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THE INFINITESIMAL COUNTERPART OF TANGENT PRESYMPLECTIC GROUPOIDS OF HIGHER ORDER

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Abstract. Let (G,ω) be a presymplectic groupoid. In this paper we characterize the infinitesimal counter part of the tangent presymplectic groupoid of higher order, $(T^rG,\omega^{(c)})$ where T^rG is the tangent groupoid of higher order and $\omega^{(c)}$ is the complete lift of higher order of presymplectic form ω.

1. Introduction

We denote by $\mathcal{LG}$ the category of Lie groupoids and by $\mathcal{LA}$ the category of Lie algebroids. For an objet G of $\mathcal{LG}$ over a manifold M, we denote the source and target map by s, $t\colon G\to M$, the multiplication $\mathfrak{m}\colon G_{(2)}\to G$ where $G_{(2)}$ is the set of composable arrows. By $\imath\colon G\to G$ and $\varepsilon\colon M\to G$ we denote respectively inversion map and unit section. There is a natural functor $A\colon\mathcal{LG}\to\mathcal{LA}$ which maps each objet $G\in\mathcal{LG}$ to the objet $AG\in\mathcal{LA}$, and every morphism of Lie groupoids $\psi\colon G_1\to G_2$ is mapped to the Lie algebroid morphism $A\psi\colon AG_1\to AG_2$ (see [13]). It is called the Lie functor and preserves fibered product. A symplectic groupoid is a pair (G,ω), where G is a Lie groupoid over M and ω is a symplectic form on G such that:

$$\mathfrak{m}^*\omega=(pr_1)^*\,\omega+(pr_2)^*\,\omega \tag{1.1}$$

where pr_1, $pr_2\colon G_{(2)}\to G$ are the natural projections. Such forms are usually called *multiplicative forms* (see [2]). Given a symplectic groupoid (G,ω), then there exists an isomorphism of Lie algebroids (see [4])

$$\begin{aligned}\sigma\colon AG&\to T^*M\\ u&\mapsto(i_u\omega)\mid_{TM}\end{aligned} \tag{1.2}$$

where the Lie algebroid structure on T^*M is the one induced by the poisson bivector π_ω on M such that the source map s is a Poisson map, and the target map t is anti-Poisson. The 1-form $(i_u\omega)\mid_{TM}$ is defined for any $x\in M$ and $h_x\in T_xM$ by:

$$(i_u\omega)(x)(h_x)=\omega(x)\big(i_{AG}(u),T_x\varepsilon(h_x)\big) \tag{1.3}$$

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and $i_{AG}\colon AG \to TG$ is a canonical injection. The anchor map induced by π_ω is given by:

$$\sharp_{\pi_\omega} = \rho_{AG} \circ \sigma^{-1} \tag{1.4}$$

where $\rho_{AG}\colon AG \to TM$ is the anchor map of the Lie algebroid AG. It follows that the Poisson manifolds can be thought as the infinitesimal counterparts of symplectic groupoids. The key point in the construction of π_ω is the isomorphism of vector bundles $\sigma\colon AG \to T^*M$, which in turn comes from the fact that ω is non degenerate. However, if $\omega \in \Omega^2(G)$ is a closed multiplicative form not necessarily symplectic the bundle map $\sigma\colon AG \to T^*M$ is not an isomorphism.

Let M be a smooth manifold, we consider the direct sum vector bundle $TM \oplus T^*M$ equipped with the nondegenerate symmetric pairing

$$\langle X \oplus \omega, Y \oplus \varpi \rangle_+ = \frac{1}{2}\big(\langle X, \varpi \rangle_M + \langle Y, \omega \rangle_M \big)$$

and the natural skew-symmetric pairing

$$\langle X \oplus \omega, Y \oplus \varpi \rangle_- = \frac{1}{2}\big(\langle X, \varpi \rangle_M - \langle Y, \omega \rangle_M \big)$$

where the bracket $\langle \cdot, \cdot \rangle_M : TM \oplus T^*M \to \mathbb{R}$ is the usual canonical duality pairing. The space of sections $\Gamma(TM \oplus T^*M) = \mathfrak{X}(M) \oplus \Omega^1(M)$ is endowed with the Courant bracket

$$[X \oplus \omega, Y \oplus \varpi] = [X, Y] \oplus (\mathcal{L}_X \varpi - i_Y d\omega)\ .$$

A Dirac structure on M (see [5]) is a sub-bundle $L \subset TM \oplus T^*M$ which is Lagrangian with respect to the non degenerate symmetric pairing $\langle\,;\rangle_+$ and involutive, in the sense that: the space of smooth sections $\Gamma(L)$ is closed under the Courant bracket. For example, a bivector field π on M induces a subbundle of $TM \oplus T^*M$ given by $L_\pi = \{\sharp_\pi(\alpha) \oplus \alpha, \alpha \in T^*M\}$. It corresponds to a Dirac structure if and only if $[\pi, \pi]_{SN} = 0$, where $[\cdot, \cdot]_{SN}$ is the Schouten bracket of multivector fields.

Let G be a Lie groupoid over a manifold M, we have the following result established in [2].

Theorem 1. *If $\omega \in \Omega^2(G)$ is a multiplicative closed form, then the associated bundle map $\sigma\colon AG \to T^*M$ satisfies the following conditions*

(1) *For any u, $v \in \Gamma(AG)$,*

$$\big\langle \rho_{AG} \oplus \sigma(u), \rho_{AG} \oplus \sigma(v) \big\rangle_+ = 0\,. \tag{1.5}$$

(2) *For any u, $v \in \Gamma(AG)$,*

$$\sigma([u, v]) = \mathcal{L}_{\rho(u)} v - \mathcal{L}_{\rho(v)} u - d\left(\langle \rho_{AG} \oplus \sigma(u), \rho_{AG} \oplus \sigma(v) \rangle_- \right)\ . \tag{1.6}$$

The bundle maps $\sigma\colon AG \to T^*M$ satisfying properties (1) and (2) in Theorem 1 are called *the infinitesimal counterparts of closed multiplicative 2-forms* (see [2], [1]).

Consider the bundle map $\sigma\colon AG \to T^*M$ associated to a presymplectic groupoid (G,ω), the set $(\rho_{AG}\oplus\sigma)(AG)\subset TM\oplus T^*M$ is not a bundle in general. It is a bundle if the following conditions are satisfied (see [1]): for any $x\in G$

$$\begin{cases} \text{(i)} & \dim G = 2\dim M \\ \text{(ii)} & \ker(T_x s)\cap\ker(T_x t)\cap\ker\left(\omega^\sharp(x)\right)=\{0\}\end{cases}$$

where $\omega^\sharp(x) : T_xG \to T^*_xG$ is a vector bundle morphism induced by $\omega(x)$. In [6], it has been proven that a presymplectic groupoid (G,ω) satisfies (i) and (ii) if and only if $\omega^\sharp\colon T^*G\to TG$ is VB-Morita map. If (i) and (ii) are satisfied, then M inherits a Dirac structure. More precisely, the bundle morphism

$$\rho_{AG}\oplus\sigma\colon AG\to TM\oplus T^*M \tag{1.7}$$

is an embedding whose image is a Dirac structure on M. Moreover, the target map $t\colon (G,\omega)\to(M,L)$ is a forward Dirac map. The bundle map (1.7) establishes an isomorphism of Lie algebroid between AG and the canonical Lie algebroid determined by the Dirac structure L. Hence, Dirac manifolds may be thought of as the infinitesimal data of presymplectic groupoids satisfying (i) and (ii) (for more details see [2]).

Let M be an m-dimensional manifold. The tangent bundle of order r of M is the $m(r+1)$-dimensional manifold T^rM of r-jets at $0\in\mathbb{R}$, of smooth map $\gamma\colon\mathbb{R}\to M$. We denote by $\pi^r_M\colon T^rM\to M$ the canonical projection defined by $\pi^r_M(j^r_0\gamma)=\gamma(0)$. Then T^rM has a bundle structure over M. If $r=1$, $T^1M=TM$ is the tangent bundle of M. However, if $r\geq 2$, $\pi^r_M\colon T^rM\to M$ is not a vector bundle (for more details see [8]). In the sequel, we adopt the notations of [14] and for the coordinates system $\left(U,x^i\right)$ in M, the local coordinates system of T^rM over T^rU is such that, the coordinate functions $\left(x^i_\gamma\right)$ with $i=1,\dots,m$ and $\gamma=0,\dots,r$ are given by:

$$\begin{cases} x^i_0\left(j^r_0 g\right) = x^i\left(g\left(0\right)\right)\,, \\ x^i_\gamma\left(j^r_0 g\right) = \frac{1}{\gamma!}\cdot\frac{d^\gamma}{dt^\gamma}\left(x^i\circ g\right)(t)\,|_{t=0}\ . \end{cases} \tag{1.8}$$

For the measure of convenience, the coordinate function x^i_0 is denoted by x^i. The differential geometry of the tangent bundles of higher order has been extensively studied by many authors, see for instance the papers [3], [9], [10], [11] and [17]. It plays an essential role in theoretical physics namely, the Lagrangian and Hamiltonian formulations of some dynamical systems of higher order (see [17]). The tangent lift of higher order of Dirac structures have been studied in [10] and some of its properties are given. This lifting generalizes the tangent lift of higher order of Poisson structures and symplectic structures (see [10]). The particular cases of symplectic groupoids is not studied and some properties induced by this lifting are not established. Therefore, in this paper we study the tangent lifts of higher order of multiplicative 2-forms on a Lie groupoid and we establish some of their properties. In particular, we describe the infinitesimal counterpart of this lifting. So, the main results of this paper are Theorems 3, 4, 5 and Corollaries 2, 3, 4. Given a presymplectic groupoid (G,ω) verifying the assertions (i) and (ii), we prove that $\left(T^rG,\omega^{(c)}\right)$ is a presymplectic groupoid verifying (i), (ii) and we characterize the Dirac structure induced by this lifting on T^rM, where T^rG is a tangent Lie groupoid of order r over T^rM obtained by applying the tangent functor of order r to each of the structure maps defining G (source, target, multiplication, inversion and identity section).

In this paper, all manifolds and mappings are assumed to be differentiable of class C^∞. We shall fix a natural integer $r \geq 1$.

2. Preliminaries

2.1. **The canonical isomorphism $A(T^rG) \cong T^r(AG)$.** For each manifold M, there is a canonical diffeomorphism (see [3])

$$\kappa_M^r \colon T^rTM \to TT^rM \tag{2.1}$$

which is an isomorphism of vector bundles

$$T^r(\pi_M) : T^rTM \to T^rM \quad \text{and} \quad \pi_{TM}^r \colon TT^rM \to T^rM$$

such that $T(\pi_M^r) \circ \kappa_M^r = \pi_{TM}^r$. Let $(x^1, \ldots, x^m)$ be a local coordinate system of M, we introduce the coordinates $(x^i, \dot{x}^i)$ in TM, $(x^i, \dot{x}^i, x^i_\beta, \dot{x}^i_\beta)$ in T^rTM and $(x^i, x^i_\beta, \dot{x}^i, \widetilde{x}^i_\beta)$ in TT^rM. We have

$$\kappa_M^r(x^i, \dot{x}^i, x^i_\beta, \dot{x}^i_\beta) = (x^i, x^i_\beta, \dot{x}^i, \widetilde{x}^i_\beta)$$

with $\widetilde{x}^i_\beta = \dot{x}^i_\beta$.

Consider now a Lie groupoid G over M with Lie algebroid AG. The vector bundle isomorphism $\kappa_G^r : T^rTG \to TT^rG$ restricted to AG induces an isomorphism of vector bundles

$$J_G^r \colon T^r(AG) \to A(T^rG)\ . \tag{2.2}$$

More precisely, we have the following commutative diagram (see [9])

$$\begin{tikzcd} T^r(AG) \arrow[r, "J^rG"] \arrow[d, "T^r(i_{AG})"'] & A(T^rG) \arrow[d, "i_{A(T^rG)}"] \\ T^rTG \arrow[r, "\kappa_G^r"'] & TT^rG \end{tikzcd} \tag{2.3}$$

where $i_{AG} \colon AG \to TG$ is a canonical injection.

Let $(E, [\cdot,\cdot], \rho)$ be a Lie algebroid over M, it exists one and only one Lie algebroid structure on T^rE, of anchor map $\rho^{(r)} = \kappa_M^r \circ T^r\rho$ such that, for any $u, v \in \Gamma(E)$ we have:

$$[T^r(u), T^r(v)] = T^r\big([u,v]\big)\,. \tag{2.4}$$

It is called tangent lift of higher order of Lie algebroid E (see [11]). It follows that, the map

$$J_G^r \colon T^r(AG) \to A(T^rG)$$

is an isomorphism of Lie algebroids, where the Lie algebroid on $T^r(AG)$ is the tangent lift of order r of Lie algebroid AG.

2.2. **The canonical isomorphism** $A(T^*G) \cong T^*(AG)$**.** Let (E, M, π) be a vector bundle, we denote by $\left(x^i, y^j\right)$ an adapted coordinate system of E, it induces the local coordinates $\left(x^i, \pi_j\right)$ in E^*, $\left(x^i, y^j, p_i, \zeta_j\right)$ in T^*E and $\left(x^i, \pi_j, p_i, \xi^j\right)$ in T^*E^*. It is defined in [13], the natural submersion $r_E\colon T^*E \to E^*$ such that locally

$$r_E\left(x^i, y^j, p_i, \zeta_j\right) = \left(x^i, \zeta_j\right)$$

and the Legendre type map

$$R_E\colon T^*E^* \to T^*E$$

which is an anti-symplectomorphism with respect to the canonical symplectic structures on T^*E^* and T^*E respectively, and locally defined by:

$$R_E\left(x^i, \pi_j, p_i, \xi^j\right) = \left(x^i, y^j, -p_i, \zeta_j\right), \quad \text{with} \quad \begin{cases} y^j = \xi^j \\ \pi_j = \zeta_j \end{cases}$$

We suppose that the vector bundle $E \to M$ carries a Lie algebroid structure, so there is a linear Poisson structure π on E^*. Since any Poisson structure on a manifold defines a Lie algebroid structure on its cotangent bundle (see [16]), we obtain a Lie algebroid structure on T^*E^* with the anchor map

$$\sharp_\pi\colon T^*E^* \to TE^*\,.$$

By the Legendre map R_E, we carry the Lie algebroid of T^*E^* on T^*E. The Lie algebroid $T^*E \to E^*$ is called *cotangent algebroid of* E.

Let G be a Lie groupoid over M, we know that T^*G is a Lie groupoid over A^*G (see [13]). The source and target maps are defined respectively by:

$$s^*(\gamma_g)(u) = \gamma_g\big(TL_g(u - Tt(u))\big) \quad \text{and} \quad t^*(\delta_g)(v) = \delta_g\big(TR_g(v)\big)$$

where $\gamma_g \in T_g^*G$, $u \in A_{s(g)}G$ and $\delta_g \in T_g^*G$, $v \in A_{t(g)}G$. The multiplication on T^*G is defined by:

$$(\beta_g \bullet \gamma_h)(X_g \centerdot X_h) = \beta_g(X_g) + \gamma_h(X_h)$$

for $(X_g, X_h) \in T_{(g,h)}G_{(2)}$. As the natural pairing $\langle\cdot,\cdot\rangle_G : TG \oplus T^*G \to \mathbb{R}$ is a groupoid morphism and applying the Lie functor, we obtain an isomorphism $\varsigma_G\colon A(TG)^* \to A(T^*G)$. On the other hand, for any manifold M, there is a canonical diffeomorphism (see [3])

$$\alpha_M^r\colon T^*T^rM \to T^rT^*M$$

which is an isomorphism of vector bundles

$$\pi_{T^rM}^*\colon T^*T^rM \to T^rM \quad \text{and} \quad T^r(\pi_M^*)\colon T^rT^*M \to T^rM$$

dual of κ_M^r with respect to pairings $\langle\cdot,\cdot\rangle'_{T^rM} = \tau_r \circ T^r(\langle\cdot,\cdot\rangle_M)$ and $\langle\cdot,\cdot\rangle_{T^rM}$, i.e. for any $(u, u^*) \in T^rTM \oplus T^*T^rM$,

$$\left\langle \kappa_M^r(u), u^*\right\rangle_{T^rM} = \left\langle u, \alpha_M^r(u^*)\right\rangle'_{T^rM}. \tag{2.5}$$

Let $\left(x^1,\ldots,x^m\right)$ be a local coordinates system of M, we introduce the coordinates $\left(x^i,p_j\right)$ in T^*M, $\left(x^i,p_j,x^i_\beta,p^\beta_j\right)$ in T^rT^*M and $\left(x^i,x^i_\beta,\pi_j,\pi^\beta_j\right)$ in T^*T^rM. We have:

$$\alpha^r_M\left(x^i,\pi_j,x^i_\beta,\pi^\beta_j\right)=\left(x^i,x^i_\beta,p_j,p^\beta_j\right)\quad\text{with}\quad\begin{cases}p_j=\pi^r_j\\ p^\beta_j=\pi^{r-\beta}_j\end{cases}$$

By ε^r_M we denote the map $\left(\alpha^r_M\right)^{-1}$ and it is called *natural isomorphism of Tulczyjew over M*.

Remark 1. Let G be a Lie groupoid. In the particular case $r=1$, the natural isomorphism of Tulczyjew, ε^1_G establishes a canonical isomorphism of Lie algebroids between the Lie algebroid of cotangent groupoid $A\left(T^*G\right)$ and the cotangent algebroid of $A(G)$ (see [12]).

2.3. **Tangent lifts of higher order of presymplectic manifolds.** In this section we recall briefly the main results of A. Morimoto [14], about the complete lifts of differential forms to the tangent bundle of higher order. These result will be used in the sequel.

2.3.1. *Prolongations of functions.* For each $s\in\{0,\ldots,r\}$, we denote by τ_s the linear form on $J^r_0(\mathbb{R},\mathbb{R})$ defined by:

$$\tau_s(j^r_0\gamma)=\frac{1}{s!}\cdot\frac{d^s}{dt^s}\left(\gamma(t)\right)\Big|_{t=0},\quad\text{where}\quad\gamma\in C^\infty(\mathbb{R},\mathbb{R})\,.$$

Let M be a smooth manifold of dimension $m>0$. For $g\in C^\infty(M)$ and $s\in\{0,\ldots,r\}$, we set: $g^{(s)}=\tau_s\circ T^rg$. The smooth map $g^{(s)}$ is called *s-prolongation of g*, and, more explicitly, it is given by:

$$g^{(s)}\left(j^r_0\varphi\right)=\frac{1}{s!}\frac{d^s\left(g\circ\varphi\right)}{dt^s}\left(t\right)|_{t=0}\quad\text{for}\quad\varphi\in C^\infty\left(\mathbb{R},M\right)\,.$$

Remark 2.

(i) By this expression, it follows that $x^i_\beta=\left(x^i\right)^{(\beta)}$ on T^rU where $\left(x^1,\ldots,x^m\right)$ are coordinates on some open subset $U\subset M$, $\beta\in\{0,\cdots,r\}$.

(ii) The mapping

$$C^\infty\left(M\right)\to C^\infty\left(T^rM\right)$$
$$g\mapsto g^{(s)}$$

is $\mathbb{R}$-linear.

2.3.2. *Prolongations of vector fields.* Let (E,M,π) be a vector bundle, consider the vector bundle morphism $\chi^{(\alpha)}_E:T^rE\to T^rE$ defined by:

$$\chi^{(\alpha)}_E(j^r_0\Psi)=j^r_0(t^\alpha\Psi)$$

where $\Psi\in C^\infty\left(\mathbb{R},E\right)$ and $t^\alpha\Psi$ is the smooth map defined for any $t\in\mathbb{R}$ by:

$$\left(t^\alpha\Psi\right)(t)=t^\alpha\Psi(t)\,.$$

Let X be a vector field on a manifold M, we define the *α-prolongation of X* denoted $X^{(\alpha)}$ by:

$$X^{(\alpha)} = \kappa_M^r \circ \chi_{TM}^{(\alpha)} \circ T^r X \,.$$

When $\alpha = 0$, it is called complete lift of X to $T^r M$ and is denoted by $X^{(c)}$. We put $X^{(\alpha)} = 0$ for $\alpha > r$ or $\alpha < 0$.

If $\left(U, x^i\right)$ is a local coordinates system of M such that

$$X = X^i \frac{\partial}{\partial x^i} \,,$$

then we have:

$$X^{(\alpha)} = \left(X^i\right)^{(\beta-\alpha)} \frac{\partial}{\partial x^i_\beta} \,.$$

Proposition 1.

(i) *For $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ and $\alpha, \beta \in \{0, \dots, r\}$, we have:*

$$X^{(\alpha)}(f^{(\beta)}) = \big(X(f)\big)^{(\beta-\alpha)} \,.$$

(ii) *For X, $Y \in \mathfrak{X}(M)$ and α, $\beta \in \{0, \dots, r\}$, we have:*

$$\left[X^{(\alpha)}, Y^{(\beta)}\right] = [X, Y]^{(\alpha+\beta)} \,.$$

(iii) *The set $\left\{X^{(\beta)},\ X \in \mathfrak{X}(M),\ \ \beta = 0, \dots, r\right\}$ generates the $C^\infty(T^r M)$-module $\mathfrak{X}(T^r M)$.*

Proof. See [7]. □

2.3.3. *Prolongations of differential forms.* Let $\omega \in \Omega^k(M)$ and $\beta \in \{0, \dots, r\}$. We have the following result establishes in [14].

Proposition 2. *It exists on $T^r M$ one and only one differential form of degree k denoted by $\omega^{(\beta)}$ verifying:*

$$\omega^{(\beta)}\big(X_1^{(\beta_1)}, \dots, X_k^{(\beta_k)}\big) = \big(\omega\left(X_1, \dots, X_k\right)\big)^{(\beta-(\beta_1+\cdots+\beta_k))} \,. \tag{2.6}$$

For all $X_1, \dots, X_k \in \mathfrak{X}(M)$ and $\beta_1, \dots, \beta_k \in \{0, \cdots, r\}$.

The differential form $\omega^{(\beta)}$ is called *β-prolongation of ω from M to $T^r M$*. When $\beta = r$, $\omega^{(\beta)}$ is called *complete lift of ω on $T^r M$* and it is denoted by $\omega^{(c)}$.

Remark 3.

(1) In local coordinates, if $\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, then

$$\omega^{(\beta)} = \sum_{\beta_1+\cdots+\beta_k+\alpha=\beta} \left(\omega_{i_1\cdots i_k}\right)^{(\alpha)} dx^{i_1}_{\beta_1} \wedge \cdots \wedge dx^{i_k}_{\beta_k} \,.$$

(2) For any $\omega \in \Omega^1(M)$, we have the following equality:

$$\omega^{(\alpha)} = \varepsilon_M^r \circ \chi_{T^*M}^{(r-\alpha)} \circ T^r \omega \,.$$

Proposition 3.

(1) *For any* $\beta \leq r$*, the map* $\omega \mapsto \omega^{(\beta)}$ *from* $\Omega^k(M)$ *to* $\Omega^k(T^r M)$ *is* $\mathbb{R}$*-linear.*

(2) *For any* $X \in \mathfrak{X}(M)$ *and* $\alpha, \beta \leq r$ *we have:*

$$i_{X^{(\alpha)}}\omega^{(\beta)} = (i_X\omega)^{(\beta-\alpha)} , \tag{2.7}$$

$$d\left(\omega^{(\beta)}\right) = (d\omega)^{(\beta)} , \tag{2.8}$$

$$\mathcal{L}_{X^{(\alpha)}}\omega^{(\beta)} = (\mathcal{L}_X\omega)^{(\beta-\alpha)} . \tag{2.9}$$

(3) *For* $f \in C^\infty(M,N)$ *and* $\varpi \in \Omega^k(N)$*, we have:*

$$(T^r f)^* \varpi^{(\beta)} = (f^*\varpi)^{(\beta)} \tag{2.10}$$

Proof. For (1) and the equalities (2.7), (2.8) and (2.9) see [7]. We prove the equality (2.10). Let (U, x^i) and $(V; y^j)$ local charts of M and N such that $f(U) \subset V$. The local expression of $T^r f$ is given by $T^r f(x^i, x^i_\alpha) = \left(f_p(x^i), f_p^{(\beta)}(x^i, x^i_\alpha)\right)$.

As $\varpi = \varpi_{i_1\ldots i_k} dy^{i_1} \wedge \cdots \wedge dy^{i_k}$, then $\varpi^{(\beta)} = \sum\limits_{\beta_1+\cdots+\beta_k+\gamma=\beta} (\varpi_{i_1\ldots i_k})^{(\gamma)} dy^{i_1}_{\beta_1} \wedge \cdots \wedge dy^{i_k}_{\beta_k}$. We have

$$\begin{aligned}
(T^r f)^* \varpi^{(\beta)} &= \left((\varpi_{i_1\ldots i_k})^{(\gamma)} \circ T^r f\right) d\left(y^{i_1}_{\beta_1} \circ T^r f\right) \wedge \cdots \wedge d\left(y^{i_k}_{\beta_k} \circ T^r f\right) \\
&= (\varpi_{i_1\cdots i_k} \circ f)^{(\gamma)}\, d\left(f^{(\beta_1)}_{i_1}\right) \wedge \cdots \wedge d\left(f^{(\beta_k)}_{i_k}\right) \\
&= (\varpi_{i_1\cdots i_k} \circ f)^{(\gamma)}\, (df_{i_1})^{(\beta_1)} \wedge \cdots \wedge (df_{i_k})^{(\beta_k)} .
\end{aligned}$$

So, $(T^r f)^* \varpi^{(\beta)} = (f^*\varpi)^{(\beta)}$. □

Remark 4. The equation (2.8) shows that if ω is closed then $\omega^{(c)}$ is also closed. In particular, for $k = 2$, the complete lift $\omega^{(c)}$ is such that,

$$\left(\omega^{(c)}\right)^\sharp = \varepsilon^r_M \circ T^r\left(\omega^\sharp\right) \circ (\kappa^r_M)^{-1} .$$

In particular, if ω is non degenerate, then $\omega^{(c)}$ is also non degenerate.

Corollary 1. *If* (M, ω) *is a symplectic manifold, then* $(T^r M, \omega^{(c)})$ *is also symplectic manifold.*

2.3.4. *Complete lift of tensor fields of type* $(0, p)$*.* Let (E, M, π) be a vector bundle and φ a tensor field of type $(0, p)$ on E. We interpret a tensor φ on E as a p-linear mapping $\varphi\colon E \times_M \cdots \times_M E \to \mathbb{R}$ on the fibered product over M of p-copies of E. Put,

$$\overline{\varphi}^{(c)} = \tau_r \circ T^r\varphi .$$

$\overline{\varphi}^{(c)}$ is a tensor field of type $(0, p)$ on the vector bundle $(T^r E \to T^r M)$, called *α-complete lift of φ from E to $T^r E$* (see [7]). On the other hand, given $s\colon M \to E$ a smooth section of E, it is defined (see [7]) the section $\overline{s}^{(\alpha)}$ of $(T^r E \to T^r M)$ by:

$$\overline{s}^{(\alpha)} = \chi^{(\alpha)}_E \circ T^r(s) , \quad 0 \leq \alpha \leq r .$$

It is called *α-prolongation of order r of s*. For the sake of convenience we put $\overline{s}^{(\alpha)} = 0$ for $\alpha < 0$ or $\alpha > r$. We have the following result established in [10].

Proposition 4.

(i) *The set* $\left\{\overline{s}^{(\alpha)},\ s \in \Gamma(E),\ 0 \leq \alpha \leq r\right\}$ *generates the* $C^{\infty}(T^r M)$*-module* $\Gamma(T^r E)$.

(ii) $\overline{\varphi}^{(c)}$ *is the only tensor field of type* $(0,p)$ *on* $T^r E$ *satisfying:*

$$\varphi^{(c)}\left(\overline{s}_1^{(\alpha_1)}, \ldots, \overline{s}_p^{(\alpha_p)}\right) = \left(\varphi\left(s_1, \ldots, s_p\right)\right)^{(r-\sum_{i=1}^{p}\alpha_i)}$$

for all $s_1, \ldots, s_p \in \Gamma(E)$ *and* $0 \leq \alpha_1, \ldots, \alpha_p \leq r$.

3. Higher order tangent lifts of twisted Dirac structures

As observed, for instance, in [2], one can use a closed 3-form ϕ on M to modify the standard Courant bracket as follows:

$$[X \oplus \omega, Y \oplus \varpi]_{\phi} = [X, Y] \oplus \left(\mathcal{L}_X \varpi - i_Y d\omega + i_{X\wedge Y}\phi\right). \tag{3.1}$$

It is called *the ϕ-twisted Courant-bracket on* $\mathfrak{X}(M) \oplus \Omega^1(M)$.

An almost Dirac structure on a manifold M is a subbundle L of vector bundle $TM \oplus T^*M$ which is maximally isotropic under the symmetric pairing $\langle \cdot, \cdot \rangle_+$. If $\Gamma(L)$ is closed under the bracket (3.1), adopting the definitions introduced in [2], [15], we will say that, the almost Dirac structure L is integrable or L is a ϕ-twisted-Dirac structure on M. It is denoted by (M, L, ϕ) and we prefer to write $L = L_\phi$. In the particular case where $\phi = 0$, we obtain the integrability condition defined in [5]. The integrability of a twisted-Dirac structure (M, L_ϕ) is also measured by the Courant 3-tensor T_ϕ defined by:

$$T_\phi(e_1, e_2, e_3) = \left\langle [e_1, e_2]_\phi, e_3 \right\rangle_+.$$

In fact, an almost Dirac structure $L \subset TM \oplus T^*M$ defines a ϕ-twisted Dirac structure on M if and only if the Courant tensor T_ϕ vanishes.

Example 1. Let ω be a 2-form on a manifold M. The vector bundle L_ω defined by the graph of the map $\omega^\sharp \colon TM \to T^*M$ defines a ϕ-twisted Dirac structure if and only if

$$d\omega + \phi = 0.$$

In this case we say that ω is closed with respect to ϕ.

Example 2. Let $\pi \in \mathfrak{X}^2(M)$, the vector bundle L_π defined by the graph of the anchor map $\sharp_\pi \colon T^*M \to TM$ defines a ϕ-twisted Dirac structure if and only if

$$\tfrac{1}{2}[\pi, \pi] = \left(\bigwedge^3 \sharp_\pi\right)(\phi)$$

where $\bigwedge^3 \sharp_\pi$ denotes the extension of the bundle map $\sharp_\pi \colon T^*M \to TM$ to higher exterior powers.

Remark 5. Given a ϕ-twisted Dirac structure L_ϕ on a manifold M, the vector bundle $L_\phi \to M$ inherits a canonical Lie algebroid structure with anchor map given by the restriction of the canonical projection $pr_1 |_{L_\phi} \colon L_\phi \to TM$, and the Lie bracket on the sections of L_ϕ is defined by the restriction of the ϕ-twisted Courant bracket. Since every Lie algebroid defines a singular foliation on its base manifold, (see [5]) it follows that L_ϕ defines a singular foliation on M. Additionally, on each leaf of M $i_{\mathcal{S}} \colon \mathcal{S} \hookrightarrow M$ there is a 2-form $\Omega_{\mathcal{S}}$ such that:

$$d\Omega_{\mathcal{S}} + i_{\mathcal{S}}^* \phi = 0$$

where for any $x \in \mathcal{S}$, u_x, $v_x \in T_x\mathcal{S} = (L_\phi)_x$, we have

$$\Omega_{\mathcal{S}}(x)(u_x, v_x) = \langle u_x, v_x \rangle_- \ .$$

So, the 2-forms $\Omega_{\mathcal{S}}$ are closed up to $i_{\mathcal{S}}^*\phi$ and we say that $\Omega_{\mathcal{S}}$ is presymplectic relative to $i_{\mathcal{S}}^*\phi$. In this way we obtain the singular presymplectic foliation related to ϕ.

Let ϕ be a closed 3-form on M and (M, L_ϕ) an almost Dirac structure. We put:

$$\mathcal{T}^r L_\phi = (\kappa_M^r \oplus \varepsilon_M^r)(T^r L_\phi) \subset TT^r M \oplus T^* T^r M \tag{3.2}$$

$\mathcal{T}^r L_\phi$ is an almost Dirac structure on $T^r M$ (see for instance [10]). On the other hand, for any $e \in \Gamma(L_\phi)$, we denote by $\overline{e}^{(\alpha)}$ the α-lift of e from L_ϕ to $T^r L_\phi$. We have:

$$\kappa_M^r \oplus \varepsilon_M^r\left(\overline{e}^{(\alpha)}\right) \in \Gamma(\mathcal{T}^r L_\phi) \ .$$

We put $e^{(\alpha)} = \kappa_M^r \oplus \varepsilon_M^r\left(\overline{e}^{(\alpha)}\right)$. In particular, if $e = X \oplus \omega$, then we have (see [10]):

$$e^{(\alpha)} = X^{(\alpha)} \oplus \omega^{(r-\alpha)} .$$

The set $\left\{e^{(\alpha)}, e \in \Gamma(L_\phi), 0 \le \alpha \le r\right\}$ generates the $C^\infty(M)$-module $\Gamma(\mathcal{T}^r L_\phi)$.

Lemma 1. *For any $e_1, e_2 \in \Gamma(L_\phi)$ we have:*

$$\left\langle e_1^{(\alpha)}, e_2^{(\beta)} \right\rangle_+ = \left(\langle e_1, e_2 \rangle_+\right)^{(r-\alpha-\beta)} \tag{3.3}$$

$$\left\langle e_1^{(\alpha)}, e_2^{(\beta)} \right\rangle_- = \left(\langle e_1, e_2 \rangle_-\right)^{(r-\alpha-\beta)} \ . \tag{3.4}$$

Proof. By calculation. □

Lemma 2. *For $e_1, e_2 \in \Gamma(L_\phi)$, we have:*

$$\left[e_1^{(\alpha)}, e_2^{(\beta)}\right]_{\phi^{(c)}} = [e_1, e_2]_\phi^{(\alpha+\beta)} \ . \tag{3.5}$$

Proof. We put $e_1 = X_1 \oplus \omega_1$, $e_2 = X_2 \oplus \omega_2$, we have:

$$\begin{aligned}
\left[e_1^{(\alpha)}, e_2^{(\beta)}\right]_{\phi^{(c)}} &= \left[X_1^{(\alpha)} \oplus \omega_1^{(r-\alpha)}, X_2^{(\beta)} \oplus \omega_2^{(r-\beta)}\right]_{\phi^{(c)}} \\
&= \left[X_1^{(\alpha)}, X_2^{(\beta)}\right] \oplus \Big((\mathcal{L}_{X_1}\omega_2)^{(r-\alpha-\beta)} - (i_{X_2} d\omega_1)^{(r-\alpha-\beta)} \\
&\qquad + (i_{X_1 \wedge X_2}\phi)^{(r-\alpha-\beta)} \Big) \\
&= [X_1, X_2]^{(\alpha+\beta)} \oplus (\mathcal{L}_{X_1}\omega_2 - i_{X_2} d\omega_1 + i_{X_1 \wedge X_2}\phi)^{(r-\alpha-\beta)} \\
&= [e_1, e_2]^{(\alpha+\beta)} \ .
\end{aligned}$$

□

Let L_ϕ be an almost ϕ-twisted Dirac structure on M. We denote by $T_{\phi^{(c)}}$ the Courant tensor of the almost Dirac structure $\mathcal{T}^r L_\phi$.

Theorem 2. *We have:*

$$\overline{T_\phi}^{(c)} = T_{\phi^{(c)}} \circ \Big[\bigoplus^{3} (\kappa_M^r \oplus \varepsilon_M^r) \Big] . \tag{3.6}$$

Proof. Let e_1, e_2, $e_3 \in \Gamma(L_\phi)$, we have:

$$\begin{aligned} T_{\phi^{(c)}} \circ \Big[\bigoplus^{3} (\kappa_M^r \oplus \varepsilon_M^r) \Big] \big(\overline{e}_1^{(\alpha)}, \overline{e}_2^{(\beta)}, \overline{e}_3^{(\gamma)}\big) &= T_{\phi^{(c)}} \big(e_1^{(\alpha)}, e_2^{(\beta)}, e_3^{(\gamma)}\big) \\ &= \big\langle \big[e_1^{(\alpha)}, e_2^{(\beta)}\big]_{\phi^{(c)}}, e_3^{(\gamma)} \big\rangle_+ = \big\langle \left[e_1, e_2\right]_\phi^{(\alpha+\beta)}, e_3^{(\gamma)} \big\rangle_+ \\ &= \big(\big\langle \left[e_1, e_2\right]_\phi, e_3 \big\rangle_+\big)^{(r-\alpha-\beta)} = \overline{T_\phi}^{(c)} \big(\overline{e}_1^{(\alpha)}, \overline{e}_2^{(\beta)}, \overline{e}_3^{(\gamma)}\big) . \end{aligned}$$

Thus, $\overline{T_\phi}^{(c)} = T_{\phi^{(c)}} \circ \Big[\bigoplus^3 (\kappa_M^r \oplus \varepsilon_M^r)\Big]$. □

For the tangent lift of Lie algebroid see [11].

Proposition 5. *Let L_ϕ be an integrable ϕ-twisted Dirac structure on M. The tangent Lift of higher order of Lie algebroid $\left(L_\phi, [\cdot,\cdot]_\phi, pr_1\right)$ is isomorphic to the Lie algebroid $\left(\mathcal{T}^r L_\phi, [\cdot,\cdot]_{\phi^{(c)}}, pr_1\right)$ over $T^r M$.*

Proof. Follows by the equation (3.2). It is such that,

$$\kappa_M^r \oplus \varepsilon_M^r \colon T^r L_\phi \to \mathcal{T}^r L_\phi$$

is an isomorphism of Lie algebroids. □

For the tangent lifts of higher order of singular foliations and their properties, see [11]. Using these results, we have:

Proposition 6. *Let L_ϕ be an integrable Dirac structure, $\mathfrak{F}_\phi$ the generalized foliation induced by L_ϕ and $\mathcal{S}$ a leaf of $\mathfrak{F}_\phi$.*

(1) *The singular foliation induced by $\mathcal{T}^r L_\phi$ is the tangent lift of order r of the generalized foliation $\mathfrak{F}_\phi$.*

(2) *If $\Omega_{\mathcal{S}}$ is the presymplectic form on $\mathcal{S}$ relative to $i_{\mathcal{S}}^*\phi$ then, $\Omega_{\mathcal{S}}^{(c)}$ is the presymplectic form on the leaf $T^r\mathcal{S}$ relative to $i_{T^r\mathcal{S}}^*\phi^{(c)}$.*

Proof. Let X, $Y \in pr_1(\Gamma(L_\phi))$ tangent to $\mathcal{S}$ such that $X \oplus \omega$, $Y \oplus \varpi \in \Gamma(L_\phi)$. We have:

$$\begin{aligned} \Omega_{T^r\mathcal{S}}\big(X^{(\alpha)}, Y^{(\beta)}\big) &= \omega^{(r-\alpha)}\big(Y^{(\beta)}\big) \\ &= \big(\omega(Y)\big)^{(r-\alpha-\beta)} \\ &= \big(\Omega_{\mathcal{S}}(X, Y)\big)^{(r-\alpha-\beta)} \\ &= \Omega_{\mathcal{S}}^{(c)}\big(X^{(\alpha)}, Y^{(\beta)}\big) . \end{aligned}$$

It follows that, $\Omega_{T^r\mathcal{S}} = \Omega_{\mathcal{S}}^{(c)}$. The equality $d\left(\Omega_{T^r\mathcal{S}}\right) + i_{T^r\mathcal{S}}^{*}\phi^{(c)} = 0$ is a now consequence of Lemma 2. □

4. The main results

Let G be a Lie groupoid over a manifold M equipped with closed multiplicative 2-form ω.

Theorem 3. *The pair $\left(T^rG, \omega^{(c)}\right)$ is a presymplectic groupoid.*

Proof. $T^r\mathfrak{m}\colon \left(T^rG\right)_{(2)} \to T^rM$ is the partial multiplication map. We have:

$$\begin{aligned}
\left(T^r\mathfrak{m}\right)^{*}\omega^{(c)} &= \left(\mathfrak{m}^{*}\omega\right)^{(c)} \\
&= \left((pr_1)^{*}\omega + (pr_2)^{*}\omega\right)^{(c)} \\
&= \left(T^r\, pr_1\right)^{*}\omega^{(c)} + \left(T^r\, pr_2\right)^{*}\omega^{(c)} \\
&= pr_1^{*}\,\omega^{(c)} + pr_2^{*}\,\omega^{(c)}
\end{aligned}$$

where $pr_1,\, pr_2\colon \left(T^rG\right)_{(2)} \to T^rG$ are the natural projections. □

We denote by $\sigma_r\colon A\left(T^rG\right) \to T^{*}T^rM$ the infinitesimal counterpart of the closed multiplicative 2-form induced by the presymplectic groupoid $\left(T^rG, \omega^{(c)}\right)$.

Theorem 4. *We have:*

$$\sigma_r = \varepsilon_M^r \circ T^r\left(\sigma\right) \circ \left(J_G^r\right)^{-1}. \tag{4.1}$$

Proof. As $J_G^r\colon T^r\left(AG\right) \to A\left(T^rG\right)$ is an isomorphism of Lie algebroids and J_G^r is the restriction of $\kappa_G^r\colon T^rTG \to TT^rG$ to $T^r\left(AG\right)$, then for any $u \in \Gamma\left(AG\right)$

$$u^{(\alpha)} = J_G^r\left(\overline{u}^{(\alpha)}\right) \in \Gamma\big(A(T^rG)\big).$$

For any $u \in \Gamma(AG)$ and $\alpha \in \{0, 1, \ldots, r\}$ we have:

$$\begin{aligned}
\sigma_r\big(u^{(\alpha)}\big) &= i_{u^{(\alpha)}}\omega^{(c)} \\
&= \left(i_u\omega\right)^{(r-\alpha)}.
\end{aligned}$$

In the same way,

$$\begin{aligned}
\varepsilon_M^r \circ T^r\left(\sigma\right) \circ \left(J_G^r\right)^{-1}\left(u^{(\alpha)}\right) &= \varepsilon_M^r \circ T^r\left(\sigma\right) \circ \chi_{AG}^{(\alpha)} \circ T^r u \\
&= \varepsilon_M^r \circ \chi_{T^{*}M}^{(\alpha)} \circ T^r\left(\sigma \circ u\right) \\
&= \varepsilon_M^r \circ \chi_{T^{*}M}^{(\alpha)} \circ T^r\left(i_u\sigma\right) \\
&= \left(i_u\sigma\right)^{(r-\alpha)}.
\end{aligned}$$

Since $\Gamma\left(A\left(T^rG\right)\right)$ is generated by the set $\left\{u^{(\alpha)}, u \in \Gamma\left(AG\right), \alpha \leq r\right\}$, then $\varepsilon_M^r \circ T^r\left(\sigma\right) \circ \left(J_G^r\right)^{-1} = \sigma_r$. □

Corollary 2. *Let (G,ω) be a symplectic groupoid and π_ω the Poisson tensor on M induced by ω. We have:*

$$\pi_{\omega^{(c)}} = (\pi_\omega)^{(c)} \;, \tag{4.2}$$

where $(\pi_\omega)^{(c)}$ is the complete lift of the Poisson tensor π_ω.

Proof. The anchor map of Poisson tensor $\pi_{\omega^{(c)}}$ is such that (see [11])

$$\sharp_{\pi_{\omega^{(c)}}} = \rho_{A(T^rG)} \circ \sigma_r^{-1} \, . \tag{4.3}$$

The anchor map of Lie algebroid $T^r(AG)$ is given by: $\rho_{AG}^{(r)} = \kappa_M^r \circ T^r(\rho_{AG})$. It follows that:

$$\rho_{A(T^rG)} = \kappa_M^r \circ T^r(\rho_{AG}) \circ (J_G^r)^{-1}$$

hence,

$$\begin{aligned}
\sharp_{\pi_{\omega^{(c)}}} &= \kappa_M^r \circ T^r(\rho_{AG}) \circ (J_G^r)^{-1} \circ \sigma_r^{-1} \\
&= \kappa_M^r \circ T^r\left(\rho_{AG} \circ \sigma^{-1}\right) \circ \alpha_M^r \\
&= \kappa_M^r \circ T^r(\sharp_{\pi_\omega}) \circ \alpha_M^r \, .
\end{aligned}$$

Therefore $\sharp_{\pi_{\omega^{(c)}}} = \sharp_{\pi_\omega^{(c)}}$, i.e. $\pi_{\omega^{(c)}} = (\pi_\omega)^{(c)}$. □

Example 3. Let G be a Lie group, it is a Lie groupoid over a point. We denote by $\mathfrak{g}$ the Lie algebra of G, T^*G is a Lie groupoid over $\mathfrak{g}^*$. Equipping T^*G with the canonical symplectic form ω_G, it becomes a symplectic groupoid and the corresponding Poisson structure on $\mathfrak{g}^*$ is the standard linear Poisson structure. So, $\left(T^rT^*G, \omega_G^{(c)}\right)$ is a symplectic groupoid. We denote by (e_i) the canonical basis of $\mathfrak{g}$ and $\left(e^i\right)$ its dual basis. By $\left(e_i^\beta\right)$ we denote the basis of $T^r\mathfrak{g}$ obtained by prolongation and by $\left(e^{\beta,i}\right)$ its dual basis. We denote by $\left(e^{i,\beta}\right)$ the basis of $T^r\mathfrak{g}^*$ obtained by prolongation. As the Poisson bracket on $\mathfrak{g}^*$ is given by:

$$\left\{e^i, e^j\right\} = c_k^{ij} e^k \, .$$

We deduce that, the Poisson bracket on $T^r\mathfrak{g}^*$ is given by:

$$\left\{e^{i,\beta}, e^{j,\gamma}\right\} = c_k^{ij} e^{k,\beta+\gamma-r} \, .$$

Consider the vector space isomorphism $I_{\mathfrak{g}^*} \colon (T^r\mathfrak{g})^* \to T^r\mathfrak{g}^*$ defined in [9]. It is such that $I_{\mathfrak{g}^*}\left(e^{\beta,i}\right) = e^{i,r-\beta}$, so the linear Poisson structure on $(T^r\mathfrak{g})^*$ is such that the canonical isomorphism $I_{\mathfrak{g}^*}$ is a Poisson map. This linear Poisson structure on $(T^r\mathfrak{g})^*$ is induced by the tangent Lie algebra $T^r\mathfrak{g}$ of order r. In the same way, $\alpha_G^r \colon T^*T^rG \to T^rT^*G$ is an isomorphism of Lie groupoids over the map $I_{\mathfrak{g}^*}$ and $(\alpha_G^r)^*\left(\omega_G^{(c)}\right) = \omega_{T^rG}$, it follows that α_G^r is an isomorphism of symplectic groupoids between (T^*T^rG, ω_{T^rG}) and $\left(T^rT^*G, \omega_G^{(c)}\right)$.

Let (G,ω) be a presymplectic groupoid such that:

$$\begin{cases} \dim G = 2\dim M \\ \ker(Ts) \cap \ker(Tt) \cap \ker(\omega^\sharp) = 0 \end{cases} \tag{4.4}$$

where $\omega^\sharp \colon TG \to T^*G$ is a vector bundle morphism induced by ω.

Theorem 5. *Let* (G,ω) *be a presymplectic groupoid satisfying the equation* (4.4), *we have:*

$$\ker\big(T\left(T^r s\right)\big) \cap \ker\big(T\left(T^r t\right)\big) \cap \ker\big((\omega^{(c)})^\sharp\big) = \{0\}\,. \tag{4.5}$$

Proof. s, $t\colon G \to M$ are source and target maps of G. Let $u \in TT^rG$ such that

$$u \in \ker\big(T\left(T^r s\right)\big) \cap \ker\big(T\left(T^r t\right)\big) \cap \ker\big((\omega^{(c)})^\sharp\big)\,.$$

There is smooth map $\gamma : \mathbb{R}\times\mathbb{R} \to G$ such that $u = \frac{d}{d\tau}\left(j_0^r\gamma\left(\cdot,\tau\right)\right)|_{\tau=0}$. As

$$\left(\kappa_G^r\right)^{-1}(u) = j_0^r\Big(\frac{d}{d\tau}\big(\gamma\left(\cdot,\tau\right)\big)\,|_{\tau=0}\Big)$$

and ε_G^r is an isomorphism of vector bundles, it follows that:

$$j_0^r\Big(\omega^\sharp\Big(\frac{d}{d\tau}\left(\gamma\left(\cdot,\tau\right)\right)|_{\tau=0}\Big)\Big) = 0\,.$$

On the other hand,

$$\begin{aligned} TT^r s\,(u) &= \frac{d}{d\tau}\big(T^r s\left(j_0^r\gamma\left(\cdot,\tau\right)\right)\big)\,|_{\tau=0} \\ &= \frac{d}{d\tau}\big(\left(j_0^r\left(s\circ\gamma\right)\left(\cdot,\tau\right)\right)\big)\,|_{\tau=0} \\ &= j_0^r\Big(\frac{d}{d\tau}\big(\left(s\circ\gamma\right)\left(\cdot,\tau\right)\big)\,|_{\tau=0}\Big)\,. \end{aligned}$$

In the same way, we have

$$TT^r t\,(u) = j_0^r\Big(\frac{d}{d\tau}\big((t\circ\gamma)(\cdot,\tau)\big)\,|_{\tau=0}\Big)\,.$$

Thus, we have:

$$j_0^r\Big(\frac{d}{d\tau}\big((t\circ\gamma)(\cdot,\tau)\big)\,|_{\tau=0}\Big) = 0$$

$$j_0^r\Big(\frac{d}{d\tau}\big((s\circ\gamma)(\cdot,\tau)\big)\,|_{\tau=0}\Big) = 0$$

$$j_0^r\Big(\omega^\sharp\Big(\frac{d}{d\tau}\big(\gamma(\cdot,\tau)\big)\,|_{\tau=0}\Big)\Big) = 0$$

Let $\left(x^i\right)$, $\left(x^i, g^j\right)$ be a local coordinates systems of M and G. For any $\alpha \leq r$, we have:

$$\frac{d}{d\tau}\big(x_\alpha^i \circ T^r t(j_0^r\gamma(\cdot,\tau))\big)\,|_{\tau=0} = 0$$

$$\frac{d}{d\tau}\big(x_\alpha^i \circ T^r s(j_0^r\gamma(\cdot,\tau))\big)\,|_{\tau=0} = 0$$

$$\frac{d}{d\tau}\big(g_\alpha^i \circ T^r(\omega^\sharp)(j_0^r\gamma(\cdot,\tau))\big)\,|_{\tau=0} = 0$$

Therefore,

$$\frac{d}{d\tau}\Big(\frac{d^\alpha}{dz^\alpha}\big(x^i\circ t\circ\gamma(\tau,z)\big)\Big)\ |_{\tau=z=0}=0$$

$$\frac{d}{d\tau}\Big(\frac{d^\alpha}{dz^\alpha}\big(x^i\circ s\circ\gamma(\tau,z)\big)\Big)\ |_{\tau=z=0}=0$$

$$\frac{d}{d\tau}\Big(\frac{d^\alpha}{dz^\alpha}\big(g^i\circ\omega^\sharp\circ\gamma(\tau,z)\big)\Big)\ |_{\tau=z=0}=0\,.$$

So for any $\alpha\neq 0$,

$$\frac{d^\alpha}{dz^\alpha}\Big(\frac{d}{d\tau}\big(x^i\circ t\circ\gamma(\tau,z)\big)\Big)\ |_{\tau=z=0}=0$$

$$\frac{d^\alpha}{dz^\alpha}\Big(\frac{d}{d\tau}\big(x^i\circ s\circ\gamma(\tau,z)\big)\Big)\ |_{\tau=z=0}=0$$

$$\frac{d^\alpha}{dz^\alpha}\Big(\frac{d}{d\tau}\big(g^i\circ\omega^\sharp\circ\gamma(\tau,z)\big)\Big)\ |_{\tau=z=0}=0\,.$$

As

$$\frac{d}{d\tau}\left(x^i\circ t\circ\gamma\left(\tau,\cdot\right)\right)|_{\tau=0}=\frac{d}{d\tau}\left(x^i\circ s\circ\gamma\left(\tau,\cdot\right)\right)|_{\tau=0}=\frac{d}{d\tau}\left(g^i\circ\omega^\sharp\circ\gamma\left(\tau,\cdot\right)\right)|_{\tau=0}=0\,.$$

Therefore, for any $0\leq\alpha\leq r$,

$$\frac{d}{d\tau}\left(\frac{d^\alpha}{dz^\alpha}\big(x^i\circ\gamma(\tau,z)\big)\ |_{z=0}\right)|_{\tau=0}=\frac{d}{d\tau}\left(\frac{d^\alpha}{dz^\alpha}\big(g^i\circ\gamma(\tau,\cdot z)\big)\ |_{z=0}\right)|_{\tau=0}=0\,.$$

More exactly

$$\frac{d}{d\tau}\big(x^i_\alpha(j^r_0\gamma(\tau,\cdot))\big)\ |_{\tau=0}=\frac{d}{d\tau}\big(g^i_\alpha(j^r_0\gamma(\tau,\cdot))\big)\ |_{\tau=0}=0\,.$$

It follows that $u=0$. □

Theorem 6. *Let (G,ω) be a presymplectic groupoid satisfying the equation (4.4) and L the Dirac structure on M induced by G. The Dirac structure induced by the presymplectic groupoid $\left(T^rG,\omega^{(c)}\right)$ is the Dirac structure $\mathcal{T}^rL$.*

Proof. The Dirac structure $L\subset TM\oplus T^*M$ induced by (G,ω) is such that:

$$L=\left(\rho_{AG}\oplus\sigma\right)\left(AG\right)\,.$$

The Dirac structure induced by $\left(T^rG,\omega^{(c)}\right)$ is given by:

$$L^r=\left(\rho_{A(T^rG)}\circ\sigma^r\right)\big(A(T^rG)\big)$$

we have:

$$\begin{aligned}L^r&=\left[\kappa^r_M\circ T^r\left(\rho_{AG}\right)\circ\left(J^r_G\right)^{-1}\right]\oplus\left[\varepsilon^r_M\circ T^r\left(\sigma\right)\circ\left(J^r_G\right)^{-1}\right]\big(A\left(T^rG\right)\big)\\&=\left(\kappa^r_M\oplus\varepsilon^r_M\right)\left(T^r\left(\rho_{AG}\oplus\sigma\right)\right)\left(T^r\left(AG\right)\right)\\&=\left(\kappa^r_M\oplus\varepsilon^r_M\right)\left(T^r\left(\left(\rho_{AG}\oplus\sigma\right)\left(AG\right)\right)\right)\\&=\mathcal{T}^rL\,.\end{aligned}$$

So, $\mathcal{T}^rL=L^r$. □

Remark 6. We have seen that every Lie groupoid G equipped with a closed multiplicative 2-form ω verifying (4.4) induces a natural Dirac structure L such that there is an isomorphism between Lie algebroid $A(G)$ and the canonical Lie algebroid determined by L. Hence, Dirac manifolds may be thought as the infinitesimal version of presymplectic groupoid. We say that (G,ω) is an integration of L (see [1], [2], [15]).

Let L be a Dirac structure on M, whose associated Lie algebroid is integrable. Let G be the source simply connected Lie groupoid integrating L, then there is a unique closed multiplicative 2-form ω on G verifying (4.4) such that (G,ω) is an integration of L (see [2]). As consequence, the presymplectic groupoid $\left(T^rG,\omega^{(c)}\right)$ is an integration of the Dirac structure $\mathcal{T}^rL$.

The infinitesimal counterpart of tangent lifts of higher order of symplectic groupoid (G,ω) verifying (4.4) is a tangent lifts of higher order of Dirac structure induced by (G,ω). Let ϕ be a closed 3-form on M, a ϕ-twisted presymplectic groupoid over M is a pair (G,ω) where G is a Lie groupoid over M and ω is a multiplicative 2-form on G satisfying:

(1) $d\omega = s^*\phi - t^*\phi$

(2) $\dim G = 2\dim M$

(3) For each $x\in G$, the following condition holds

$$\ker\left(T_xs\right)\cap\ker\left(T_xt\right)\cap\ker\left(\omega^{\sharp}\left(x\right)\right)=\{0\}\,.$$

It induces a bundle map $\sigma_\phi\colon AG\to TM$ called infinitesimal multiplicative 2-form. Using the condition (2) and (3), it follows that the image L_ϕ of the bundle morphism

$$\rho_{AG}\oplus\sigma_\phi\colon AG\to TM\oplus T^*M$$

is a ϕ- twisted Dirac structure on M.

Corollary 3. *Let (G,ω) be a ϕ-twisted presymplectic groupoid, the pair $\left(T^rG,\omega^{(c)}\right)$ is also a $\phi^{(c)}$-twisted presymplectic groupoid and the IM-2-form $\sigma_{\phi^{(c)}}$ induced by $\left(T^rG,\omega^{(c)}\right)$ is given by:*

$$\sigma_{\phi^{(c)}}=\varepsilon^r_G\circ T^r\left(\sigma_\phi\right)\circ\left(J^r_G\right)^{-1}\,. \tag{4.6}$$

Corollary 4. *The $\phi^{(c)}$-twisted Dirac structure on T^rM induced by the $\phi^{(c)}$-twisted presymplectic groupoid $\left(T^rG,\omega^{(c)}\right)$ is the tangent lift of order r of the ϕ-twisted Dirac structure induced by the ϕ-twisted presymplectic groupoid (G,ω).*

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