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ON THE DIOPHANTINE EQUATION $\sum_{j=1}^{k} jF_j^p = F_n^q$

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Abstract. Let F_n denote the n^{th} term of the Fibonacci sequence. In this paper, we investigate the Diophantine equation $F_1^p + 2F_2^p + \cdots + kF_k^p = F_n^q$ in the positive integers k and n, where p and q are given positive integers. A complete solution is given if the exponents are included in the set $\{1,2\}$. Based on the specific cases we could solve, and a computer search with $p,q,k \le 100$ we conjecture that beside the trivial solutions only $F_8 = F_1 + 2F_2 + 3F_3 + 4F_4$, $F_4^2 = F_1 + 2F_2 + 3F_3$, and $F_4^3 = F_1^3 + 2F_2^3 + 3F_3^3$ satisfy the title equation.

1. Introduction

As usual, let $(F_n)_{n\ge 0}$ and $(L_n)_{n\ge 0}$ denote the sequences of Fibonacci and Lucas numbers, respectively, given by the initial values $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, and by the recurrence relations

$$F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n \quad \text{for all} \quad n \ge 0\,, \tag{1}$$

respectively. Putting $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2 = -1/\alpha$ for the two roots of the common characteristic equation $x^2 - x - 1 = 0$ of the two sequences, the formulae

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n$$

hold for all $n \ge 0$. These numbers are well-known for possessing amazing and wonderful properties (consult, for instance, [13] and [5] together with their very rich annotated bibliography for history and additional references). Observing

$$\begin{aligned} F_1 &= F_2\,,\\ F_1 + 2F_2 &= F_4\,,\\ F_1 + 2F_2 + 3F_3 &= F_4^2\,,\\ F_1 + 2F_2 + 3F_3 + 4F_4 &= F_8\,, \end{aligned}$$

the question arises naturally: is there any rule for $F_1 + 2F_2 + 3F_3 + \cdots + kF_k$? We study this question more generally, according to the title equation.

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Diophantine equations among the terms of Fibonacci numbers have a very extensive literature. Here we quote a few results that partially motivated us.

By the defining equality (1) of the Fibonacci numbers and the identity $F_n^2 + F_{n+1}^2 = F_{2n+1}$ (Lemma 1.8), we see that $F_n^s + F_{n+1}^s$ $(n \geq 0)$ is a Fibonacci number for $s \in \{1, 2\}$. For larger s Marques and Togbé [8] proved in 2010 that if $F_n^s + F_{n+1}^s$ is a Fibonacci number for all sufficiently large n then $s = 1$ or 2. Next year Luca and Oyono [6] completed the solution of the question by showing that apart from $F_1^s + F_2^s = F_3$ there is no solution $s \geq 3$ to the equation $F_n^s + F_{n+1}^s = F_m$.

Let $l, s_1, \ldots, s_l, a_1, \ldots, a_l$ be integers with $l \geq 1$ and $s_j \geq 1$. Suppose that there exists $1 \leq t \leq l$ such that $a_t \neq 0$ and $s_t > s_j$, for all $j \neq t$. Chaves, Marques and Togbé [4], showed that if either s_t is even or a_t is not a positive power of 5, then the sum

$$a_1 F_{n+1}^{s_1} + a_2 F_{n+2}^{s_2} + \cdots + a_l F_{n+l}^{s_l}$$

does not belong to the Fibonacci sequence for all sufficiently large n.

A balancing problem having similar flavor has been considered by Behera et al. [3]. They studied the equation

$$F_1^p + F_2^p + \cdots + F_{k-1}^p = F_{k+1}^q + \cdots + F_{k+r}^q\,, \tag{2}$$

and solved it for the cases $(p, q) = (2, 1), (3, 1), (3, 2)$, and for $2 \leq p \leq q$ by showing the non-existence of any solution. Further the authors conjectured that only the quadruple $(k, r, p, q) = (4, 3, 8, 2)$ of positive integers satisfies (2). The conjecture was completely justified by Alvarado et al. [1]. Note that if $(p, q) = (1, 1)$ we obtain the problem of sequence balancing numbers handled by Panda [9].

Recalling the formulae $F_1 + F_2 + \cdots + F_k = F_{k+2} - 1$ and $F_1^2 + F_2^2 + \cdots + F_k^2 = F_k F_{k+1}$, it is obvious that the problems

$$F_1 + F_2 + \cdots + F_k = F_n^q\,, \quad \text{and} \quad F_1^2 + F_2^2 + \cdots + F_k^2 = F_n^q$$

are rather simple. Indeed, the equations above lead to the lightsome ones

$$F_{k+2} - 1 = F_n^q\,, \qquad F_k F_{k+1} = F_n^q\,.$$

However the equation $F_1^p + F_2^p + \cdots + F_k^p = F_n^q$ might be taken an interest if $p \geq 3$.

The last motivation of our examination was the Diophantine equation

$$x^2 + 2(x+1)^2 + \cdots + n(x+n-1)^2 = y^2 \tag{3}$$

to determine the values of n for which it has finitely or infinitely many positive integer solutions (x, y) (see Wulczyn [14], and for details, see also [2]). For variations of the equation (3), we refer the reader to [12].

In this paper, we investigate the Diophantine equation

$$F_1^p + 2F_2^p + \cdots + kF_k^p = F_n^q \tag{4}$$

in the positive integers k and n, where p and q are fixed positive integers. We consider

$$F_1^p = 1 = F_1^q = F_2^q, \qquad \text{and} \qquad F_1^p + 2F_2^p = 3 = F_4$$

as trivial solutions to (4). We have the following conjecture based upon the specific cases we could solve, and a computer search with p, q, $k \leq 100$.

Conjecture 1. The non-trivial solutions to (4) are only

$$\begin{aligned} F_4^2 &= \ 9 = F_1 + 2F_2 + 3F_3\,, \\ F_8 &= 21 = F_1 + 2F_2 + 3F_3 + 4F_4\,, \\ F_4^3 &= 27 = F_1^3 + 2F_2^3 + 3F_3^3\,. \end{aligned}$$

This work handles the particular cases $p,\, q \in \{1, 2\}$ (hence the first two solutions above will be obtained), the precise results proved are described as follow.

Theorem 1. *If*

$$F_1 + 2F_2 + \cdots + kF_k = F_n\,, \tag{5}$$

then $(k, n) = (1, 1),\ (1, 2),\ (2, 4),\ (4, 8)$*, among them only the last one is non-trivial solution.*

Theorem 2. *The Diophantine equation*

$$F_1^2 + 2F_2^2 + \cdots + kF_k^2 = F_n^2 \tag{6}$$

possesses only the trivial solutions $(k, n) = (1, 1),\ (1, 2)$.

Theorem 3. *If*

$$F_1 + 2F_2 + \cdots + kF_k = F_n^2\,, \tag{7}$$

then $(k, n) = (1, 1),\ (1, 2),\ (3, 4)$*, among them only the last one is non-trivial solution.*

Theorem 4. *The Diophantine equation*

$$F_1^2 + 2F_2^2 + \cdots + kF_k^2 = F_n \tag{8}$$

possesses only the trivial solutions $(k, n) = (1, 1),\ (1, 2),\ (2, 4)$.

2. Lemmata

In this section, we present the lemmata that are needed in the proofs of the theorems. The first lemma is a collection of a few well-known results, we state them without proof, and in the proof of the theorems sometimes we do not refer to them.

Lemma 1. *Let k and n be arbitrary integers.*

(i) $\sum_{j=1}^{k} jF_j = kF_{k+2} - F_{k+3} + 2$.

(ii) $\sum_{j=1}^{k} jF_j^2 = F_k(kF_{k+1} - F_k) + \tau$*, where* $\tau = 0$ *if k is even, and* $\tau = 1$ *otherwise.*

(iii) *For* $k \ge 0$ *we have* $F_{-k} = (-1)^{k+1}F_k$*, further* $L_{-k} = (-1)^k L_k$ *(extension of the sequences for negative subscripts).*

(iv) $\gcd(F_k, F_n) = F_{\gcd(k,n)}$.

(v) $\gcd(F_k, L_n) = 1$ *or* 2 *or* $L_{\gcd(k,n)}$.

(vi) $F_k \mid F_n$ *if and only if* $k \mid n$.

(vii) $F_{k+1}F_n - F_kF_{n+1} = (-1)^{n+1}F_{k-n}$ *(d' Ocagne's identity).*

(viii) $F_{k+n} = F_k F_{n+1} + F_{k-1} F_n$.

(ix) $F_{2k} = F_k L_k$.

(x) $F_{k+n}^2 - F_{k-n}^2 = F_{2k} L_{2n}$.

Lemma 2.

$$F_k - 1 = \begin{cases} F_{(k+2)/2} L_{(k-2)/2}, & \text{if} \quad k \equiv 0 \pmod 4, \\ F_{(k-1)/2} L_{(k+1)/2}, & \text{if} \quad k \equiv 1 \pmod 4, \\ F_{(k-2)/2} L_{(k+2)/2}, & \text{if} \quad k \equiv 2 \pmod 4, \\ F_{(k+1)/2} L_{(k-1)/2}, & \text{if} \quad k \equiv 3 \pmod 4, \end{cases}$$

$$F_k + 1 = \begin{cases} F_{(k-2)/2} L_{(k+2)/2}, & \text{if} \quad k \equiv 0 \pmod 4, \\ F_{(k+1)/2} L_{(k-1)/2}, & \text{if} \quad k \equiv 1 \pmod 4, \\ F_{(k+2)/2} L_{(k-2)/2}, & \text{if} \quad k \equiv 2 \pmod 4, \\ F_{(k-1)/2} L_{(k+1)/2}, & \text{if} \quad k \equiv 3 \pmod 4. \end{cases}$$

Proof. See, for instance, [7] and [11]. □

Lemma 3.

$$F_k^2 - 1 = \begin{cases} F_{k-1} F_{k+1}, & \text{if} \quad k \equiv 1 \pmod 2, \\ F_{k-2} F_{k+2}, & \text{if} \quad k \equiv 0 \pmod 2, \end{cases}$$

$$F_k^2 + 1 = \begin{cases} F_{k-1} F_{k+1}, & \text{if} \quad k \equiv 0 \pmod 2, \\ F_{k-2} F_{k+2}, & \text{if} \quad k \equiv 1 \pmod 2. \end{cases}$$

Proof. See Lemma 3 in [10]. □

Lemma 4. *If $j \geq 4$ is even, then*

$$2F_{j-1}^{\varphi(F_j)-1} \equiv F_{j-3} \pmod{F_j}.$$

Proof. Since $\gcd(F_{j-1}, F_j) = 1$, and $F_{j-1}^{\varphi(F_j)} \equiv 1 \pmod{F_j}$, it is sufficient to show that

$$2 \equiv F_{j-3} F_{j-1} \pmod{F_j}.$$

But

$$F_{j-1} F_{j-3} = (F_{j+1} - F_j) F_{j-3} \equiv F_{j+1} F_{j-3} = F_j F_{j-2} + (-1)^{j-2} F_3 \equiv 2 \pmod{F_j}$$

follows from the definition of the Fibonacci numbers, d' Ocagne's identity (Lemma 1 (1)), and the parity of j. □

Lemma 5. *If $j \geq 3$ is odd, then*

$$F_{j-1}^{\varphi(F_j)-1} \equiv F_{j-2} \pmod{F_j}.$$

Proof. Similarly to the proof of the previous lemma, the statement is equivalent to

$$1 \equiv F_{j-2} F_{j-1} \pmod{F_j}.$$

And it is easy to see that

$$F_{j-2} F_{j-1} = (F_j - F_{j-1}) F_{j-1} \equiv -F_{j-1}^2 = -(F_{j-2} F_j + (-1)^j F_{-1}) \equiv 1 \pmod{F_j}.$$

$\square$

Lemma 6. *Let k_0 be a positive integer, and for $i \in \{0,1\}$ put*

$$\delta_i = \log_\alpha\left(\frac{1+(-1)^{i-1}\left(|\beta|/\alpha\right)^{k_0}}{\sqrt{5}}\right),$$

where $\log_\alpha$ is the logarithm in base $\alpha = (1+\sqrt{5})/2$. Then for all integers $k \geq k_0$, the two inequalities

$$\alpha^{k+\delta_0} \leq F_k \leq \alpha^{k+\delta_1}$$

hold.

Proof. This is a part of Lemma 5 in [7]. $\square$

In order to make the application of Lemma 6 more convenient, we shall suppose that $k_0 \geq 1$. Then we have

Corollary 5. *If $k \geq 1$, then*

$$\alpha^{k-2} \leq F_k \leq \alpha^{k-1},$$

and equality holds if and only if $k=2$, and $k=1$, respectively.

Now, we are ready to justify the theorems.

3. Proofs

Proof of Theorem 1.

Verifying the cases $k = 1, \ldots, 5$ by hand we found the solutions listed in Theorem 1. Put $\kappa = k+2$, and suppose that $\kappa \geq 8$. Consequently, $F_{\kappa-3} \geq 5$ and $F_\kappa \geq 21$. If equation (5) holds, then $n > \kappa$, and then by Lemma 1 (1) we conclude

$$k = \frac{F_n + F_{\kappa+1} - 2}{F_\kappa} = \frac{F_n + F_{\kappa-1} - 2}{F_\kappa} + 1 \in \mathbb{N}. \tag{9}$$

In the sequel, we study the sequence $(F_u)_{u=0}^{\infty}$ modulo F_κ if κ is fixed. Note that we indicate a suitable value congruent to F_u modulo F_κ, not always the smallest non-negative remainders. The period can be deduced from the range

$$\overbrace{0,1,1,2,\ldots,F_{\kappa-2},F_{\kappa-1}}^{\kappa},\overbrace{0,F_{\kappa-1},F_{\kappa-1},2F_{\kappa-1},\ldots,F_{\kappa-2}F_{\kappa-1},F_{\kappa-1}F_{\kappa-1}}^{\kappa},$$

of length 2κ if κ is even, since then, by Lemma 1 (1) we have $F_{\kappa-1}^2 \equiv 1 \pmod{F_\kappa}$ and then

$$F_{\kappa-2}F_{\kappa-1} = (F_\kappa - F_{\kappa-1})F_{\kappa-1} \equiv -1 \pmod{F_\kappa}.$$

In case of odd κ we have $F_{\kappa-1}^2 \equiv -1 \pmod{F_\kappa}$, therefore the length of the period is 4κ coming from

$$\overbrace{0,1,1,2,\dots,F_{\kappa-2},F_{\kappa-1}}^{\kappa},\overbrace{0,F_{\kappa-1},F_{\kappa-1},2F_{\kappa-1},\dots,F_{\kappa-2}F_{\kappa-1},F_{\kappa-1}F_{\kappa-1}}^{\kappa},$$
$$\overbrace{0,-1,-1,-2,\dots,-F_{\kappa-2},-F_{\kappa-1}}^{\kappa},$$
$$\overbrace{0,-F_{\kappa-1},-F_{\kappa-1},-2F_{\kappa-1},\dots,-F_{\kappa-2}F_{\kappa-1},-F_{\kappa-1}F_{\kappa-1}}^{\kappa}.$$

Based on the length of the period we distinguish two cases.

Case I: κ is even. Either $F_n \equiv F_j$ or $F_n \equiv F_jF_{\kappa-1}$ modulo F_κ holds for some $j = 0,1,\dots,\kappa-1$. Hence

$$F_n + F_{\kappa-1} - 2 \equiv \begin{cases} F_j + F_{\kappa-1} - 2, & \text{or} \\ F_jF_{\kappa-1} + F_{\kappa-1} - 2 \end{cases} \pmod{F_\kappa}. \tag{10}$$

We will show that none of them is congruent to 0 modulo F_κ. In the first branch

$$F_j + F_{\kappa-1} - 2 \geq F_{\kappa-1} - 2 \geq 11,$$

further if $j \neq \kappa - 1$, then

$$F_j + F_{\kappa-1} - 2 \leq F_{\kappa-2} + F_{\kappa-1} - 2 \leq F_\kappa - 2.$$

Thus $F_j + F_{\kappa-1} - 2 \not\equiv 0 \pmod{F_\kappa}$, hence (9) does not hold. Assume now, that $j = \kappa - 1$. Then, together with the definition of the Fibonacci sequence we have

$$F_j + F_{\kappa-1} - 2 = F_{\kappa-1} + (F_\kappa - F_{\kappa-2}) - 2 \equiv F_{\kappa-3} - 2 \pmod{F_\kappa}.$$

But $3 \leq F_{\kappa-3} - 2 < F_\kappa$ contradicts to (9).

Choosing the second branch of (10), suppose that $F_{\kappa-1}(F_j+1) - 2$ is congruent to 0 modulo F_κ. Then

$$F_{\kappa-1}^{\varphi(F_\kappa)}(F_j+1) \equiv 2F_{\kappa-1}^{\varphi(F_\kappa)-1} \pmod{F_\kappa}.$$

Subsequently, by Lemma 4, it leads to

$$F_j + 1 \equiv F_{\kappa-3} \pmod{F_\kappa}.$$

Since $j = 0,1,\dots,\kappa-1$, ($\kappa \geq 8$) it follows that $F_j = F_{\kappa-3} - 1$, a contradiction.

Case II: κ is odd. Now $\kappa \geq 9$, and either $F_n \equiv \pm F_j \pmod{F_\kappa}$ or $F_n \equiv \pm F_jF_{\kappa-1} \pmod{F_\kappa}$ holds for some $j = 0,1,\dots,\kappa-1$. Hence

$$F_n + F_{\kappa-1} - 2 \equiv \begin{cases} \pm F_j + F_{\kappa-1} - 2 \\ \pm F_jF_{\kappa-1} + F_{\kappa-1} - 2 \end{cases} \pmod{F_\kappa}.$$

First, obviously, if $j \neq \kappa - 1$, then

$$6 \leq F_{\kappa-3} - 2 \leq \pm F_j + F_{\kappa-1} - 2 \leq F_\kappa - 2,$$

so dividing $\pm F_j + F_{\kappa-1} - 2$ by F_κ, the result is not an integer. If $j = \kappa - 1$, then the treatment of the "+" case coincides the treatment when κ was even. The "−" case leads to $F_n + F_{\kappa-1} - 2 \equiv -2 \pmod{F_\kappa}$, a contradiction.

Assume now that $F_n + F_{\kappa-1} - 2 \equiv \pm F_j F_{\kappa-1} + F_{\kappa-1} - 2 \pmod{F_\kappa}$. Thus $F_{\kappa-1}(1 \pm F_j) \equiv 2 \pmod{F_\kappa}$. Multiplying both sides by $F_{\kappa-1}^{\varphi(F_\kappa)-1}$, by Lemma 5 it gives

$$1 \pm F_j \equiv 2F_{\kappa-2} \pmod{F_\kappa}.$$

First let $F_j = 2F_{\kappa-2} - 1$, which leads immediately a contradiction via $0 < 2F_{\kappa-2} - 1 = F_{\kappa-1} + F_{\kappa-4} - 1 < F_\kappa$. If $F_\kappa - F_j + 1 = 2F_{\kappa-2}$, then $F_j = F_{\kappa-3} + 1$ follows, a contradiction again. The proof of Theorem 1 is complete.

Proof of Theorem 2.

For the range $k = 1, 2, \ldots, 20$ we checked (6) by hand. From now we assume $k \geq 21$. Based on Lemma 1 (1), we must distinguish two cases.

Case I: k is even. Consider the equation

$$F_k(kF_{k+1} - F_k) = F_n^2\,.$$

Trivially, $n > k$. Put $\nu = \gcd(k, n)$.

If $\nu = k$, then $F_k \mid F_n$ by Lemma 1 (1). Consequently,

$$\left(\frac{F_n}{F_k}\right)^2 = \frac{kF_{k+1} - F_k}{F_k} = \frac{kF_{k+1}}{F_k} - 1$$

is integer. But F_k and F_{k+1} are coprime, hence $F_k \mid k$, and it results $k \leq 5$, a contradiction.

Examine the possibility $\nu = k/2$. Put $\kappa = k/2$. Now $F_\kappa L_\kappa(kF_{k+1} - F_\kappa L_\kappa) = F_n^2$ leads to

$$\frac{L_\kappa(kF_{k+1} - F_\kappa L_\kappa)}{F_\kappa} = \left(\frac{F_n}{F_\kappa}\right)^2.$$

This is an equality of integers, which together with $\gcd(F_\kappa, F_{k+1})$ and $\gcd(F_\kappa, L_\kappa) = 1, 2$ (see Lemma 1 (1)) shows that $2k/F_\kappa$ is integer. Thus $k \leq 14$, a contradiction.

Finally, we have $3 \leq \nu \leq k/3$. Since $\gcd(F_k/F_\nu, F_n/F_\nu) = 1$, then from the equation

$$\frac{F_k}{F_\nu}(kF_{k+1} - F_k) = \frac{F_n}{F_\nu} F_n$$

we conclude

$$\frac{F_k}{F_\nu} \mid F_n \qquad \text{and} \qquad \frac{F_n}{F_\nu} \mid kF_{k+1} - F_k\,.$$

Subsequently, $F_k \mid F_\nu F_n$ and $F_n \mid F_\nu(kF_{k+1} - F_k)$. Thus $F_k \mid F_\nu^2(kF_{k+1} - F_k)$, and then $F_k \mid kF_\nu^2$ holds since $\gcd(F_k, F_{k+1}) = 1$. Applying Corollary 5, we obtain

$$\alpha^{k-2} \leq F_k \leq kF_\nu^2 \leq k\alpha^{2\nu-2} \leq k\alpha^{2/3k-2}\,,$$

which implies $k < 19$.

Case II: k is odd. In this part, we follow the idea of the previous case. Recall that $k \geq 21$. Now

$$F_k(kF_{k+1} - F_k) = F_{n-\varepsilon}F_{n+\varepsilon}\,,$$

where $\varepsilon = 1$ or 2 depending on the parity of n (see Lemma 3). Clearly, $\gcd(n-\varepsilon, n+\varepsilon) = 2$ or 4. Thus $\gcd(F_{n-\varepsilon}, F_{n+\varepsilon}) = 1$ or 3, respectively.

Put $\nu_1 = \gcd(k, n-\varepsilon)$ and $\nu_2 = \gcd(k, n+\varepsilon)$. Obviously, $\gcd(\nu_1, \nu_2)$ divides $\gcd(n-\varepsilon, n+\varepsilon)$. Hence $\nu = \gcd(\nu_1, \nu_2) = 1$ or 2 or 4, and then $F_\nu = \gcd(F_{\nu_1}, F_{\nu_2}) = 1$ or 3. Thus $F_{\nu_1}F_{\nu_2} \mid F_\nu F_k$. The terms of both the left and right sides of

$$\frac{F_k}{F_{\nu_1}}(kF_{k+1} - F_k) = \frac{F_{n-\varepsilon}}{F_{\nu_1}}F_{n+\varepsilon} \qquad \text{and} \qquad \frac{F_k}{F_{\nu_2}}(kF_{k+1} - F_k) = F_{n-\varepsilon}\frac{F_{n+\varepsilon}}{F_{\nu_2}}$$

are integers, and

$$\frac{F_{n-\varepsilon}}{F_{\nu_1}} \mid kF_{k+1} - F_k\,, \qquad \frac{F_k}{F_{\nu_2}} \mid F_{n-\varepsilon}\,.$$

Combining them, $F_k \mid F_{\nu_1}F_{\nu_2}(kF_{k+1} - F_k)$ follows, and then $F_k \mid kF_{\nu_1}F_{\nu_2}$. The remaining part of the proof consists of three cases.

Suppose first that $\nu_1 = k$, i.e. $F_k \mid F_{n-\varepsilon}$. Observe, that $n-\varepsilon$ and $n+\varepsilon$ are even, and $k \mid (n-\varepsilon)/2$. Thus $k+1 \leq (n-\varepsilon)/2 + 1$, which does not exceed $(n+\varepsilon)/2$. Then

$$\begin{aligned} kF_{k+1} > \frac{F_{n-\varepsilon}}{F_{\nu_1}}F_{n+\varepsilon} &= \frac{F_{(n-\varepsilon)/2}}{F_{\nu_1}}L_{(n-\varepsilon)/2}F_{n+\varepsilon} \geq L_{(n-\varepsilon)/2}L_{(n+\varepsilon)/2}F_{(n+\varepsilon)/2} \\ &\geq L_{(n-\varepsilon)/2}L_{(n+\varepsilon)/2}F_{k+1}\,. \end{aligned}$$

Simplifying by F_{k+1} we conclude

$$\frac{n-\varepsilon}{2} \geq k > L_{(n-\varepsilon)/2}L_{(n+\varepsilon)/2}\,,$$

and we arrived at a contradiction since $21 \leq k < n$. Note that the same machinery works when $\nu_2 = k$, i.e. $F_k \mid F_{n+\varepsilon}$.

If none of the two conditions above holds, we can assume $\nu_1 \leq k/3$ and $\nu_2 \leq k/3$. Indeed, k is odd, so the largest non-trivial divisor of k is at most $k/3$. The application of Corollary 5 gives

$$\alpha^{k-2} \leq F_k \leq kF_{\nu_1}F_{\nu_2} \leq k\alpha^{\nu_1 - 1}\alpha^{\nu_2 - 1} \leq k\alpha^{2k/3-2}\,,$$

and then $k < 19$.

The proof of the theorem is complete.

Proof of Theorem 3.

The proof partially follows the proof of Theorem 1. The small cases of (7) can be verified by hand. Suppose $\kappa = k + 2 \geq 9$. Similarly to (9), we have

$$k = \frac{F_n^2 + F_{\kappa+1} - 2}{F_\kappa} = \frac{F_n^2 + F_{\kappa-1} - 2}{F_\kappa} + 1 \in \mathbb{N}\,. \tag{11}$$

Now we study the sequence $(F_u^2)_{u=0}^\infty$ modulo F_κ, and we again indicate the most suitable values by modulo F_κ, not always the smallest non-negative remainders. Lemma 1 (1), together with Lemma 1(1) implies

$$F_{2\kappa \pm j}^2 \equiv F_j^2 \pmod{F_\kappa}\,,$$

where $j = 0, 1, \ldots, \kappa - 1$. Hence the period having length 2κ can be given by

$$\overbrace{0, 1, 1, 2^2, \ldots, F_{\kappa-2}^2, F_{\kappa-1}^2}^{\kappa}, \overbrace{0, F_{\kappa-1}^2, F_{\kappa-2}^2, \ldots, 2^2, 1, 1}^{\kappa}.$$

Let us distinguish two cases according to the parity of κ.

Case I: κ is even. Put $\kappa = 2\ell$. Again by Lemma 1 (1), together with $\kappa - i = 2\ell - i = \ell + (\ell - i)$ and $i = \ell - (\ell - i)$ admits $F_{\kappa-i}^2 \equiv F_i^2 \pmod{F_\kappa}$. It reduces the possibilities to $j = 0, 1, \ldots, \ell$.

If $j \le \ell - 1 = (\kappa - 2)/2$, then

$$0 < F_j^2 + F_{\kappa-1} - 2 \le F_{(\kappa-2)/2}^2 + F_{\kappa-1} - 2 \le F_{\kappa-2} + F_{\kappa-1} - 2 < F_\kappa$$

hold since $F_{(\kappa-2)/2}^2 \le F_{(\kappa-2)/2} L_{(\kappa-2)/2} = F_{\kappa-2}$.

Suppose now that $j = \ell = \kappa/2$. Repeating the previous idea we find $F_{\kappa/2}^2 + F_{\kappa-1} - 2 \le F_\kappa + F_{\kappa-1} - 2 < 2F_\kappa$. Consequently, $F_{\kappa/2}^2 + F_{\kappa-1} - 2 = F_\kappa$ might be fulfilled. Thus $F_{\kappa/2}^2 - 2 = F_{\kappa-2}$. Recalling $\kappa = 2\ell$ we equivalently obtain

$$F_\ell^2 - 1 = F_{2\ell-2} + 1\,.$$

Both sides have decomposition described in Lemma 3 and Lemma 2, respectively, providing

$$F_{\ell-1}F_{\ell+1} = F_{\ell-2}L_\ell \qquad \text{or} \qquad F_{\ell-2}F_{\ell+2} = F_\ell L_{\ell-2}$$

if ℓ is odd or even, respectively. Firstly, $F_{\ell-2} \mid F_{\ell-1}F_{\ell+1}$, and then $F_{\ell-2} \mid F_{\ell+1}$ holds only for small ℓ values. Secondly, $F_\ell \mid F_{\ell-2}F_{\ell+2}$ contradicts to $\ell \ge 5$.

Case II: κ is odd. Now we have $F_{\kappa-i}^2 \equiv -F_i^2 \pmod{F_\kappa}$. Indeed, Lemma 1 (1) admits

$$F_{\kappa-i}^2 = (F_\kappa F_{-i+1} + F_{\kappa-1}F_{-i})^2 \equiv F_{\kappa-1}^2 F_i^2 \pmod{F_\kappa}\,,$$

and then Lemma 3 justifies the statement. It makes possible to split the proof into a few parts.

If $j = (\kappa - 1)/2$, then $F_{(\kappa-1)/2}^2 + F_{\kappa-1} - 2 < 2F_{\kappa-1} - 2 < 2F_\kappa$. Thus $F_{(\kappa-1)/2}^2 + F_{\kappa-1} - 2 = F_\kappa$ is the only one chance to fulfill (11). Then we apply Lemma 6 with $k_0 = 4 \le (\kappa - 1)/2$ for $F_{\kappa-2} < F_{(\kappa-1)/2}^2$ to reach a contradiction.

If $j \le (\kappa - 3)/2$, then

$$0 < F_j^2 + F_{\kappa-1} - 2 \le F_{(\kappa-3)/2}^2 + F_{\kappa-1} - 2 \le F_{\kappa-3} + F_{\kappa-1} - 2 < F_\kappa$$

holds since $F_{(\kappa-3)/2}^2 \le F_{(\kappa-3)/2} L_{(\kappa-3)/2} = F_{\kappa-3}$.

Finally, if $(\kappa + 1)/2 \le j \le \kappa - 1$, then

$$F_j^2 + F_{\kappa-1} - 2 \equiv -F_{\kappa-j}^2 + F_{\kappa-1} - 2 \pmod{F_\kappa}\,,$$

where $1 \le J = \kappa - j \le (\kappa - 1)/2$. Thus we need to check the equation $F_J^2 + 2 = F_{\kappa-1}$, since $-2 \le -F_J^2 + F_{\kappa-1} - 2 < F_{\kappa-1} < F_\kappa$. When $J \le (\kappa - 3)/2$ holds then $F_J^2 + 2 < F_{\kappa-3} + 2 < F_{\kappa-1}$. Lastly, $J = (\kappa - 1)/2$ leads to $F_{(\kappa-1)/2}^2 + 2 = F_{\kappa-1}$, and then to $2 = F_J\,(L_J - F_J)$. Obviously, it gives $J \le 3$. Thus $\kappa \le 7$, a contradiction.

Proof of Theorem 4.

The statement for (8) is obviously true if $k \leq 12$. So we may assume $k \geq 13$. Let $\tau \in \{0,1\}$. The formula of the summation, together with Corollary 5 implies

$$\alpha^{n-2} \leq F_n = F_k(kF_{k+1} - F_k) + \tau < kF_kF_{k+1} \leq \alpha^{\log_\alpha k + k - 1 + (k+1) - 1},$$

and then

$$n - k < k + 1 + \log_\alpha k < \frac{3}{2}k\,.$$

Similarly,

$$\begin{aligned}\alpha^{n-1} \geq F_n = F_k(kF_{k+1} - F_k) + \tau > F_k(kF_{k+1} - F_{k+1}) &= (k-1)F_kF_{k+1} \\ &> \alpha^{\log_\alpha (k-1) + k - 2 + (k+1) - 2},\end{aligned}$$

subsequently

$$n - k > k - 2 + \log_\alpha (k-1) > k + 3\,,$$

that is $n - k \geq k + 4$. Putting together the two estimates it gives

$$2k + 4 \leq n < \frac{5}{2}k. \tag{12}$$

Case I: k is even. Clearly, $k + 2 < n - k - 1 \leq 3k/2 - 2$ holds. By Lemma 1 (1), we conclude

$$F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1} = kF_kF_{k+1} - F_k^2\,,$$

and equivalently

$$F_k(F_k + F_{n-k-1}) = F_{k+1}(kF_k - F_{n-k})\,.$$

Thus $\gcd(F_k, F_{k+1}) = 1$ admits $F_{k+1} \mid F_k + F_{n-k-1}$. The periodicity of $(F_u)_{u=0}^{\infty}$ modulo F_{k+1} guarantees, together with the bounds on $n - k - 1$ that

$$F_k + F_{n-k-1} \equiv F_k + F_jF_k = F_k(F_j + 1) \pmod{F_{k+1}}$$

holds for some $j = 1, 2, \dots, k/2 - 3$. Consequently, $F_{k+1} \mid F_j + 1$, a contradiction.

Case II: k is odd. Again $k + 2 < n - k - 1 \leq 3k/2 - 2$ holds. Lemma 2 and Lemma 1 (1) imply

$$F_k(kF_{k+1} - F_k) = F_{(n-\varepsilon)/2}L_{(n+\varepsilon)/2} = (F_{k+1}F_{(n-\varepsilon)/2-k} + F_kF_{(n-\varepsilon)/2-k-1})L_{(n+\varepsilon)/2}\,,$$

where $\varepsilon \in \{\pm 1, \pm 2\}$ according to the modular property of n. It leads to

$$F_{k+1}(kF_k - F_{(n-\varepsilon)/2-k}L_{(n+\varepsilon)/2}) = F_k(F_k + F_{(n-\varepsilon)/2-k-1}L_{(n+\varepsilon)/2})\,.$$

Thus $F_k \mid F_{(n-\varepsilon)/2-k}L_{(n+\varepsilon)/2}$.

By (12) it is obvious that

$$k < k + 1 \leq \frac{n-2}{2} \leq \frac{n+\varepsilon}{2} \leq \frac{n+2}{2} \leq \frac{5}{4}k + 1 < 2k\,,$$

which exclude $\gcd(k, (n+\varepsilon)/2) = k$. Thus $\gcd(k, (n+\varepsilon)/2) \leq k/3$ since k is odd, subsequently $\gcd(F_k, L_{(n+\varepsilon)/2}) \leq L_{k/3}$. On the other hand

$$\frac{n-\varepsilon}{2} - k \leq \frac{n+2}{2} - k \leq \frac{1}{4}k + 1\,,$$

which implies $\gcd(F_k, F_{(n-\varepsilon)/2-k}) \leq F_{k/4+1}$.

Thus, $F_k \mid F_{(n-\varepsilon)/2-k}L_{(n+\varepsilon)/2}$, together with the previous arguments entails $F_k \leq F_{k/4+1}L_{k/3}$, but it leads to a contradiction since $L_{k/3} = F_{k/3-1} + F_{k/3+1} < 2F_{k/3+1}$, and the application of Corollary 5 on $F_k \leq 2F_{k/4+1}F_{k/3+1}$ returns with $k < 9$.

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