\,. \end{aligned} \tag{5.10}$$

It follows from (5.5) and (5.10) that setting

$$\begin{aligned} v_{11} &:= u_{11} - s_{12}\,, \quad & v_{12} &:= u_{12} + s_{12}\,, \\ v_{21} &:= u_{21} + s_{21}\,, \quad & v_{22} &:= u_{22} - s_{21}\,, \end{aligned}$$

we get a refinement of (5.4) in $\mathcal{M}$. □

Let n be a non-negative integer. Let

$$\boldsymbol{\mathcal{A}}_n := (\mathbf{0} \times \mathbb{N}_0^n) \cup (\mathbb{N} \times \mathbb{Z}^n) \tag{5.11}$$

be a submonoid of the Cartesian power $\mathbb{Z}^{n+1}$. Note that being a submonoid of a group, the monoid $\boldsymbol{\mathcal{A}}_n$ is cancellatice. We denote by $\mathcal{O}_n$ the o-ideal of $\boldsymbol{\mathcal{A}}_n$ defined by $\mathcal{O}_n := \mathbf{0} \times \mathbb{N}_0^n$, and we set $\mathcal{U}_n := \boldsymbol{\mathcal{A}}_n \setminus \mathcal{O}_n = \mathbb{N} \times \mathbb{Z}^n$.

Corollary 5.2. *The monoid $\boldsymbol{\mathcal{A}}_n$ is a refinement monoid, for every non-negative integer n.*

Proof. It is straightforward to see that $o \leq_{\boldsymbol{\mathcal{A}}_n} x$ for every $o \in \mathcal{O}_n$ and every $x \in \mathcal{U}_n$. Therefore property (5.1) of Lemma 5.1 is satisfied. Clearly $\mathcal{O}_n$, being a Cartesian product of refinement monoids, is a refinement monoid. Observing that

$$\boldsymbol{\mathcal{A}}_n/\Theta_{\mathcal{O}_n}^{\boldsymbol{\mathcal{A}}_n} \simeq \mathbb{N}_0\,,$$

which is a refinement monoid as well, we conclude from Lemma 5.1 that $\boldsymbol{\mathcal{A}}_n$ is a refinement monoid. □

Lemma 5.3. *Let n be a non-negative integer and $\iota\colon \boldsymbol{\mathcal{A}}_n \to \mathbb{Z}^{n+1}$ the inclusion map. Then $\boldsymbol{\mathcal{A}}_n/\Theta_{\mathcal{O}_n}^{\mathcal{F}}$ is a refinement monoid for every submonoid $\mathcal{F}$ of $\mathbb{Z}^{n+1}$. Moreover*

(a) *if $\boldsymbol{\mathcal{A}}_n \cap \mathcal{F}^{\natural} = \mathbf{0}$ holds true, then $\boldsymbol{\mathcal{A}}_n/\Theta_{\mathcal{O}_n}^{\mathcal{F}}$ is directly finite;*

(b) *if* $\mathfrak{O}_n^{\natural} \cap \mathfrak{F}^{\natural} \neq \mathbf{0}$, *then* $\mathcal{A}_n/\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}$ *is not cancellative.*

Proof. Firstly note that according to Lemma 4.5 we can without loss of generality assume that $\mathfrak{F}$ is a subgroup of $\mathbb{Z}^{n+1}$, i.e, that $\mathfrak{F} = \mathfrak{F}^{\natural}$. Put $\mathfrak{F}_+ := \mathfrak{F} \cap (\mathbb{N}_0 \times \mathbb{Z}^n)$ and observe that $\iota(\mathfrak{U}_n) + \mathfrak{F}_+ \subseteq \iota(\mathfrak{U}_n)$. Applying Corollary 4.9 we conclude that $\mathcal{A}_n/\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}$ is a refinement monoid.

Being a submonoid of $\mathbb{Z}^{n+1}$, the monoid $\mathcal{A}_n$ is cancellative and, *a fortiori*, directly finite. Then (a) follows readily from Lemma 4.6.

The assumption $\mathfrak{O}_n^{\natural} \cap \mathfrak{F}^{\natural} \neq \mathbf{0}$ implies that there are $x \neq y$ in $\mathfrak{O}_n$ and $p, q \in \mathfrak{F}$ such that $x - y = q - p$, and so, equivalently, $x + p = y + q$. Since ι is an inclusion map, the monoid $\mathcal{A}_n/\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}$ is not cancellative due to Lemma 4.3. □

Although the monoid $\mathcal{A}_n/\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}$ might not be cancellative we can cancel the elements from $\mathfrak{O}_n$ due to Lemma 4.4.

Lemma 5.4. *Let $\mathfrak{F}$ be a non-trivial submonoid of $\mathbb{Z}^{n+1}$. If $x, y \in \mathcal{A}_n$ and $o \in \mathfrak{O}_n$ satisfy*

$$[x]_{\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}} + [o]_{\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}} = [y]_{\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}} + [o]_{\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}}\,. \tag{5.12}$$

Then $[x]_{\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}} = [y]_{\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}}$.

Fix a positive integer n. For an element $x = \langle x_0, x_1, \dots, x_n \rangle \in \mathbb{Z}^{n+1}$ we set

$$\sigma x := x_0 + x_1 + \cdots + x_n\,.$$

We put $\Sigma_n^0 = \{x \in \mathbb{Z}^{n+1} \mid x_0 = 0 \text{ and } \sigma x = 0\}$. Observe that Σ_n^0 is a subgroup of $\mathbb{Z}^{n+1}$.

Corollary 5.5. *Let $\mathfrak{F}$ be a non-trivial subgroup of Σ_n^0. Then $\mathcal{A}_n/\Theta_{\mathfrak{O}_n}^{\mathfrak{F}}$ is a non-cancellative directly finite refinement monoid.*

Proof. Observe that $\mathcal{A}_n \cap \Sigma_n^0 = \mathbf{0}$, ${\mathfrak{O}_n}^{\natural} \cap \Sigma_n^0 = \Sigma_n^0$, and apply Lemma 5.3. □

Given a positive integer n, let $\mathfrak{F}_{2n}$ denote a subgroup of $\mathbb{Z}^{2n+1}$ generated by $\langle 0, 1, -1, \dots, 1, -1 \rangle$. We set

$$\mathfrak{B}_{2n} := \mathcal{A}_{2n}/\Theta_{\mathfrak{O}_{2n}}^{\mathfrak{F}_{2n}}.$$

As $\mathfrak{F}_{2n}$ is clearly a non-trivial subgroup of Σ_{2n}^0, $\mathfrak{B}_{2n}$ is a non-cancellative directly finite refinement monoid. We are going to realize the monoids $\mathfrak{B}_{2n}$ as $\mathcal{V}(\mathbf{R}_{2n})$ of regular rings $\boldsymbol{R}_{2n}$.

Before that, we prove that the monoid $\mathfrak{B}_4$ (and consequently the monoids $\mathfrak{B}_{2n}$ for all $n \geq 2$) does not satisfy the Riesz interpolation property.

Proposition 5.6. *The monoid $\mathfrak{B}_4$ does not satisfy the Riesz interpolation property.*

Proof. Let $x = \langle x_0, x_1, \dots, x_4 \rangle$ and $y = \langle y_0, y_1, \dots, y_4 \rangle$ be elements of $\mathcal{A}_4$. We observe readily from the definitions that if $x_0 = y_0$, then

$$[x]_{\Theta_{\mathfrak{O}_2}^{\mathfrak{F}_2}} <_{\mathfrak{B}_4} [y]_{\Theta_{\mathfrak{O}_2}^{\mathfrak{F}_2}} \implies \sigma x < \sigma y \quad \text{and} \quad [x]_{\Theta_{\mathfrak{O}_2}^{\mathfrak{F}_2}} = [y]_{\Theta_{\mathfrak{O}_2}^{\mathfrak{F}_2}} \implies \sigma x = \sigma y\,. \tag{5.13}$$

We set

$$x^1 := \langle 1,1,1,0,0\rangle\,,\quad x^2 := \langle 1,1,0,1,0\rangle\,,$$
$$y^1 := \langle 1,1,1,1,0\rangle\,,\quad y^2 := \langle 1,1,1,0,1\rangle\,.$$

We see that x^1, $x^2 \leq_{\boldsymbol{A}_4} y^1$ and $x^1 \leq_{\boldsymbol{A}_4} y^2$. Since $\sigma x^1 = \sigma x^2 = 3 < 4 = \sigma y^1 = \sigma y^2$, we get that $[x^1]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}}$, $[x^2]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}} <_{\mathfrak{B}_4} [y^1]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}}$ and $[x^1]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}} <_{\mathfrak{B}_4} [y^2]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}}$. Since

$$y^2 \equiv \langle 1,2,0,1,0\rangle\,(\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4})\,,$$

we have that also $[x^2]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}} <_{\mathfrak{B}_4} [y^2]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}}$. Suppose that there is $z = \langle z_0, z_1, \dots, z_4\rangle$ with

$$[x^1]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}}, [x^2]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}} <_{\mathfrak{B}_4} [z]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}} <_{\mathfrak{B}_4} [y^1]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}}, [y^2]_{\Theta^{\mathcal{F}_4}_{\mathfrak{O}_4}}\,. \tag{5.14}$$

Since $x^i_0 = y^i_0 = 1$, for all $i = 1, 2$, we get that $z_0 = 1$. From (5.13) and (5.14) we get that $3 = \sigma x^i < \sigma z < \sigma y^j = 4$, $i, j = 1, 2$. This is absurd. □

Let n be a positive integer. For each $i, j \in \{1, 2, \dots, n\}$ we set $x_{\{2i-1,2j\}} := x_{2i-1} + x_{2j}$ and we define

$$\boldsymbol{\mathcal{V}}_{2n} := \{\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\leq n} \mid x_0 \in \mathbb{N} \text{ and } x_{\{2i-1,2j\}} \in \mathbb{Z}\ \forall\ i, j \in \{1, 2, \dots, n\}\}$$

and we set $\mathfrak{C}_{2n} := \mathfrak{O}_{2n} \cup \boldsymbol{\mathcal{V}}_{2n}$. Observe that $\boldsymbol{\mathcal{V}}_{2n}$ is a semigroup isomorphic to $\mathbb{N}\times\mathbb{Z}^{n^2}$ and that $\mathfrak{C}_{2n}$ is a monoid with the operation of addition defined coordinate-wise on the two components $\mathfrak{O}_{2n}$ and $\boldsymbol{\mathcal{V}}_{2n}$ and by

$$p + x := \langle x_0, p_{2i-1} + p_{2j} + x_{\{2i-1,2j\}}\rangle_{i,j\in\{1,2,\dots,n\}}$$

for all $p = \langle 0, p_1, \dots, p_{2n}\rangle \in \mathfrak{O}_{2n}$ and $x = \langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\leq n} \in \boldsymbol{\mathcal{V}}_{2n}$.

Let $\varphi_{2n} \colon \boldsymbol{A}_{2n} \to \mathfrak{C}_{2n}$ be a map corresponding to the identity on $\mathfrak{O}_{2n}$ and sending

$$\langle x_0, x_1, x_2 \dots, x_{2n}\rangle \mapsto \langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\leq n} \in \boldsymbol{\mathcal{V}}_{2n}\,,$$

whenever $x_0 > 0$. It is straightforward to see that φ_{2n} is a monoid homomorphism.

Let $x = \langle x_0, \dots, x_{2n}\rangle$, $y = \langle y_0, \dots, y_{2n}\rangle$ be elements from $\boldsymbol{A}_{2n}$ satisfying $\varphi_{2n}(x) = \varphi_{2n}(y)$. Readily from the definition we see that $x_0 = y_0$. If $x_0 = y_0 = 0$, then necessarily $x = y$. Suppose that $x_0 = y_0 > 0$. In this case the equality $\varphi_{2n}(x) = \varphi_{2n}(y)$ is equivalent to

$$x_{2i-1} + x_{2j} = y_{2i-1} + y_{2j} \tag{5.15}$$

for all $i, j \in \{1, 2, \dots, n\}$. This is equivalent to

$$x_1 - y_1 = y_2 - x_2 = \cdots = x_{2n-1} - y_{2n-1} = y_{2n} - x_{2n}\,,$$

which happens if and only if

$$x = y + \lambda\langle 1, -1, \dots, 1, -1\rangle\,,$$

for some $\lambda \in \mathbb{Z}$. Therefore the kernel of the homomorphism φ_{2n} coincides with the congruence $\Theta^{\mathcal{F}_{2n}}_{\mathfrak{O}_{2n}}$, and so φ_{2n} factors through an embedding $\psi_{2n} \colon \mathfrak{B}_{2n} \to \mathfrak{C}_{2n}$. This one is given by

$$\psi_{2n}([x]_{\Theta^{\mathcal{F}_{2n}}_{\mathfrak{O}_{2n}}}) = \begin{cases} x = \langle 0, x_1, x_2, \dots, x_{2n}\rangle & \text{if } x \in \mathfrak{O}_{2n}\,, \\ \langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\leq n} & \text{if } x \in \mathfrak{U}_{2n}\,, \end{cases} \tag{5.16}$$

for every $x = \langle x_0, x_1, \dots, x_{2n}\rangle \in \boldsymbol{\mathcal{A}}_{2n}$.

We say that a tuple $\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\le n} \in \boldsymbol{\mathcal{V}}_{2n}$ is *balanced* provided that

$$x_{\{2i-1,2j\}} + x_{\{2k-1,2l\}} = x_{\{2k-1,2j\}} + x_{\{2i-1,2l\}} \tag{5.17}$$

holds true for all i, j, k, $l \in \{1, 2, \dots, n\}$. We denote by $\boldsymbol{\mathcal{W}}_{2n}$ the set of all balanced tuples from $\boldsymbol{\mathcal{V}}_{2n}$ and we set

$$\boldsymbol{\mathcal{D}}_{2n} := \boldsymbol{\mathcal{O}}_{2n} \cup \boldsymbol{\mathcal{W}}_{2n}\,.$$

It is straightforward to show that $\boldsymbol{\mathcal{D}}_{2n}$ is a submonoid of $\boldsymbol{\mathcal{C}}_{2n}$. Observe also that $\boldsymbol{\mathcal{D}}_2 = \boldsymbol{\mathcal{C}}_2$.

Lemma 5.7. *The monoid $\boldsymbol{\mathcal{D}}_{2n}$ corresponds to $\varphi_{2n}(\boldsymbol{\mathcal{A}}_{2n})$, the image of $\boldsymbol{\mathcal{A}}_{2n}$ under the monoid homomorphism $\varphi_{2n}\colon \boldsymbol{\mathcal{A}}_{2n} \to \boldsymbol{\mathcal{C}}_{2n}$.*

Proof. As $\varphi_{2n} \restriction \boldsymbol{\mathcal{O}}_{2n}$ is the identity map, we have that $\varphi_{2n}(\boldsymbol{\mathcal{O}}_{2n}) = \boldsymbol{\mathcal{O}}_{2n}$. We are going to prove that $\varphi_{2n}(\boldsymbol{\mathcal{U}}_{2n}) = \boldsymbol{\mathcal{W}}_{2n}$.

Let $x = \langle x_0, x_1, \dots, x_{2n}\rangle \in \boldsymbol{\mathcal{U}}_{2n}$. By the definition, $\varphi_{2n}(x) = \langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\le n}$, where

$$x_{\{2i-1,2j\}} = x_{2i-1} + x_{2j} \quad \forall \quad i, j \in \{1, 2, \dots, n\}\,.$$

Given $i, j, k, l \in \{1, 2, \dots, n\}$, we get straightaway that

$$x_{\{2i-1,2j\}} + x_{\{2k-1,2l\}} = x_{2i-1} + x_{2j} + x_{2k-1} + x_{2l} = x_{\{2i-1,2l\}} + x_{\{2k-1,2j\}}\,,$$

and so $\varphi_{2n}(x)$ is a balanced tuple. Therefore we have the inclusion $\varphi_{2n}(\boldsymbol{\mathcal{U}}_{2n}) \subseteq \boldsymbol{\mathcal{W}}_{2n}$.

Let $\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\le n} \in \boldsymbol{\mathcal{W}}_{2n}$ be a balanced tuple. We set

$$x_{2i-1} := x_{\{2i-1,2n\}} \text{ and } x_{2j} := x_{\{2j-1,2j\}} - x_{\{2j-1,2n\}} \tag{5.18}$$

for all i, $j = \{1, 2, \dots, n\}$ and we put $x := \langle x_0, x_1, \dots, x_{2n}\rangle$. Since the tuple $\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\le n}$ is balanced, we have the equality

$$x_{\{2i-1,2j\}} + x_{\{2j-1,2n\}} = x_{\{2i-1,2n\}} + x_{\{2j-1,2j\}}\,,$$

hence

$$x_{2i-1,2j} = x_{\{2i-1,2n\}} + x_{\{2j-1,2j\}} - x_{\{2j-1,2n\}} = x_{2i-1} + x_{2j}\,,$$

for all i, $j \in \{1, 2, \dots, n\}$. It follows that $\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\le n} = \varphi_{2n}(x)$. Since $\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\le n} \in \boldsymbol{\mathcal{W}}_{2n}$, we have that $x_0 > 0$, and so $x \in \boldsymbol{\mathcal{U}}_{2n}$. Therefore $\boldsymbol{\mathcal{W}}_{2n} \subseteq \varphi_{2n}(\boldsymbol{\mathcal{U}}_{2n})$. ☐

Corollary 5.8. *The map defined by correspondence* (5.16) *is an isomorphism*

$$\psi_{2n}\colon \boldsymbol{\mathcal{B}}_{2n} \to \boldsymbol{\mathcal{D}}_{2n}\,.$$

It is easy to gain insight into the algebraic preorder on $\boldsymbol{\mathcal{A}}_{2n}$. Indeed,

$$x = \langle x_0, x_1, \dots, x_{2n}\rangle \le_{\boldsymbol{\mathcal{A}}_{2n}} y = \langle y_0, y_1, \dots, y_{2n}\rangle$$

if and only if either $x_0 < y_0$ or $x_0 = y_0$ and $x_i \le y_i$ for all $i \in \{1, 2, \dots, n\}$. We are going to show that the algebraic preorder on the monoid $\boldsymbol{\mathcal{D}}_{2n}$ behaves analogously.

Let $x = \langle x_0, \dots \rangle$ and $y = \langle y_0, \dots \rangle$ be elements of $\mathfrak{D}_{2n}$. We set

$$x \ll y \iff \begin{cases} x_0 < y_0 \\ x_0 = y_0 = 0 \quad \text{and} \quad x_i \leq y_i \,, \quad \forall\, i \in \{1, 2, \dots, 2n\} \,, \\ x_0 = y_0 > 0 \quad \text{and} \quad x_{\{2i-1,2j\}} \leq y_{\{2i-1,2j\}} \,, \quad \forall\, i, j \in \{1, 2, \dots, n\} \,. \end{cases}$$

It is easy to see that $\ll$ is a partial order on the set $\mathfrak{D}_{2n}$.

Lemma 5.9. *Let $x = \langle x_0, x_{\{2i-1,2j\}} \rangle_{i,j \leq n} \in \boldsymbol{\mathcal{V}}_{2n}$ and $z = \langle z_0, z_1, \dots, z_{2n} \rangle \in \boldsymbol{\mathfrak{U}}_{2n}$ be such that $x \ll \varphi_{2n}(z)$. There is $w \in \boldsymbol{\mathfrak{U}}_{2n}$ such that $w \leq_{\boldsymbol{\mathcal{A}}_{2n}} z$ and $x = \varphi_{2n}(w)$.*

Proof. We set

$$\mu := \min\{z_{2j} - x_{\{1,2j\}} + z_1 \mid j = 1, 2, \dots, n\}$$

and

$$w_0 := x_0, \;\; w_{2j} := x_{\{1,2j\}} - z_1 + \mu \,, \textit{and } w_{2i-1} := x_{\{2i-1,2\}} + z_1 - x_{\{1,2\}} - \mu \,,$$

for every $i, j \in \{1, 2, \dots, n\}$. Since the tuple x is balanced, we have that

$$x_{\{1,2j\}} + x_{\{2i-1,2\}} = x_{\{1,2\}} + x_{\{2i-1,2j\}} \,,$$

hence

$$x_{\{2i-1,2\}} - x_{\{1,2\}} = x_{\{2i-1,2j\}} - x_{\{1,2j\}} \,,$$

whence

$$w_{2i-1} = x_{\{2i-1,2j\}} + z_1 - x_{\{1,2j\}} - \mu \,, \tag{5.19}$$

for all $i, j \in \{1, 2, \dots, n\}$. It follows that

$$w_{2i-1} + w_{2j} = x_{\{2i-1,2j\}} + z_1 - x_{\{1,2j\}} - \mu + x_{\{1,2j\}} - z_1 + \mu = x_{\{2i-1,2j\}} \,,$$

for all $i, j \in \{1, 2, \dots, n\}$. Since $x_0 = w_0$ by definition, we conclude that $x = \varphi_{2n}(w)$. Let $j \in \{1, 2, \dots, n\}$. From $\mu \leq z_{2j} - x_{\{1,2j\}} + z_1$ we get that

$$w_{2j} = x_{\{1,2j\}} - z_1 + \mu \leq x_{\{1,2j\}} - z_1 + z_{2j} - x_{\{1,2j\}} + z_1 = z_{2j} \,. \tag{5.20}$$

Let $k \in \{1, 2, \dots, n\}$ be such that $\mu = z_{2k} - x_{\{1,2k\}} + z_1$. Then, with regard to (5.19), we compute that

$$\begin{aligned} w_{2i-1} &= x_{\{2i-1,2k\}} + z_1 - x_{\{1,2k\}} - \mu \\ &= x_{\{2i-1,2k\}} + z_1 - x_{\{1,2k\}} - z_{2k} - x_{\{1,2k\}} + z_1 \\ &= x_{\{2i-1,2k\}} - z_{2k} \leq x_{\{2i-1,2k\}} - z_{2k} \,. \end{aligned} \tag{5.21}$$

Since $x \ll \varphi_{2n}(z)$, we have that $x_{\{2i-1,2k\}} \leq z_{2i-1} + z_{2k}$. Substituting to (5.21), we conclude that

$$w_{2i-1} \leq x_{\{2i-1,2k\}} - z_{2k} \leq z_{2i-1} + z_{2k} - z_{2k} = z_{2i-1} \,, \tag{5.22}$$

for all $i = \{1, 2, \dots, n\}$. Since $x \ll \varphi_{2n}(z)$, we have $w_0 = x_0 \leq z_0$. This together with (5.20) and (5.22) implies that $w \leq_{\boldsymbol{\mathcal{A}}_{2n}} z$, which was to prove. $\square$

Proposition 5.10. *Let $x = \langle x_0, \dots \rangle$ and $y = \langle y_0, \dots \rangle$ be elements of $\mathfrak{D}_{2n}$. Then $x \ll y$ if and only if $x \leq_{\mathfrak{D}_{2n}} y$.*

Proof. If $x \leq_{\mathfrak{D}_{2n}} y$, there are $x' \leq_{\mathcal{A}_{2n}} y'$ in $\mathcal{A}_{2n}$ sutisfying $x = \varphi_{2n}(x')$ and $y = \varphi_{2n}(y')$. Using the description of the algebraic preoreder in $\mathcal{A}_{2n}$, it is easy to see that the relation $x \ll y$ holds true. On the other hand, suppose that $x \ll y$. If $x_0 \leq y_0$ or $x_0 = y_0 = 0$, then $x \ll y$ clearly implies that $x \leq_{\mathfrak{D}_{2n}} y$. In the remaining case when $0 < x_0 = y_0$, the implication $x \ll y \implies x \leq_{\mathfrak{D}_{2n}} y$ follows from Lemma 5.9. □

6. Some linear algebra

We fix an arbitrary field $\mathbb{F}$. All vector spaces are supposed to be over $\mathbb{F}$. Let $\boldsymbol{U}$, $\boldsymbol{V}$ be vector spaces and $f\colon \boldsymbol{U} \to \boldsymbol{V}$ a linear map. We define a *dimension* and a *codimension* of the map f by

(i) $\dim f := \operatorname{codim} \ker f + \dim \operatorname{im} f$,

(ii) $\operatorname{codim} f := \dim \ker f + \operatorname{codim} \operatorname{im} f$.

Observe that $\dim f = 2 \dim \operatorname{im} f$ and $\dim f + \operatorname{codim} f = \dim U + \dim V$. In particular, if $\dim f$ is finite, it is even.

Lemma 6.1. *Let $\boldsymbol{U}$ be a vector space. Let f, $g\colon \boldsymbol{U} \to \boldsymbol{V}$ be linear maps such that* $\dim f$ *and* $\operatorname{codim} g$ *are finite, and let $h := f + g$ be the sum of the linear maps. Then* $\operatorname{codim} h$ *is finite and*

$$\dim \ker h - \operatorname{codim} \operatorname{im} h = \dim \ker g - \operatorname{codim} \operatorname{im} g\,. \tag{6.1}$$

Proof. We decompose $\boldsymbol{U} = \ker g \oplus \boldsymbol{X}$ and we put $\boldsymbol{Y} := \boldsymbol{X} \cap \ker f$. Now we set $\boldsymbol{Z} := h(\boldsymbol{Y}) = g(\boldsymbol{Y})$ and we use $g', h'\colon \boldsymbol{U}/\boldsymbol{Y} \to \boldsymbol{V}/\boldsymbol{Z}$ to denote the quotients of the maps g, h, respectively.

Observe that $\ker h' = \ker h + \boldsymbol{Y}$ and $\ker g' = \ker g + \boldsymbol{Y}$. Since $\boldsymbol{Y} \subseteq \boldsymbol{X}$, we have that $\boldsymbol{Y} \cap \ker g = 0$. Since $\boldsymbol{Y} \subseteq \ker f$, we have that $h \restriction \boldsymbol{Y} = g \restriction \boldsymbol{Y}$, and so $\boldsymbol{Y} \cap \ker h = 0$. It follows that

$$\dim \ker h' = \dim \ker h \quad \text{and} \quad \dim \ker g' = \dim \ker g\,. \tag{6.2}$$

Clearly $\operatorname{im} h' = \operatorname{im} h + \boldsymbol{Z}$ and $\operatorname{im} g' = \operatorname{im} g + \boldsymbol{Z}$. Since $\boldsymbol{Z} \subseteq \operatorname{im} h \cap \operatorname{im} g$, we conclude that

$$\operatorname{codim} \operatorname{im} h' = \operatorname{codim} \operatorname{im} h \quad \text{and} \quad \operatorname{codim} \operatorname{im} g' = \operatorname{codim} \operatorname{im} g\,. \tag{6.3}$$

Since both $\operatorname{codim} \ker f$ and $\operatorname{codim} \boldsymbol{X} = \dim \ker g$ are finite, we have that $\operatorname{codim} \boldsymbol{Y}$ is finite. As $\operatorname{codim} \operatorname{im} g = \operatorname{codim} g(\boldsymbol{X})$ is finite, and the codimension of $\boldsymbol{Y}$ in $\boldsymbol{X}$ is finite (cf. $\operatorname{codim} \boldsymbol{Y}$ is finite), $\operatorname{codim} \boldsymbol{Z} = \operatorname{codim} g(\boldsymbol{Y})$ is finite.

Clearly $\dim \ker h' + \operatorname{codim} \ker h' = \operatorname{codim} \boldsymbol{Y}$ and $\dim \operatorname{im} h' + \operatorname{codim} \operatorname{im} h' = \operatorname{codim} \boldsymbol{Z}$. Since $\operatorname{codim} \ker h' \leq \operatorname{codim} \boldsymbol{Y}$ is finite, we have that $\operatorname{codim} \ker h' = \dim \operatorname{im} h'$. We conclude that

$$\dim \ker h' - \operatorname{codim} \operatorname{im} h' = \operatorname{codim} \boldsymbol{Y} - \operatorname{codim} \boldsymbol{Z}\,.$$

Similarly we prove that

$$\dim \ker g' - \operatorname{codim} \operatorname{im} g' = \operatorname{codim} \boldsymbol{Y} - \operatorname{codim} \boldsymbol{Z}\,,$$

and so

$$\dim \ker h' - \operatorname{codim} \operatorname{im} h' = \dim \ker g' - \operatorname{codim} \operatorname{im} g'\,. \tag{6.4}$$

Equation (6.4) together with equalities (6.2) and (6.3) give (6.1). □

Lemma 6.2. *Let* $f\colon \boldsymbol{U} \to \boldsymbol{V}$ *and* $g\colon \boldsymbol{V} \to \boldsymbol{W}$ *be homomorphisms of vector spaces and* $h := g \circ f$ *their composition. Let* $\boldsymbol{X}$ *be a subspace of* $\ker g$ *such that* $\ker g$ *decomposes as* $\ker g = \boldsymbol{X} \oplus (\operatorname{im} f \cap \ker g)$*. Then*

$$\operatorname{codim} f + \operatorname{codim} g = \operatorname{codim} h + 2(\dim \boldsymbol{X})\,. \tag{6.5}$$

Proof. The lemma follows from these straightforward equalities:

$$\begin{aligned}
\dim\ker h &= \dim\ker f + \dim(\operatorname{im} f \cap \ker g)\,,\\
\operatorname{codim}\operatorname{im} h &= \operatorname{codim}\operatorname{im} g + \operatorname{codim}(\operatorname{im} f + \ker g)\,,\\
\operatorname{codim}\operatorname{im} f &= \dim \boldsymbol{X} + \operatorname{codim}(\operatorname{im} f + \ker g)\,,\\
\dim\ker g &= \dim \boldsymbol{X} + \dim(\operatorname{im} f \cap \ker g)\,.
\end{aligned}$$

□

The next lemma is "the reason why it works". It is a crucial part of Lemma 7.10.

Lemma 6.3. *Let* $\boldsymbol{U}$ *be a vector space. Let* x, u, y_i, v_i*,* $i = 1, 2$*, be endomorphisms of the vector space* $\boldsymbol{U}$ *such that both* $\operatorname{codim} x$ *and* $\operatorname{codim} u$ *are finite as well as all* $\dim y_i$ *and* $\dim v_i$*,* $i = 1, 2$*, are finite. Put* $f_i := x + y_i$ *and* $g_i := u + v_i$*,* $i = 1, 2$*, and set*

$$\begin{aligned}
h_1 &:= g_1 \circ f_1 = (u + v_1) \circ (x + u_1)\,,\\
h_2 &:= f_2 \circ g_2 = (x + u_2) \circ (u + v_2)\,.
\end{aligned}$$

Then

$$\operatorname{codim} h_1 + \operatorname{codim} h_2 \geq \max\{\operatorname{codim} f_1 + \operatorname{codim} f_2, \operatorname{codim} g_1 + \operatorname{codim} g_2\}\,. \tag{6.6}$$

Proof. We are going to prove that

$$\operatorname{codim} h_1 + \operatorname{codim} h_2 \geq \operatorname{codim} g_1 + \operatorname{codim} g_2\,. \tag{6.7}$$

The other inequality, namely $\operatorname{codim} h_1 + \operatorname{codim} h_2 \geq \operatorname{codim} f_1 + \operatorname{codim} f_2$, is symmetric. We choose decompositions

$$\begin{aligned}
\ker g_2 &= \boldsymbol{X} \oplus (\operatorname{im} f_1 \cap \ker g_2) \qquad \text{and}\\
\ker f_2 &= \boldsymbol{Y} \oplus (\operatorname{im} g_1 \cap \ker f_2)\,.
\end{aligned} \tag{6.8}$$

Applying Lemma 6.2, we get that

$$\begin{aligned}
\operatorname{codim} h_1 + 2\dim \boldsymbol{X} &= \operatorname{codim} f_1 + \operatorname{codim} g_1 \qquad \text{and}\\
\operatorname{codim} h_2 + 2\dim \boldsymbol{Y} &= \operatorname{codim} f_2 + \operatorname{codim} g_2\,.
\end{aligned}$$

Since, by the initial assumptions, $\operatorname{codim} u$ is finite and both $\dim v_i$, $i = 1, 2$, are finite, the co-dimensions $\operatorname{codim} g_i$, $i = 1, 2$, are finite due to Lemma 6.1. Thus it suffices to prove that

$$2(\dim \boldsymbol{X} + \dim \boldsymbol{Y}) \leq \operatorname{codim} g_1 + \operatorname{codim} g_2\,. \tag{6.9}$$

Applying Lemma 6.1 again we get that

$$\dim\ker g_1 - \operatorname{codim}\operatorname{im} g_1 = \dim\ker u - \operatorname{codim}\operatorname{im} u = \dim\ker g_2 - \operatorname{codim}\operatorname{im} g_2\,,$$

hence

$$\dim\ker g_1 + \operatorname{codim}\operatorname{im} g_2 = \dim\ker g_2 + \operatorname{codim}\operatorname{im} g_1\,,$$

whence

$$\operatorname{codim} g_1 + \operatorname{codim} g_2 = 2(\dim\ker g_2 + \operatorname{codim}\operatorname{im} g_1)\,. \tag{6.10}$$

It follows from (6.8) that $\dim \boldsymbol{X} \leq \dim\ker g_2$ and $\dim \boldsymbol{Y} \leq \operatorname{codim}\operatorname{im} g_1$. This together with previous equality (6.10) implies inequality (6.9), and, consequently, inequality (6.7). This concludes the proof. □

Lemma 6.4. *Let $f\colon \boldsymbol{U} \to \boldsymbol{U}$ be an endomorphism of a vector space $\boldsymbol{U}$ of a finite dimension. We denote by 1 the identity endomorphism of $\boldsymbol{U}$. Then* $\operatorname{codim}(1+f)$ *is finite and even.*

Proof. We apply Lemma 6.1 putting $g := 1$ and $h := g + f = 1 + f$. Note that $\dim\ker 1 = \operatorname{codim}\operatorname{im} 1 = 0$. Thus it follows from (6.1) that $\dim\ker(1+f) = \operatorname{codim}\operatorname{im}(1+f)$, hence $\operatorname{codim}(1+f) = \dim\ker(1+f) + \operatorname{codim}\operatorname{im}(1+f)$ is even. □

7. The example of Bergman and Goodearl

In this section we recall the Goodearl's modification [9, Example 5.10] of the Bergman's example [9, Example 4.26] of a regular ring $\boldsymbol{R}_2$ which is not unit-regular but the matrix rings $\mathbb{M}_n(\boldsymbol{R}_2)$ are directly finite for all positive integers. The ring $\boldsymbol{R}_2$ is constructed as follows: Let $\boldsymbol{T}$ denote the ring $\mathbb{F}[[\,t\,]]$ of all formal power series over a field $\mathbb{F}$ in an indeterminate t, and let $\mathbb{K}$ denote the quotient field of $\boldsymbol{T}$. Denote by $\boldsymbol{S}$ the ring of all $a \in \operatorname{End}_{\mathbb{F}}(\boldsymbol{T})$ such that there is a positive integer n and $b \in \mathbb{K}$ with $(a-b)t^n\boldsymbol{T} = 0$ (i.e., $b\,t^n\boldsymbol{T} \subseteq \boldsymbol{T}$ and the restriction $a \restriction t^n\boldsymbol{T}$ coincides with the multiplication by b). It turns out that the element $b \in \mathbb{K}$ is unique and the correspondence $a \mapsto b := \varphi(a)$ determines an $\mathbb{F}$-algebra homomorphism $\varphi\colon \boldsymbol{S} \to \mathbb{K}$ (cf. [9, Example 4.26]). Finally let us denote by $\boldsymbol{S}^{\mathrm{op}}$ the opposite ring of the ring $\boldsymbol{S}$ and set

$$\boldsymbol{R}_2 := \{\langle a_1, a_2\rangle \in \boldsymbol{S} \times \boldsymbol{S}^{\mathrm{op}} \mid \varphi(a_1) = \varphi(a_2)\}\}\,.$$

Observe that every nonzero element a of $\boldsymbol{T} = \mathbb{F}[[t]]$ is a product $a = t^n a'$ for some $n \in \mathbb{N}_0$ and some invertible $a' \in \boldsymbol{T}$. Moreover, every nonzero $b \in \mathbb{K}$ is of the form $b = t^z b'$ for a unique integer z and $b' \in \boldsymbol{T}$ invertible in $\boldsymbol{T}$. Denote the unique exponent z by $\nu(b)$ and set $\nu(0) := 0$. Let $b \neq 0$ be an element of $\mathbb{K}$. Observe that whenever $n + \nu(b) \geq 0$ for a positive integer n, the left multiplication by b determines a bijection $t^n\boldsymbol{T} \to t^{n+\nu(b)}\boldsymbol{T}$.

Given an element $\boldsymbol{a} = \langle a_1, a_2\rangle \in \boldsymbol{R}_2$, we define $\varphi(\boldsymbol{a}) := \varphi(a_1) = \varphi(a_2)$. For elements $a \in \boldsymbol{S}$ and $\boldsymbol{a} \in \boldsymbol{R}_2$ we set $\nu(a) := \nu(\varphi(a))$ and $\nu(\boldsymbol{a}) = \nu(\varphi(\boldsymbol{a}))$, respectively. Finally given an element $\boldsymbol{a} = \langle a_1, a_2\rangle \in \boldsymbol{R}_2$, we define $\dim \boldsymbol{a} := \dim a_1 + \dim a_2$ and $\operatorname{codim} \boldsymbol{a} := \operatorname{codim} a_1 + \operatorname{codim} a_2$.[2]

Let $\boldsymbol{a} \in \boldsymbol{R}_2$. Observe that $\varphi(\boldsymbol{a}) = 0$ implies that $\dim \boldsymbol{a}$ is finite while $\varphi(\boldsymbol{a}) \neq 0$ implies that $\operatorname{codim} \boldsymbol{a}$ is finite; the latter follows from the first statement of Lemma 6.1.

[2] Note that this is consistent with the notation introduced in Section 6.

Lemma 7.1. *Let a be an element of the ring $\boldsymbol{S}$. Then*

$$\begin{aligned} \varphi(a) = 0 &\implies \dim a \quad \text{is even}\,, \\ \varphi(a) = 1 &\implies \operatorname{codim} a \quad \text{is even}\,. \end{aligned} \tag{7.1}$$

Proof. It follows from the finiteness of $\dim a$, $\operatorname{codim} a$, respectively, and Lemma 6.4. ☐

Corollary 7.2. *Let $\boldsymbol{a} = (a_1, a_2)$ be an element of the ring $\boldsymbol{R}_2$. Then*

(1) *$\varphi(\boldsymbol{a}) = 0$ implies that both the dimensions $\dim a_1$ and $\dim a_2$ are even;*

(2) *$\varphi(\boldsymbol{a}) \neq 0$ implies that $\operatorname{codim} \boldsymbol{a}$ is even.*

Proof. If $\varphi(\boldsymbol{a}) = 0$, then both the dimensions $\dim a_1$ and $\dim a_2$ are finite and (1) follows readily from Lemma 7.1. Suppose that $\boldsymbol{a} \in \boldsymbol{R}_2$ satisfy $\varphi(\boldsymbol{a}) \neq 0$. Since $\boldsymbol{R}_2$ is a regular ring, there is an idempotent $\boldsymbol{e} \in \boldsymbol{R}_2$ such that $\boldsymbol{e}\boldsymbol{R}_2 = \boldsymbol{a}\boldsymbol{R}_2$. Then clearly $\operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{e}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{a})$. From $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{e})$ and $\boldsymbol{e} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{a})$ we get that $\operatorname{codim} \boldsymbol{a} \leq \operatorname{codim} \boldsymbol{e}$ and $\operatorname{codim} \boldsymbol{e} \leq \operatorname{codim} \boldsymbol{a}$, due to Lemma 6.3. Therefore $\operatorname{codim} \boldsymbol{a} = \operatorname{codim} \boldsymbol{e}$. Since $\boldsymbol{e}$ is an idempotent of a finite codimension, $\varphi(\boldsymbol{e}) = 1$, and so the codimension of $\boldsymbol{e}$ is even due to Lemma 7.1. ☐

Lemma 7.3. *Let $\boldsymbol{U}_i$, $\boldsymbol{V}_i$, $i = 1, 2$, be finite-dimensional vector spaces over a common field $\mathbb{F}$, let $a\colon \boldsymbol{U}_1 \to \boldsymbol{U}_2$ and $b\colon \boldsymbol{V}_1 \to \boldsymbol{V}_2$ be linear maps. Then $\dim a \leq \dim b$ if and only if there are linear maps $r\colon \boldsymbol{U}_1 \to \boldsymbol{V}_1$ and $s\colon \boldsymbol{V}_2 \to \boldsymbol{U}_2$ such that $a = sbr$.*

Proof. Folklore. ☐

Lemma 7.4. *Let $\boldsymbol{U}$ be a vector space,*

$$\boldsymbol{U} = \boldsymbol{U}_0 \supseteq \boldsymbol{U}_1 \supseteq \boldsymbol{U}_2 \supseteq \dots$$

be a decreasing sequence of subspaces of $\boldsymbol{U}$, and $\boldsymbol{V}$ a finite-dimensional subspace of $\boldsymbol{U}$. Suppose that

$$\boldsymbol{V} \cap \left(\bigcap_{i \in \mathbb{N}} \boldsymbol{U}_i \right) = \boldsymbol{0}\,,$$

then there is a positive integer n such that $\boldsymbol{V} \cap \boldsymbol{U}_n = \boldsymbol{0}$.

Proof. For every positive integer n we set $\boldsymbol{V}_n := \boldsymbol{V} \cap \boldsymbol{U}_n$. Note that

$$\boldsymbol{V} = \boldsymbol{V}_0 \supseteq \boldsymbol{V}_1 \supseteq \boldsymbol{V}_2 \supseteq \dots \tag{7.2}$$

is a decreasing sequence of subspaces of $\boldsymbol{V}$ such that $\bigcap_{i \in \mathbb{N}} \boldsymbol{V}_i = \boldsymbol{0}$. Since $\boldsymbol{V}$ is finite-dimensional, the sequence (7.2) is eventually stationary. Therefore there is $n \in \mathbb{N}$ such that $\boldsymbol{0} = \boldsymbol{V}_n = \boldsymbol{V} \cap \boldsymbol{U}_n$. ☐

We set

$$\boldsymbol{I} := \{ a \in \boldsymbol{S} \mid \varphi(a) = 0 \}\,.$$

It is straightforward to see that $\boldsymbol{I}$ is an ideal of the ring $\boldsymbol{S}$.

Lemma 7.5. *For all $a, b \in \boldsymbol{I}$, the following properties are equivalent:*

(1) $\dim a \leq \dim b$.

(2) $a \in \operatorname{tr}_{\boldsymbol{I}}(b)$.

(3) $a \in \operatorname{tr}_{\boldsymbol{S}}(b)$.

Proof. ($1 \Rightarrow 2$) Let $\boldsymbol{U}_1$ and $\boldsymbol{V}_1$ denote complements of $\ker a$ and $\ker b$, respectively, in $\boldsymbol{T}$ (viewed as a vector space over the field $\mathbb{F}$). We set $\boldsymbol{U}_2 := \operatorname{im} a$ and $\boldsymbol{V}_2 = \operatorname{im} b$ and we denote by $a' \colon \boldsymbol{U}_1 \to \boldsymbol{U}_2$, resp. $b' \colon \boldsymbol{V}_1 \to \boldsymbol{V}_2$ the restrictions $a' := a \restriction \boldsymbol{U}_1$, resp. $b' := b \restriction \boldsymbol{V}_1$. Observe that $\dim a' \leq \dim b'$. Applying Lemma 7.3, we find homomorphisms $r' \colon \boldsymbol{U}_1 \to \boldsymbol{V}_1$ and $s' \colon \boldsymbol{V}_2 \to \boldsymbol{U}_2$ such that $a' = s'b'r'$. There are positive integers m and n such that $\boldsymbol{U}_1 \cap t^m\boldsymbol{T} = \boldsymbol{0} = \boldsymbol{V}_1 \cap t^n\boldsymbol{T}$ due to Lemma 7.4. It follows that there are r and s in $\operatorname{End}_{\mathbb{F}}(\boldsymbol{T})$ extending r' and s', satisfying $t^m\boldsymbol{T} \subseteq \ker r$ and $t^nT \subseteq \ker s$, respectively. It follows that $r, s \in \boldsymbol{I}$ and that $a = sbr$, hence $a \in \operatorname{tr}_{\boldsymbol{I}}(b)$. The implication (2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (1) follows from Lemma 7.3. $\square$

We set

$$\boldsymbol{J}_2 := \{\boldsymbol{a} \in \boldsymbol{R}_2 \mid \varphi(\boldsymbol{a}) = 0\}\,,$$

and observe that $\boldsymbol{J}_2$ is an ideal of the ring $\boldsymbol{R}_2$. The next corollary follows readily from Lemma 7.5.

Corollary 7.6. *Let $\boldsymbol{a} = \langle a_1, a_2\rangle$ and $\boldsymbol{b} = \langle b_1, b_2\rangle$ be elements of the ideal $\boldsymbol{J}_2$. The following properties are equivalent:*

(1) $\dim a_i \leq \dim b_i$, *for all* $i = 1, 2$.

(2) $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{J}_2}(\boldsymbol{b})$.

(3) $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b})$.

For each ordered pair $m \leq n$ of non-negative integers we denote by $e_{m,n} \colon \boldsymbol{T} \to \boldsymbol{T}$ the projection onto $\bigoplus_{i=m}^{n} t^i\mathbb{F}$ given by

$$\sum_{i=0}^{\infty} a_i t^i \mapsto \sum_{i=m}^{n} a_i t^i\,.$$

Lemma 7.7. *Let λ be a positive integer and $\boldsymbol{e} = \langle e_1, e_2\rangle$ an idempotent of the ring $\boldsymbol{R}_2$ such that $\varphi(\boldsymbol{e}) \neq 0$. Then the following hold true:*

(1) *If* $\operatorname{codim} e_1 \geq 2\lambda$, *there is* $\boldsymbol{f} = \langle f_1, f_2\rangle \in \operatorname{Idem}(\boldsymbol{R}_2)$ *with*

$$\operatorname{codim} f_1 = \operatorname{codim} e_1 - 2\lambda$$
$$\operatorname{codim} f_2 = \operatorname{codim} e_2 + 2\lambda$$

and elements $\boldsymbol{r}$, $\boldsymbol{s} \in \boldsymbol{R}_2$ *such that* $\varphi(\boldsymbol{r}) = t^{\lambda}$, $\varphi(\boldsymbol{s}) = t^{-\lambda}$, *and* $\boldsymbol{f} = \boldsymbol{ser}$.

(2) *If* $\operatorname{codim} e_2 \geq 2\lambda$, *there is* $\boldsymbol{f} = \langle f_1, f_2\rangle \in \operatorname{Idem}(\boldsymbol{R}_2)$ *with*

$$\operatorname{codim} f_1 = \operatorname{codim} e_1 + 2\lambda$$
$$\operatorname{codim} f_2 = \operatorname{codim} e_2 - 2\lambda$$

and elements $\boldsymbol{r}$, $\boldsymbol{s} \in \boldsymbol{R}_2$ *such that* $\varphi(\boldsymbol{r}) = t^{-\lambda}$, $\varphi(\boldsymbol{s}) = t^{\lambda}$, *and* $\boldsymbol{f} = \boldsymbol{ser}$.

Proof. We prove property (1). Property (2) is symmetric. Since $\varphi(\boldsymbol{e}) \neq 0$ and $\boldsymbol{e}$ is an idempotent, we have that $\varphi(\boldsymbol{e}) = 1$. By the definition (of the ring $\boldsymbol{R}_2$) there is a natural number n such that $(e_i - 1)t^n\boldsymbol{T} = 0$, in particular $\ker e_i \cap t^n\boldsymbol{T} = \mathbf{0}$, for all $i = 1, 2$. For each $i = 1, 2$ we pick a complement $\boldsymbol{U}_i$ of $t^n\boldsymbol{T} \oplus \ker e_i$ in $\boldsymbol{T}$.

Observe that the restrictions $e_i \restriction (t^n\boldsymbol{T} \oplus \boldsymbol{U}_i)$ are one-to-one. Since $e_i \restriction t^n\boldsymbol{T}$ coincides with identity, we conclude that $e_i\boldsymbol{U}_i \cap t^n\boldsymbol{T} = \mathbf{0}$, for all $i = 1, 2$. Since $\operatorname{codim} t^n\boldsymbol{T} = n$ is finite, we get that

$$\operatorname{codim}\operatorname{im} e_i = \operatorname{codim}(e_i\boldsymbol{U}_i \oplus t^n\boldsymbol{T}) = \operatorname{codim}(\boldsymbol{U}_i \oplus t^n\boldsymbol{T}) = \dim\ker e_i\,,$$

hence $\operatorname{codim} e_i = 2\dim\ker e_i$, for both $i = 1, 2$. Since $\operatorname{codim} e_1 \geq 2\lambda$, we get that $\dim\ker e_1 \geq \lambda$, hence $\dim \boldsymbol{U}_1 = \operatorname{codim} t^n\boldsymbol{T} - \dim\ker e_1 \leq n - \lambda$. It follows that there are $\mathbb{F}$-linear maps

$$r_1' \colon \bigoplus_{i=0}^{n-\lambda-1} t^i\mathbb{F} \to \boldsymbol{U}_1 \text{ and } s_1' \colon e_1\boldsymbol{U}_1 \to \bigoplus_{i=0}^{n-\lambda-1} t^i\mathbb{F}$$

such that the composition $s_1'e_1r_1'$ is an idempotent linear map with $\dim s_1'e_1r_1' = 2\dim\boldsymbol{U}_1$. Clearly $\dim\boldsymbol{U}_2 \leq \operatorname{codim} t^n\boldsymbol{T} = n$. Therefore there are

$$r_2' \colon \bigoplus_{i=0}^{n+\lambda-1} t^i\mathbb{F} \to \boldsymbol{U}_2 \text{ and } s_2' \colon e_2\boldsymbol{U}_2 \to \bigoplus_{i=0}^{n+\lambda-1} t^i\mathbb{F}$$

such that $s_2'e_2r_2'$ is idempotent and $\dim s_2'e_2r_2' = 2\dim\boldsymbol{U}_2$. We extend the linear maps r_i', s_i', $i = 1, 2$, to $\mathbb{F}$-endomorphisms of $\boldsymbol{T}$ by setting

$$r_1'(t^{n-\lambda}\boldsymbol{T}) = \mathbf{0},\ r_2'(t^{n+\lambda}\boldsymbol{T}) = \mathbf{0},\ \text{and } s_i'(t^n\boldsymbol{T} \oplus \boldsymbol{W}_i) = \mathbf{0}\,,$$

where $\boldsymbol{W}_i$ are complements of $t^n\boldsymbol{T} \oplus e_i\boldsymbol{U}_i$, for both $i = 1, 2$. Observe that r_i', s_i' belong to $\boldsymbol{I}$.

Let us define $r \colon t^{n-\lambda}\boldsymbol{T} \to \boldsymbol{T}^n$, resp. $s \colon t^n\boldsymbol{T} \to \boldsymbol{T}^{n-\lambda}$, to be the $\mathbb{F}$-liner maps corresponding to multiplications by t^λ, resp. $t^{-\lambda}$, and we extend these maps to $\mathbb{F}$-endomorphisms of $\boldsymbol{T}$ by setting $r(\bigoplus_{i=0}^{n-\lambda-1} t^i\mathbb{F}) = s(\bigoplus_{i=0}^{n} t^i\mathbb{F}) = \mathbf{0}$. Observe that both r and s belong to $\boldsymbol{S}$.

We set $r_i := r + r_i'$ and $s_i = s + s_i'$, for both $i = 1, 2$. Then it is straightforward from the constructions of the endomorphisms r, r_i, s, and s_i that $\varphi(r_i) = \varphi(r) = t^\lambda$, $\varphi(s_i) = \varphi(s) = t^{-\lambda}$, and that $f_i = s_i'e_ir_i' = s_ie_ir_i$ are idempotents, for all $i = 1, 2$. Furthermore we have that

$$\operatorname{codim} f_1 = \operatorname{codim} e_1 - 2\lambda \text{ and } \operatorname{codim} f_2 = \operatorname{codim} e_2 + 2\lambda\,.$$

Finally setting $\boldsymbol{f} := \langle f_1, f_2\rangle$, $\boldsymbol{r} = \langle r_1, r_2\rangle$, and $\boldsymbol{s} = \langle s_1, s_2\rangle$, we get the desired idempotent and elements of $\boldsymbol{R}_2$ such that $\boldsymbol{f} = \boldsymbol{ser}$. $\square$

Observe that $\boldsymbol{f} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{e})$ and since $\operatorname{codim}\boldsymbol{f}$ is finite, and it is an idempotent, we have that $\varphi(\boldsymbol{f}) = 1$.

Lemma 7.8. *Let e, $f \in \boldsymbol{S} \setminus \boldsymbol{I}$ be idempotents. Then $\operatorname{codim} e \geq \operatorname{codim} f$ if and only if there are elements r, $s \in \boldsymbol{S}$ such that $\varphi(r) = \varphi(s) = 1$ and $e = sfr$. In particular, if any of the equivalent properties is satisfied, then $e \in \operatorname{tr}_{\boldsymbol{S}}(f)$.*

Proof. ($\Leftarrow$) First suppose that $e = sfr$ for some r, $s \in \boldsymbol{T}$ with $\varphi(r) = \varphi(s) = 1$. Since e, $f \in \operatorname{Idem}(\boldsymbol{S}) \setminus \boldsymbol{I}$, there is a positive integer n such that $(e-1)t^n\boldsymbol{T} = (s-1)t^n\boldsymbol{T} = (f-1)t^n\boldsymbol{T} = (r-1)t^n\boldsymbol{T} = \boldsymbol{0}$. It follows that $et^n\boldsymbol{T} = ft^n\boldsymbol{T} = rt^n\boldsymbol{T} = st^n\boldsymbol{T} = t^n\boldsymbol{T}$, hence e, f, r, $s \in \operatorname{End}_{\mathbb{F}}(\boldsymbol{T})$ induce endomorphisms e', f', r', and s' of the finite-dimensional $\mathbb{F}$-vector space $\boldsymbol{T}/t^n\boldsymbol{T}$. From $\operatorname{codim} e = \operatorname{codim} e'$, $\operatorname{codim} f = \operatorname{codim} f'$, and $\dim e' = \dim s'f'r' \leq \dim f'$, we deduce that

$$\operatorname{codim} e = \operatorname{codim} e' = 2n - \dim e' \geq 2n - \dim f' = \operatorname{codim} f' = \operatorname{codim} f\,.$$

($\Rightarrow$) Suppose now that $\operatorname{codim} e \geq \operatorname{codim} f$. Since e and f are idempotents not in $\boldsymbol{I}$, we have that $\varphi(e) = \varphi(f) = 1$. It follows that there is a positive integer n such that $(e-1)t^n\boldsymbol{T} = (f-1)t^n\boldsymbol{T} = \boldsymbol{0}$. Therefore $\ker e \cap t^n\boldsymbol{T} = \ker f \cap t^n\boldsymbol{T} = \boldsymbol{0}$. We pick subspaces $\boldsymbol{U}$ and $\boldsymbol{V}$ of the $\mathbb{F}$-vector space $\boldsymbol{T}$ such that

$$\boldsymbol{T} = \boldsymbol{U} \oplus \ker e \oplus t^n\boldsymbol{T} = \boldsymbol{V} \oplus \ker f \oplus t^n\boldsymbol{T}$$

and we set $e' := e \restriction \boldsymbol{U}$, $f' := f \restriction \boldsymbol{V}$. Since $\ker e \cap (\boldsymbol{U} \oplus t^n\boldsymbol{T}) = \boldsymbol{0}$ and the restriction $e \restriction t^n\boldsymbol{T}$ coincides with the identity map, we have that $e\boldsymbol{T} = e\boldsymbol{U} \oplus t^n\boldsymbol{T}$. Similarly we prove that $f\boldsymbol{T} = f\boldsymbol{V} \oplus t^n\boldsymbol{T}$. It follows that

$$\dim \boldsymbol{U} = \dim e\boldsymbol{U} = n - \frac{\operatorname{codim} e}{2} \leq n - \frac{\operatorname{codim} f}{2} = \dim f\boldsymbol{V} = \dim \boldsymbol{V}$$

and there are linear maps $r' \colon \boldsymbol{U} \to \boldsymbol{V}$ and $s' \colon f\boldsymbol{V} \to e\boldsymbol{U}$ such that $e' = s'f'r'$. There are r, $s \in \operatorname{End}_{\mathbb{F}}(\boldsymbol{W})$ such that

$$\begin{aligned} r \restriction \boldsymbol{U} = r',\ \ker r \geq \ker e,\quad \text{and}\quad (r-1)t^n\boldsymbol{T} = \boldsymbol{0}\,,\\ s \restriction \boldsymbol{V} = s',\ \ker s \geq \ker f,\quad \text{and}\quad (s-1)t^n\boldsymbol{T} = \boldsymbol{0}\,. \end{aligned}$$

We conclude that r and s are elements of $\boldsymbol{S}$ satisfying $\varphi(r) = \varphi(s) = 1$ and $e = sfr$. As an immediate consequence we have that $e \in \operatorname{tr}_{\boldsymbol{S}}(f)$. ☐

The next corollary will be applied in the forthcoming section.

Corollary 7.9. *Let λ be a positive integer and $\boldsymbol{e} = \langle e_1, e_2\rangle$ an idempotent in $\boldsymbol{R}_2 \setminus \boldsymbol{J}_2$. Then the following hold true:*

(1) *Suppose that $\operatorname{codim} e_1 \geq 2\lambda$ and let $\boldsymbol{f} = \langle f_1, f_2\rangle$ be the idempotent constructed in Lemma 7.7. Then there are elements $\boldsymbol{r}^*, \boldsymbol{s}^* \in \boldsymbol{R}_2$ with $\varphi(\boldsymbol{r}^*) = t^{-\lambda}$ and $\varphi(\boldsymbol{s}^*) \in t^{\lambda}$ such that $\boldsymbol{e} = \boldsymbol{s}^*\boldsymbol{f}\boldsymbol{r}^*$.*

(2) *Suppose that $\operatorname{codim} e_2 \geq 2\lambda$ and let $\boldsymbol{f} = \langle f_1, f_2\rangle$ be the idempotent constructed in Lemma 7.7. Then there are elements $\boldsymbol{r}^*$, $\boldsymbol{s}^* \in \boldsymbol{R}_2$ with $\varphi(\boldsymbol{r}^*) = t^{\lambda}$ and $\varphi(\boldsymbol{s}^*) \in t^{-\lambda}$ such that $\boldsymbol{e} = \boldsymbol{s}^*\boldsymbol{f}\boldsymbol{r}^*$.*

Proof. Both the cases are symmetric, we only prove (1). Suppose that $\operatorname{codim} e_1 \geq 2\lambda$. Then $\operatorname{codim} f_2 = \operatorname{codim} e_2 + 2\lambda \geq 2\lambda$, and so there is an idempotent $\boldsymbol{g} = \langle g_1, g_2\rangle \in \boldsymbol{R}_2$ with

$$\begin{aligned} &\operatorname{codim} g_1 = \operatorname{codim} f_1 + 2\lambda = \operatorname{codim} e_1 \qquad \text{and}\\ &\operatorname{codim} g_2 = \operatorname{codim} f_2 - 2\lambda = \operatorname{codim} e_2\,, \end{aligned} \tag{7.3}$$

and element $\boldsymbol{r}'$, $\boldsymbol{s}' \in \boldsymbol{R}_2$ such that $(\boldsymbol{r}') = t^{-\lambda}$, $\varphi(\boldsymbol{s}') = t^{\lambda}$, and $\boldsymbol{g} = \boldsymbol{s}'\boldsymbol{f}\boldsymbol{r}'$ due to Lemma 7.7. Applying Lemma 7.8, we get elements $\boldsymbol{r}'', \boldsymbol{s}'' \in \boldsymbol{R}_2$ with $\varphi(\boldsymbol{r}'') = \varphi(\boldsymbol{s}'') = 1$ and

$$\boldsymbol{e} = \boldsymbol{s}''\boldsymbol{g}\boldsymbol{r}'' = \boldsymbol{s}''\boldsymbol{s}'\boldsymbol{f}\boldsymbol{r}'\boldsymbol{r}'' .$$

We put $\boldsymbol{r}^* = \boldsymbol{r}'\boldsymbol{r}''$ and $\boldsymbol{s}^* = \boldsymbol{s}''\boldsymbol{s}'$. It is straightforward to compute that

$$\varphi(\boldsymbol{r}^*) = \varphi(\boldsymbol{r}'\boldsymbol{r}'') = \varphi(\boldsymbol{r}')\varphi(\boldsymbol{r}'') = t^{-\lambda} \text{ and } \varphi(\boldsymbol{s}^*) = \varphi(\boldsymbol{s}''\boldsymbol{r}') = \varphi(\boldsymbol{s}'')\varphi(\boldsymbol{s}') = t^{\lambda} .$$

□

Lemma 7.10. *Let $\boldsymbol{a}$, $\boldsymbol{b} \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$. Then $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b})$ if and only if* $\operatorname{codim} \boldsymbol{a} \geq \operatorname{codim} \boldsymbol{b}$.

Proof. (⇒) It follows from Lemma 6.3 that

$$\operatorname{codim} \boldsymbol{c}\boldsymbol{d} \geq \max\{\operatorname{codim} \boldsymbol{c}, \operatorname{codim} \boldsymbol{d}\} , \tag{7.4}$$

for all $\boldsymbol{c}$, $\boldsymbol{d} \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$. If $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b})$, then $\boldsymbol{a} = \boldsymbol{s}\boldsymbol{b}\boldsymbol{r}$ for some $\boldsymbol{s}$, $\boldsymbol{r} \in \boldsymbol{R}_2$. Observe that $\boldsymbol{s}$, $\boldsymbol{r} \notin \boldsymbol{J}_2$, for otherwise $\boldsymbol{a} \in \boldsymbol{J}_2$. Applying (7.4) twice, we get that

$$\operatorname{codim} \boldsymbol{a} = \operatorname{codim} \boldsymbol{s}\boldsymbol{b}\boldsymbol{r} \geq \operatorname{codim} \boldsymbol{b}\boldsymbol{r} \geq \operatorname{codim} \boldsymbol{b} .$$

(⇐) Suppose that $\operatorname{codim} \boldsymbol{a} \geq \operatorname{codim} \boldsymbol{b}$. Since $\boldsymbol{R}_2$ is regular there are idempotents $\boldsymbol{e} = \langle e_1, e_2 \rangle$ and $\boldsymbol{f} = \langle f_1, f_2 \rangle$ such that $\boldsymbol{e}\boldsymbol{R}_2 = \boldsymbol{a}\boldsymbol{R}_2$ and $\boldsymbol{f}\boldsymbol{R}_2 = \boldsymbol{b}\boldsymbol{R}_2$, respectively. As a consequence we get that

$$\operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{e}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{a}) \text{ and } \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{f}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b}) . \tag{7.5}$$

By the already proved implication we have that

$$\operatorname{codim} \boldsymbol{e} = \operatorname{codim} \boldsymbol{a} \geq \operatorname{codim} \boldsymbol{b} = \operatorname{codim} \boldsymbol{f} .$$

By Lemma 7.7, there is an idempotent $\boldsymbol{g} = \langle g_1, g_2 \rangle \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{f})$ such that $\operatorname{codim} e_1 \geq \operatorname{codim} g_1$ and $\operatorname{codim} e_2 \geq \operatorname{codim} g_2$. By Lemma 7.8, there are elements r_i, $s_i \in \boldsymbol{S}$, $i = 1, 2$, such that $\varphi(r_i) = \varphi(s_i) = 1$ and $e_i = s_i g_i r_i$. It follows that $\boldsymbol{e} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{g}) \subseteq \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{f})$, and so $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{a}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{e}) \subseteq \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{f}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b})$ due to (7.5). □

Lemma 7.11. *Let $\boldsymbol{a} = \langle a_1, a_2 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2 \rangle$ be elements of the ring $\boldsymbol{R}_2$. Then $\operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{a}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b})$ if and only if either both $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to $\boldsymbol{J}_2$ and $\dim a_i = \dim b_i$ for both $i = 1, 2$, or none of the elements $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to $\boldsymbol{J}_2$ and then* $\operatorname{codim} \boldsymbol{a} = \operatorname{codim} \boldsymbol{b}$.

Proof. (⇒) Assume that $\operatorname{tr}_{\boldsymbol{R}_2}(a) = \operatorname{tr}_{\boldsymbol{R}_2}(b)$. Since $\boldsymbol{J}_2$ is a two-sided ideal of $\boldsymbol{R}_2$, either both the elements $\boldsymbol{a}$ and $\boldsymbol{b}$ or none of them belong to $\boldsymbol{J}_2$. If $\boldsymbol{a}$, $\boldsymbol{b} \in \boldsymbol{J}_2$, then both $\dim a_i = \dim b_i$, $i = 1, 2$, due to Corollary 7.6. In the other case when $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$, the equality $\operatorname{codim} \boldsymbol{a} = \operatorname{codim} \boldsymbol{b}$ holds true due to Lemma 7.10.
(⇐) This implication follows readily from Corollary 7.6 and Lemma 7.10. □

Lemma 7.12. *Let $g \in \boldsymbol{I}$ be an idempotent, λ and μ non-negative integers such that* $\dim g = 2\lambda + 2\mu$. *Then there is a pair e, $f \in \boldsymbol{I}$ of orthogonal idempotents such that* $\dim e = 2\lambda$, $\dim f = 2\mu$, *and* $g = e + f$.

Proof. Since $g \in \boldsymbol{I}$, it is of a finite dimension, and so $\dim \operatorname{im} g = (\dim g)/2 = \lambda + \mu$. We pick a decomposition $\operatorname{im} g = \boldsymbol{U} \oplus \boldsymbol{V}$ with $\dim \boldsymbol{U} = \lambda$ and $\dim \boldsymbol{V} = \mu$. Let e be an endomorphism of $\boldsymbol{T}$ such that $\ker e = \ker g \oplus \boldsymbol{U}$ and $e \restriction \boldsymbol{V} = g \restriction \boldsymbol{V}$. Putting $f = g - e$, we get a pair e, f of orthogonal idempotents with the desired properties. □

Lemma 7.13. *Let $g \in \boldsymbol{S} \setminus \boldsymbol{I}$ be an idempotent, λ and μ non-negative integers such that $2\lambda = \operatorname{codim} g + 2\mu$. Then there is a pair $e \in \boldsymbol{S} \setminus \boldsymbol{I}$ and $f \in \boldsymbol{I}$ of orthogonal idempotents such that $\operatorname{codim} e = 2\lambda$, $\dim f = 2\mu$, and $g = e + f$.*

Proof. From $g \in \boldsymbol{S} \setminus \boldsymbol{I}$ we infer that $\dim \operatorname{im} g$ is infinite. We find a decomposition $\operatorname{im} g = \boldsymbol{U} \oplus \boldsymbol{V}$ such that $\dim \boldsymbol{U} = \mu$. Let $f \in \operatorname{End}_{\mathbb{F}}(\boldsymbol{T})$ be such that $\ker f = \ker g \oplus \boldsymbol{V}$ and $f \restriction \boldsymbol{U} = g \restriction \boldsymbol{U}$. Putting $e = g - f$, we get a pair of orthogonal idempotents $e \in \boldsymbol{S} \setminus \boldsymbol{I}$ and $f \in \boldsymbol{I}$ satisfying the desired properties. □

Applying Lemma 7.13 we get that

Corollary 7.14. *Let $\boldsymbol{g} \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$ be an idempotent. Let λ, μ_1, μ_2 be non-negative integers such that $2\lambda = \operatorname{codim} \boldsymbol{g} + 2\mu_1 + 2\mu_2$. Then there are orthogonal idempotents $\boldsymbol{e} \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$ and $\boldsymbol{f} = \langle f_1, f_2 \rangle \in \boldsymbol{J}_2$ such that $\operatorname{codim} \boldsymbol{e} = 2\lambda$, $\dim f_i = 2\mu_i$, for all $i = 1, 2$, and $\boldsymbol{g} = \boldsymbol{e} + \boldsymbol{f}$.*

Theorem 7.15. *The monoid $\mathcal{V}(\boldsymbol{R}_{2n})$ is isomorphic to $\mathfrak{C}_2 = \mathfrak{D}_2$ and, via the isomorphism $\psi_2^{-1} \colon \mathfrak{D}_2 \to \mathfrak{B}_2$, also to $\mathfrak{B}_2$.*

Proof. We define a map $\gamma \colon \operatorname{Idem}(\boldsymbol{R}_2) \to \mathfrak{C}_2$ by

$$\boldsymbol{e} = \langle e_1, e_2 \rangle \mapsto \begin{cases} \langle 0, \frac{\dim e_1}{2}, \frac{\dim e_2}{2} \rangle & \text{if } \ \varphi(\boldsymbol{e}) = 0\,, \\ \langle 1, -\frac{\operatorname{codim} \boldsymbol{e}}{2} \rangle & \text{if } \ \varphi(\boldsymbol{e}) = 1\,, \end{cases}$$

and we verify that the properties (1)–(3) of Corollary 3.9 are satisfied. Property (1) follows from Lemma 7.11.

Property (2) is a consequence of Lemma 7.12 in the case that $\varphi(\boldsymbol{e}) = 0$ and Corollary 7.14 if $\varphi(\boldsymbol{e}) = 1$. Observe that in the latter case, when $\varphi(\boldsymbol{e}) = 1$, if $\gamma(\boldsymbol{e}) = u + v$ for some u, $v \in \mathfrak{C}_2$, one of them belongs to $\mathfrak{O}_2$. This is because

$$\gamma(\boldsymbol{e}) = \langle 1, -\frac{\operatorname{codim} \boldsymbol{e}}{2} \rangle\,,$$

and so $u_0 + v_0 = 1$.

By the definition, $\gamma(1) = \langle 1, 0 \rangle$ which is an order-unit in $\mathfrak{C}_2$, thus property (3) holds true as well.

Since the ring $\boldsymbol{R}_2$ is directly finite due to [9, Example 5.10], the map γ extends to a unique isomorphism $\beta \colon \mathcal{V}(\boldsymbol{R}_2) \to \mathfrak{C}_2$, due to Corollary 3.9. □

8. Representing the monoids $\mathfrak{B}_{2n}$

Let $\boldsymbol{R}_2$ and $\boldsymbol{S}$ be the rings defined in the previous section. Given a positive integer n, we set

$$\boldsymbol{R}_{2n} := \{ \langle a_1, a_2, \dots, a_{2n} \rangle \mid a_{2i-1} \in \boldsymbol{S}, a_{2i} \in \boldsymbol{S}^{\mathrm{op}}, \text{ and } \ \varphi(a_1) = \dots = \varphi(a_{2n}) \}\,.$$

Observe that $\boldsymbol{R}_{2n}$ is a sub-direct product of copies of the ring $\boldsymbol{R}_2$. Therefore it is regular and directly finite (cf. [9, Proposition 1.4] and [9, Lemma 5.1], respectively). Further, we set

$$\boldsymbol{J}_{2n} := \{\langle a_1, a_2, \dots, a_{2n}\rangle \in \boldsymbol{R}_{2n} \mid \varphi(a_1) = \cdots = \varphi(a_{2n}) = 0\}\,.$$

Clearly, the set $\boldsymbol{J}_{2n}$ forms a two-sided ideal of the ring $\boldsymbol{R}_{2n}$. Applying Lemma 7.5 we get, similarly as in the previous section, that

Lemma 8.1. *For a pair of elements $\boldsymbol{a} = \langle a_1, \dots, a_{2n}\rangle$ and $\boldsymbol{b} = \langle b_1, \dots, b_{2n}\rangle$ from $\boldsymbol{J}_{2n}$, the following properties are equivalent:*

(1) $\dim a_i \leq \dim b_i$ *for all* $i = 1, \dots, 2n$.

(2) $\boldsymbol{a} \in \mathrm{tr}_{\boldsymbol{J}_{2n}}(\boldsymbol{b})$.

(3) $\boldsymbol{a} \in \mathrm{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b})$.

Let $\boldsymbol{a} = \langle a_1, \dots, a_{2n}\rangle$ be an element of the ring $\boldsymbol{R}_{2n}$. For each $i, j \in \{1, 2, \dots, n\}$ we set $\boldsymbol{a}_{\{2i-1,2j\}} := \langle a_{2i-1}, a_{2j}\rangle$. Observe that $\boldsymbol{a}_{\{2i-1,2j\}}$ is an element of the ring $\boldsymbol{R}_2$.

Lemma 8.2. *Let $n \in \mathbb{N}$ and a_i, b_i, $i \in \{1, 2, \dots, 2n\}$, integers such that*

$$a_{2i-1} + a_{2j} \geq b_{2i-1} + b_{2j}\,. \tag{8.1}$$

for all $i, j \in \{1, 2, \dots, n\}$. Then there is an integer λ such that

$$a_{2i-1} + \lambda \geq b_{2i-1} \quad \text{and} \quad a_{2j} - \lambda \geq b_{2j}\,, \tag{8.2}$$

for all $i, j = \{1, 2, \dots, n\}$.

Proof. The equations (8.1) are equivalent to

$$a_{2j} - b_{2j} \geq b_{2i-1} - a_{2i-1}\,,$$

for all i, $j \in \{1, 2, \dots, n\}$, hence

$$\min\{a_{2j} - b_{2j} \mid j = 1, \dots, n\} \geq \max\{b_{2i-1} - a_{2i-1} \mid i = 1, \dots, n\}\,.$$

We pick any integer λ with

$$\min\{a_{2j} - b_{2j} \mid j = 1, \dots, n\} \geq \lambda \geq \max\{b_{2i-1} - a_{2i-1} \mid i = 1, \dots, n\}$$

and observe that (8.2) holds true. □

Lemma 8.3. *Let $\boldsymbol{a} = \langle a_1, \dots, a_{2n}\rangle$ and $\boldsymbol{b} = \langle b_1, \dots, b_{2n}\rangle$ be elements of $\boldsymbol{R}_{2n} \setminus \boldsymbol{J}_{2n}$. Then $\boldsymbol{a} \in \mathrm{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b})$ if and only if $\operatorname{codim} \boldsymbol{a}_{\{2i-1,2j\}} \geq \operatorname{codim} \boldsymbol{b}_{\{2i-1,2j\}}$ for all $i, j \in \{1, 2, \dots, n\}$.*

Proof. ($\Rightarrow$) Suppose that $\boldsymbol{a} \in \mathrm{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b})$. Then $\boldsymbol{a}_{\{2i-1,2j\}} \in \mathrm{tr}_{\boldsymbol{R}_2}(\boldsymbol{b}_{\{2i-1,2j\}})$, which implies that $\operatorname{codim} \boldsymbol{a}_{\{2i-1,2j\}} \geq \operatorname{codim} \boldsymbol{b}_{\{2i-1,2j\}}$, for all $i, j \in \{1, 2, \dots, n\}$, due to Lemma 7.10.

($\Leftarrow$) Since the ring $\boldsymbol{R}_{2n}$ is regular, it contains idempotents $\boldsymbol{e} = \langle e_1, e_2, \dots, e_{2n}\rangle$ and $\boldsymbol{f} = \langle f_1, f_2, \dots, f_{2n}\rangle$ such that $\mathrm{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{a}) = \mathrm{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{e})$ and $\mathrm{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b}) = \mathrm{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{f})$. As we have just proved, this implies that $\operatorname{codim} \boldsymbol{a}_{\{2i-1,2j\}} = \operatorname{codim} \boldsymbol{e}_{\{2i-1,2j\}}$ and

$\operatorname{codim} \boldsymbol{b}_{\{2i-1,2j\}} = \operatorname{codim} \boldsymbol{f}_{\{2i-1,2j\}}$, for all i, $j \in \{1, 2, \dots, n\}$. According to the assumption we have that

$$\operatorname{codim} e_{2i-1} + \operatorname{codim} e_{2j} \geq \operatorname{codim} f_{2i-1} + \operatorname{codim} f_{2j},$$

for all i, $j \in \{1, 2, \dots, n\}$. By Lemma 7.1 all $\operatorname{codim} e_i$ and $\operatorname{codim} f_i$, $i = 1, \dots, 2n$, are even. Applying Lemma 8.2, there is an integer 2λ such that

$$\operatorname{codim} e_{2i-1} + 2\lambda \geq \operatorname{codim} f_{2i-1} \quad \text{and} \quad \operatorname{codim} e_{2j} - 2\lambda \geq \operatorname{codim} f_{2j}\,,$$

for all i, $j \in \{1, 2, \dots, n\}$. Applying Corollary 7.9 we find idempotents $\boldsymbol{g}_{\{2i-1,2i\}} = (g_{2i-1}, g_{2i}) \in \boldsymbol{R}_2 \backslash \boldsymbol{J}_2$, and elements $\boldsymbol{r}^*_{\{2i-1,2i\}} = \langle r^*_{2i-1}, r^*_{2i}\rangle$, $\boldsymbol{s}^*_{\{2i-1,2i\}} = \langle s^*_{2i-1}, s^*_{2i}\rangle \in \boldsymbol{R}_2$ with $\varphi(\boldsymbol{r}_{\{2i-1,2i\}}) = t^{\lambda}$, $\varphi(\boldsymbol{s}^*_{\{2i-1,2i\}}) = t^{-\lambda}$, for all $i \in \{1, 2, \dots, n\}$, satisfying

$$\begin{aligned} \operatorname{codim} g_{2i-1} &= \operatorname{codim} e_{2i-1} + 2\lambda\,, \\ \operatorname{codim} g_{2i} &= \operatorname{codim} e_{2i} - 2\lambda\,, \end{aligned}$$

and

$$\boldsymbol{e}_{\{2i-1,2i\}} = \boldsymbol{s}^*_{\{2i-1,2i\}} \boldsymbol{g}_{\{2i-1,2i\}} \boldsymbol{r}^*_{\{2i-1,2i\}}\,,$$

for all $i \in \{1, 2, \dots, n\}$. Putting $\boldsymbol{g} := \langle g_1, g_2, \dots, g_{2n}\rangle$, $\boldsymbol{r}^* := \langle r^*_1, r^*_2, \dots, r^*_{2n}\rangle$, and $\boldsymbol{s}^* := \langle s^*_1, s^*_2, \dots, s^*_{2n}\rangle$, we get elements of $\boldsymbol{R}_{2n}$ with $\varphi(\boldsymbol{g}) = 1$, $\varphi(\boldsymbol{r}^*) = t^{\lambda}$, and $\varphi(\boldsymbol{s}^*) = t^{-\lambda}$, satisfying $\boldsymbol{e} = \boldsymbol{s}^* \boldsymbol{g} \boldsymbol{r}^*$. Since $\operatorname{codim} g_i \geq \operatorname{codim} f_i$ for all $i = 1, \dots, 2n$, we have $\boldsymbol{r}$, $\boldsymbol{s} \in \boldsymbol{R}_{2n}$ with $\varphi(\boldsymbol{r}) = \varphi(\boldsymbol{s}) = 1$ satisfying $\boldsymbol{g} = \boldsymbol{s}\boldsymbol{f}\boldsymbol{r}$ due to Lemma 7.8. It follows that

$$\boldsymbol{e} = \boldsymbol{s}^* \boldsymbol{g} \boldsymbol{r}^* = \boldsymbol{s}^* \boldsymbol{s} \boldsymbol{f} \boldsymbol{r} \boldsymbol{r}^*\,,$$

hence $\boldsymbol{e} \in \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{f})$. Therefore $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b})$. □

The next lemma is an analogy of Lemma 7.11. It follows readily as a combination of Lemmas 8.1 and 8.3.

Lemma 8.4. *Let $\boldsymbol{a} = \langle a_1, a_2, \dots, a_{2n}\rangle$ and $\boldsymbol{b} = \langle b_1, b_2, \dots, b_{2n}\rangle$ be elements of the ring $\boldsymbol{R}_{2n}$. Then $\operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{a}) = \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b})$ if and only if either both $\boldsymbol{a}$, $\boldsymbol{b} \in \boldsymbol{J}_{2n}$ and*

$$\dim a_i = \dim b_i$$

for all $i \in \{1, 2, \dots, 2n\}$, or both $\boldsymbol{a}$, $\boldsymbol{b} \in \boldsymbol{R}_{2n} \setminus \boldsymbol{J}_{2n}$ and

$$\operatorname{codim} \boldsymbol{a}_{\{2i-1,2j\}} = \operatorname{codim} \boldsymbol{b}_{\{2i-1,2j\}}$$

for all i, $j \in \{1, 2, \dots, n\}$.

Theorem 8.5. *Let n be a positive integer. The monoid $\boldsymbol{\mathfrak{V}}(\boldsymbol{R}_{2n})$ is isomorphic to $\boldsymbol{\mathfrak{D}}_{2n}$ and, via the isomorphism $\psi^{-1}_{2n}\colon \boldsymbol{\mathfrak{D}}_{2n} \to \boldsymbol{\mathfrak{B}}_{2n}$, also to $\boldsymbol{\mathfrak{B}}_{2n}$.*

Proof. We define a map $\gamma\colon \operatorname{Idem}(\boldsymbol{R}_{2n}) \to \boldsymbol{\mathfrak{D}}_{2n}$ by

$$\boldsymbol{e} = \langle e_1, e_2, \dots, e_{2n}\rangle \mapsto \begin{cases} \langle 0, \frac{\dim e_1}{2}, \frac{\dim e_2}{2}, \dots, \frac{\dim e_{2n}}{2}\rangle \in \boldsymbol{\mathfrak{O}}_{2n} & \text{if } \varphi(\boldsymbol{e}) = 0\,, \\ \langle 1, -\frac{\operatorname{codim} \boldsymbol{e}_{\{i,j\}}}{2}\rangle_{\{i,j\}} \in \boldsymbol{\mathfrak{V}}_{2n} & \text{if } \varphi(\boldsymbol{e}) = 1\,, \end{cases}$$

and we verify that the properties (1–3) of Corollary 3.9 are satisfied. Property (1) follows from Lemma 8.4.

We are going to prove that (2) holds true. Let $x = \langle x_0, \dots \rangle, y = \langle y_0, \dots \rangle \in \boldsymbol{\mathfrak{D}}_{2n}$ and $\boldsymbol{g} = \langle g_1, g_2, \dots, g_{2n} \rangle \in \operatorname{Idem}(\boldsymbol{R}_{2n})$. The implication $(\Leftarrow)$ is trivial. In order to prove the opposite one, $(\Rightarrow)$, assume that $\gamma(g) = x + y$. We are going to discuss two cases.

The first case is when $\boldsymbol{g} \in \boldsymbol{J}_{2n}$. Then $0 = \varphi(\boldsymbol{g}) = x_0 + y_0$, hence $x_0 = y_0 = 0$ and both x_0 and y_0 belong to $\boldsymbol{\mathfrak{O}}_{2n}$. Applying Lemma 7.12, we find, for each $i \in \{1, 2, \dots, 2n\}$, a pair of orthogonal idempotents e_i, $f_i \in \boldsymbol{I}$ such that $\dim e_i = x_i$, $\dim f_i = y_i$, and $g_i = e_i + f_i$. Putting $\boldsymbol{e} = \langle e_1, e_2, \dots, e_{2n} \rangle$ and $\boldsymbol{f} = \langle f_1, f_2, \dots, f_{2n} \rangle$, we get a pair of orthogonal idempotents $\boldsymbol{e}$, $\boldsymbol{f}$ such that $\gamma(\boldsymbol{e}) = x$, $\gamma(\boldsymbol{f}) = y$, and $\boldsymbol{g} = \boldsymbol{e} + \boldsymbol{f}$.

The latter case is when $\boldsymbol{g}$ is an idempotent from $\boldsymbol{R}_{2n} \setminus \boldsymbol{J}_{2n}$. We can without loss of generality assume that $x_0 \geq y_0$. Since $x_0 + y_0 = z_0 = 1$, we get that $x_0 = 1$, hence $x \in \boldsymbol{\mathfrak{V}}_{2n}$, and $y_0 = 0$, hence $y \in \boldsymbol{\mathfrak{O}}_{2n}$. Applying Lemma 7.13 we find for each $i \in \{1, 2, \dots, 2n\}$ a pair of orthogonal idempotents $e_i \in \boldsymbol{S} \setminus \boldsymbol{I}$, and $f_i \in \boldsymbol{I}$ such that

$$\frac{\operatorname{codim} e_i}{2} = \frac{\operatorname{codim} g_i}{2} + y_i\,, \quad \frac{\dim f_i}{2} = y_i, \text{ and } g_i = e_i + f_i\,.$$

Set $\boldsymbol{e} := \langle e_1, e_2, \dots, e_{2n} \rangle$ and $\boldsymbol{f} := \langle f_1, f_2, \dots, f_{2n} \rangle$. Then $\boldsymbol{e} \in \boldsymbol{R}_{2n} \setminus \boldsymbol{J}_{2n}$ and $\boldsymbol{f} \in \boldsymbol{R}_{2n}$ are orthogonal idempotents such that $\boldsymbol{g} = \boldsymbol{e} + \boldsymbol{f}$ and $\gamma(\boldsymbol{f}) = y$. It follows that

$$\gamma(\boldsymbol{e}) + y = \gamma(\boldsymbol{g}) = x + y\,.$$

Applying Lemma 5.4, we infer from $y \in \boldsymbol{\mathfrak{O}}_{2n}$ that $\gamma(\boldsymbol{e}) = x$. Therefore property (2) is satisfied.

By the definition

$$\gamma(1) = \langle 1, \underbrace{0, \dots, 0}_{n^2 \times} \rangle\,,$$

which is an order-unit in $\boldsymbol{\mathfrak{D}}_{2n}$, thus property (3) holds true as well.

Since the ring $\boldsymbol{R}_{2n}$ is directly finite, the map γ extends to a unique isomorphism $\beta\colon \boldsymbol{\mathfrak{V}}(\boldsymbol{R}_{2n}) \to \boldsymbol{\mathfrak{D}}_{2n}$, due to Corollary 3.9. $\square$

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