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WEAK NORMAL AND QUASINORMAL FAMILIES OF HOLOMORPHIC CURVES

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Abstract. In this paper we introduce the notion of weak normal and quasinormal families of holomorphic curves from a domain in $\mathbb{C}$ into projective spaces. We will prove some criteria for the weak normality and quasinormality of at most a certain order for such families of holomorphic curves.

1. Introduction

In the sense of Montel, a family $\mathcal{F}$ of meromorphic functions defined on a domain Ω of the complex plane $\mathbb{C}$ is said to be normal if from every sequences in $\mathcal{F}$ we may extract a subsequence which converges compactly with respect to the spherical metric to a meromorphic function or ∞ on Ω. The family $\mathcal{F}$ is said to be quasinormal (of order v) if the above extracted sequences converge compactly on $\Omega \setminus \{\text{a disceret set}\}$ (of at most v points).

One of the earliest criterion for normality of families of meromorphic functions is given by Montel. He showed that a family $\mathcal{F}$ of meromorphic functions on Ω is normal if all $f \in \mathcal{F}$ omit three distinct values a_1, a_2, $a_3 \in \mathbb{C}$. Moreover, he showed a more general result on the quasinormality as follows.

Theorem A (see [3]). *Let $\mathcal{F}$ be a family of meromorphic functions on a domain in $\mathbb{C}$ which do not take a value a_1 more than p times, a value a_2 more than q times, nor a value a_3 more than r times, with $p \le q \le r$. Then $\mathcal{F}$ is quasinormal of order at most q.*

Over the last few decades, there have been many results generalizing and improving the above result of Montel. The theory on the normality and quasinormality of meromorphic functions had grown into a huge theory with many contributions. We refer readers to the articles [2, 7, 10, 11] and references therein for the development of related subjects. Specially, Zalcmann [15] gave a famous criterion for the normality of the families of meromorphic functions. And then, his criterion is generalized to the case of holomorphic mappings by Aladro-Krantz [1] and Thai-Trang-Huong [14]. These criteria give the necessary and sufficient condition for the non-normality

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of a family of holomorphic mappings and play an essential role in the development of the normal and quasinormal theory.

On the other hand, Fujimoto [5] introduced the notion of the meromorphic convergence for meromorphic mappings as follows: *"A sequence of meromorphic mappings $\{f_n\}_{n=1}^{\infty}$ from a domain Ω of $\mathbb{C}^m$ into $\mathbb{P}^N(\mathbb{C})$ is said to meromorphically converge to a meromorphic mapping g if for each $p \in \Omega$, there exist an open neighborhood U of p, reduced representations $\tilde{f}_n = (f_{n0}, \dots, f_{nN})$ of f_n, a representation $\tilde{g} = (g_0, \dots, g_N)$ of g on U such that $\{f_{nk}\}_{n=1}^{\infty}$ converges compactly (i.e., converges uniformly on every compact subsets) on U to g_k for all $0 \leq k \leq N$".* With respect to the notion of meromorphic convergence, there are many results on the meromorphic normality of meromorphic mappings established in some recent years, e.g., [4, 6, 12].

Our purpose in this paper is to generalize the above result of Montel to the case of holomorphic curves from $\mathbb{C}$ into projective spaces, and we also give a criterion for the quasinormality of such curves with respect to the convergent notion of Fujimoto. In order to state our results, we give the following definition.

Definition 1. Let $\mathcal{F}$ be a family of holomorphic curves from a domain $\Omega \subset \mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$.

1) The family $\mathcal{F}$ is said to be weak normal if every sequence in $\mathcal{F}$ has a subsequence which converges compactly (with respect to the Fubini-Study metric on $\mathbb{P}^N(\mathbb{C})$) to a holomorphic curve on $\Omega \setminus S$, where S is a discrete subset of Ω and the limit curve is holomorphically extendable on Ω.

2) The family $\mathcal{F}$ is said to be quasinormal (resp. meromorphically quasinormal) of order at most v (v may be $+\infty$) if every sequence in $\mathcal{F}$ has a subsequence which converges compactly (resp. meromorphically converges) to a holomorphic curve on a domain $\Omega \setminus S$, where S is a discrete subset of at most v elements in Ω. If $v = +\infty$ we will say that $\mathcal{F}$ is quasinormal (resp. meromorphically quasinormal).

Throughout this paper, we fix homogeneous coordinates $(\omega_0 : \cdots : \omega_N)$ on $\mathbb{P}^N(\mathbb{C})$. Let H be a hypersurface of degree d in $\mathbb{P}^N(\mathbb{C})$ defined by the equation

$$\sum_{I \in \mathcal{T}_d} a_I \omega^I = 0\,,$$

where $\mathcal{T}_d = \{(i_0, \dots, i_N) \in \mathbb{Z}_+^{N+1}\ ;\ i_0 + \cdots + i_N = d\}$, $\omega^I = \omega_0^{i_0} \dots \omega_N^{i_N}$ for $I = (i_0, \dots, i_N)$. We define $||H|| = (\sum_{I \in \mathcal{T}_d} |a_I|^2)^{1/2}$. Throughout this paper, we always assume that the coefficients a_I are chosen so that $||H|| = 1$. Sometimes, we identify the hypersurface H with its defining polynomial, i.e., we will write

$$H(\omega_0, \dots, \omega_N) = \sum_{I \in \mathcal{T}_d} a_I \omega^I\,.$$

Let $H_1, \dots, H_q (q \geq N+1)$ be q hypersurfaces of $\mathbb{P}^N(\mathbb{C})$, which may be of different degrees. We define

$$D(H_1, \dots, H_q) = \prod_{1 \leq i_1 < i_2 < \cdots < i_{N+1} \leq q} \inf_{z \in \mathbb{C}^{N+1}, \|z\|=1} \left(|H_{i_1}(z)|^2 + \cdots + |H_{i_{N+1}}(z)|^2\right)$$

where $\| z \| = \sqrt{\sum_{i=0}^{N} |z_i|^2}$.

Our results are stated as follows.

Main theorem. *Let $\mathcal{F}$ be a family of holomorphic mappings of a subdomain Ω of $\mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$. Let $q \geq 2N+1$ be a non-negative integer and let $p_1, \ldots, p_{N+1}$ be $N+1$ positive integers with $p_1 \leq \cdots \leq p_{N+1}$. For each $f \in \mathcal{F}$, let $H_1(f), \ldots, H_q(f)$ (depending on f) be hypersurfaces satisfying $\inf\{D(H_1(f), \ldots, H_q(f)) \colon f \in \mathcal{F}\} > 0$. Assume that for any compact subset K of Ω,*

$$\sup_{f \in \mathcal{F}} \sharp\{z \in K \ ; \ \nu_{f,H_i(f)}(z) > 0\} < +\infty, \ \forall 1 \leq i \leq q \,.$$

Then the following assertions hold:

(a) *$\mathcal{F}$ is weak normal family.*

(b) *If for any compact subsets K of Ω, there exists an positive integer c_K satisfying*

$$\sup_{f \in \mathcal{F}} \sharp\{z \in K \ ; \ \nu_{f,H_i(f)}(z) \geq c_k\} \leq p_i, \ \forall 1 \leq i \leq N+1 \,,$$

then $\mathcal{F}$ is meromorphically quasinormal of order at most p_{N+1} in Ω.

(c) *If $\sup_{f \in \mathcal{F}} \sharp\{f^{-1}(H_i(f_n))\} \leq p_i, \ \forall 1 \leq i \leq N+1$, then $\mathcal{F}$ is quasinormal of order at most p_{N+1} in Ω.*

We would like to note that, in almost all recent results on the normality of families of holomorphic mappings into $\mathbb{P}^N(\mathbb{C})$ with $2N+1$ hypersurfaces, the authors always assume that the inverse images of at least $N+1$ hypersurfaces counted with multiplicities are compactly bounded from above. This is an esential condition in their proofs. In our above result, this condition is omitted.

2. Some definitions and lemmas

For $p \in \mathbb{C}$ and $r > 0$, we set

$$\Delta(p, r) = \{z \in \mathbb{C} \ ; \ |z - p| < r\}$$

and

$$\Delta^*(p, r) = \Delta(p, r) \setminus \{p\} \,.$$

Let Ω be a domain in $\mathbb{C}$. A divisor ν on Ω is a formal sum

$$\nu = \sum_{i \in \Lambda} a_i \{z_i\} \,,$$

where $a_i \in \mathbb{Z}$ and $\{z_i\}_{i \in \Lambda}$ is a discrete subset of Ω. We may regard the divisor ν as a function with values in $\mathbb{Z}$ by setting

$$\nu(z) = \begin{cases} a_i & \text{if} \quad z = z_i \,, \\ 0 & \text{if} \quad z \neq z_i \ \forall i \,. \end{cases}$$

The support of ν will be defined by $\mathrm{supp}(\nu) = \{z_i \ ; \ a_i \neq 0\}$. Divisor ν is said to be non-negative if $a_i \geq 0 \ \ \forall i \in \Lambda$, and we write $\nu \geq 0$. If $\nu \geq 0$ then for every compact subset K of Ω, $\sharp\{z \ ; \ \nu(z) > 0\} \cap K \leq \sum_{z \in K} \nu(z)$.

Definition 2. A sequence of non-negative divisor $\{\nu_i\}_{i=1}^{\infty}$ is said to be bounded compactly on Ω if for every compact subset K of Ω, there exits a positive constant M_K such that $\sum_{z\in K}\nu_i(z)\le M$ for all $i\ge 1$.

Let ω be the Fubini-Study form on $\mathbb{P}^N(\mathbb{C})$. Then ω is defined by

$$\omega = dd^c\log\left(\sum_{j=0}^{N}\left|\frac{\omega_j}{\omega_i}\right|^2\right)^{1/2}$$

on the affine open set $U_i=\mathbb{P}^N(\mathbb{C})\setminus\{\omega_i=0\}$, where $(\omega_0:\cdots:\omega_n)$ is a homogeneous coordinates system of $\mathbb{P}^N(\mathbb{C})$. Then for two distinct points $p=(p_0:\cdots:p_N)$ and $q=(q_0:\cdots:q_N)$, the distance between p and q with respect to is given by

$$d(p,q)=\frac{\sum_{i,j=0}^{N}|p_iq_j-p_jq_i|}{\sqrt{\sum_{i=0}^{N}|p_i|^2}\cdot\sqrt{\sum_{i=0}^{N}|q_i|^2}}\,.$$

For two hyperplanes H and G in $\mathbb{P}^N(\mathbb{C})$ defined by the linear forms:

$$H: a_0\omega_0+\cdots a_N\omega_n\,,$$
$$G: b_0\omega_0+\cdots b_N\omega_n\,,$$

we regard H and G as two points in the dual space $\mathbb{P}^N(\mathbb{C})^*$ and define the "*distance* $d'(H,G)$ *in* $\mathbb{P}^N(\mathbb{C})^*$" between them by

$$d'(H,G)=\frac{\sum_{i,j=0}^{N}|a_ib_j-a_jb_i|}{\sqrt{\sum_{i=0}^{N}|a_i|^2}\cdot\sqrt{\sum_{i=0}^{N}|b_i|^2}}\,.$$

Definition 3 (see [13])**.** Let M be a locally compact Hausdorff space. A point a of M is called a limit point of the sequence $\{E_k\}_{k=1}^{\infty}$ of closed subsets of M if there exist a positive integer k_0 and points $a_k\in E_k$ $(k\ge k_0)$ such that $a=\lim a_k$. A point of M is called a cluster point of $\{E_k\}_{k=1}^{\infty}$ if it is a limit point of some subsequence of $\{E_k\}_{k=1}^{\infty}$. If the set E of limit points coincides with the set of cluster points, $\{E_k\}_{k=1}^{\infty}$ is said to converge to E and write $\lim E_k=E$.

The following two lemmas on the convergence of closed subsets in Hausdorff space are due to Stoll [13]. In this paper, we only state these lemmas for the case of subsets of $\mathbb{C}$.

Lemma 4 ([13, Proposition 4.11])**.** *Let $\{N_i\}_{i=1}^{\infty}$ be a sequence of discrete subsets of a domain Ω in $\mathbb{C}$. Assume that the numbers of elements of $N_i\cap K$ $(i=1,2,\ldots)$ regardless of multiplicities are bounded above for any fixed compact subset K of Ω and $\{N_i\}_{i=1}^{\infty}$ converges to N as a sequence of closed subsets of Ω. Then N is either empty or a discrete subset of Ω.*

Lemma 5 ([13, Proposition 4.12])**.** *Let $\{N_i\}_{i=1}^{\infty}$ be a sequence of discrete subsets of a domain Ω in $\mathbb{C}$. The numbers of elements of $N_i\cap K$ $(i=1,2,\ldots)$ regardless of multiplicities are bounded above for any fixed compact subset K of Ω, then $\{N_i\}$ is normal in the sense of the convergence of closed subsets in Ω.*

Definition 6 ([5, Definition 2.5]). Let $\{\nu_i\}_{i=1}^{\infty}$ be a sequence of non-negative divisors on a domain Ω in $\mathbb{C}$. It is said to converge to a non-negative divisor ν on Ω if and only if any $a \in \Omega$ has a neighborhood U such that there exist nonzero holomorphic functions h and h_i on U with $\nu_i = \nu_{h_i}$ and $\nu = \nu_h$ on U such that $\{h_i\}_{i=1}^{\infty}$ converges compactly to h on U.

Lemma 7 ([13, Theorem 2.24]). *A sequence $\{\nu_i\}_{i=1}^{\infty}$ of non-negative divisors on a domain Ω in $\mathbb{C}$ is normal in the sense of the convergence of divisors on Ω if and only if $\{\nu_i\}_{i=1}^{\infty}$ is compactly bounded on Ω.*

Lemma 8 ([14, Theorem 2.5]). *Let Ω be a domain in $\mathbb{C}$ and M be a compact complex Hermitian space. Let $\mathcal{F} \subset \mathrm{Hol}(\Omega, M)$. Then the family $\mathcal{F}$ is not normal if and only if there exist sequences $\{p_j\} \subset \Omega$ with $\{p_j\} \to p_0 \in \Omega$, $\{f_j\} \subset \mathcal{F}$, $\{\rho_j\} \subset \mathbb{R}$ with $\rho_j > 0$ and $\{\rho_j\} \searrow 0$ such that the sequence*

$$g_j(z) := f_j(p_j + \rho_j z), z \in \mathbb{C}$$

converges compactly on $\mathbb{C}$ to a nonconstant holomorphic map $g\colon \mathbb{C} \to M$.

Lemma 9 (see [4]). *Let δ be a positive number and let $q \geq N+1$ be an integer. Then there exists a positive constant $M(\delta, q, N)$ such that for any q hypersurfaces $H_1, \ldots, H_q (q \geq N+1)$ in $\mathbb{P}^N(\mathbb{C})$ with*

$$D(H_1, \ldots, H_q) > \delta\,,$$

we have $\max\{\deg H_1, \ldots, \deg H_q\} \leq M(\delta, q, N)$.

Let f be a holomorphic mapping from a domain $\Omega \subset \mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$ and let H be a hypersurface in $\mathbb{P}^n(\mathbb{C})$. We define the divisor $\nu_{f,H}$ as follows: for each point $p \in \Omega$, take an open neighborhood U of p in Ω such that f have a reduced representation $\tilde{f} = (f_0, \ldots, f_N)$ on U, and define $\nu_{f,H}|_U$ to be the zero divisor of the function $H(\tilde{f})$ on U.

We have the following proposition due to Fujimoto.

Lemma 10 ([5, Proposition 3.5]). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic mappings from $\Delta = \{z\ ;\ |z| < r\}$ into $\mathbb{P}^N(\mathbb{C})$. Suppose that f_n meromorphically converges on $\Delta^* = \{|z| < r\} - \{0\}$ to a holomorphic mapping g. Let H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$. Assume that $g(\Delta^*) \not\subset H$. If the sequence $\{\nu_{f_n,H}\}_{n=1}^{\infty}$ converges to a divisor then $\{f_n\}_{n=1}^{\infty}$ meromorphic converges to a holomorphic mapping g^* on Δ such that $g^*|_{\Delta^*} = g$.*

3. Proof of Main theorem

In order to prove the Main theorem, we need the following lemmas.

Lemma 11. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic mappings of a domain Ω in $\mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$. If $\{f_n\}_{n=1}^{\infty}$ holomorphically converges to a holomorphic mapping g of Ω into $\mathbb{P}^N(\mathbb{C})$ then $\{f_n\}_{n=1}^{\infty}$ meromorphically converges to g on Ω.*

Proof. We fix a homogeneous coordinates $(\omega_0 : \cdots : \omega_N)$ on $\mathbb{P}^N(\mathbb{C})$. Suppose that $\{f_n\}_{n=1}^{\infty}$ holomorphically converges to a holomorphic mapping g on Ω.

Let p be an arbitrary point of Ω. Then, we may choose a neighborhood $\Delta(p,r) := \{z\ ;\ |z-p|<r\} \subset \Omega$, on which g has a reduced representation $(g_0 : \cdots : g_N)$, where $r>0$. Since g is holomorphic, there exists an index $i_0 \in \{0,1,\dots,N\}$, for instance $i_0 = 0$, such that $g_0(p) \neq 0$. Thus $g(p) \notin H$, where H is hyperplane defined by $H := \{w_0 = 0\}$. We choose an open neighborhood W of $g(p)$ in $\mathbb{P}^N(\mathbb{C})$ such that $\overline{W} \subset \mathbb{P}^N(\mathbb{C})\backslash H$. Since g is holomorphic, by choosing r small enough, we may assume that $g(\Delta(p,r)) \subset W$. Then, g has a reduced representation $\tilde{g} = (1, \frac{g_1}{g_0}, \dots, \frac{g_N}{g_0})$ on $\Delta(p,r)$.

We now note that, the functions $\frac{\omega_i}{\omega_0}$, $1 \leq i \leq N$ are continuous on $\mathbb{P}^N(\mathbb{C}) \setminus H$. Hence, there exists

$$M = \sup_{\omega\in W}\Big(1 + |\frac{\omega_1}{\omega_0}|^2 + \cdots + |\frac{\omega_N}{\omega_0}|^2\Big),$$

where $\omega = (\omega_0 : \cdots : \omega_N)$.

Since $\{f_n\}_{n=1}^{\infty}$ converges compactly to g on Ω, by reducing r if necessary we may assume that $f_n(\Delta(p,r)) \subset W$, and hence f_n has a reduce representation $\tilde{f}_n = (1, f_{n1}, \dots, f_{nN})$ on $\Delta(p,r)$. Then for every $\epsilon > 0$, there exists a positive integer n_0 such that for all $n > n_0$ we have

$$\max_{z\in\overline{\Delta}(p,r_1)} \frac{\sum_{k,l=0}^{N} |f_{nk}(z)g_l(z) - f_{nl}(z)g_k(z)|}{\| \tilde{f}_n(z) \| \cdot \| g(z) \|} < \frac{\epsilon}{M}\,, \tag{12}$$

where $f_{n0} = g_0 = 1$. By the definition of M, we have $\| \tilde{f}_n(z) \| < M$ and $\| g(z) \| < M$ for all $z \in \Delta(p,r_1)$. Therefore, (12) implies that

$$\max_{z\in\overline{\Delta}(p,r_1)} \sum_{k,l=0}^{N} |f_{nk}(z)g_0(z) - f_{n0}(z)g_k(z)| < \epsilon,\ \forall n > n_0\,.$$

This yields that

$$\max_{z\in\overline{\Delta}(p,r_1)} |f_{nk} - g_k(z)| < \epsilon\,, \quad 1 \leq k \leq N,\ \forall n > n_0.$$

This inequality shows that the sequence of holomorphic functions $\{f_{nk}\}$ converges uniformly to g_k on $\Delta(p,r_1)$ for each $1 \leq k \leq N$.

Hence, by the definition, we have that the sequence $\{f_n\}_{n=1}^{\infty}$ meromorphically converges to g on Ω. The lemma is proved. □

Lemma 13. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic mappings from $\Delta = \Delta(0,r)$ into $\mathbb{P}^N(\mathbb{C})$. Suppose that f_n meromorphically converges on $\Delta^* = \Delta^*(0,r)$ to a holomorphic mapping g. Let H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$. Assume that $g(\Delta^*) \not\subset H$. If $f_n(\Delta) \cap H = \emptyset$ $(\forall n \geq 1)$ then $\{f_n\}_{n=1}^{\infty}$ holomorphically converges to a holomorphic mapping g^* on Δ such that $g^*|_{\Delta^*} = g$.*

Proof. By the assumption of the lemma, from Lemma 10, we have $\{f_n\}_{n=1}^{\infty}$ meromorphically converges on Δ to a holomorphic mapping g^* with $g^*|_{\Delta^*} = g$. In order to prove that $\{f_n\}_{n=1}^{\infty}$ holomorphically converges to g^* on Δ, it suffices for us to prove that $\{f_n\}_{n=1}^{\infty}$ holomorphically converges to g^* on a neighbourhood of every point $p \in \Delta$.

For an arbitrary point $p \in \Delta$, there exist an open neighborhood U of p in Δ, reduced representations $\tilde{f}_n = (f_{n0}, \ldots, f_{nN})$ of f_n $(n \geq 1)$ and a representation $\tilde{g}^* = (g_0^*, \ldots, g_n^*)$ of g^* on U such that $\{f_{nk}\}_{n=1}^{\infty}$ converges compactly to g_k^* on U. Then the sequence of functions $\{H(\tilde{f}_n)\}$ converges compactly to $H(\tilde{g}^*)$ on U. Thus, by the Hurwitz's theorem, one of the following two assertions holds:

(1) $H(\tilde{g}^*) \equiv 0$ on U, i.e., $g^*(U) \subset H$, and hence it implies that $g(\Delta^*) \subset H$,

(2) $H(\tilde{g}^*) \neq 0$ on U, i.e., $g^*(U) \cap H = \emptyset$.

By the assumption of the lemma, the first assertion does not hold. Therefore, $g^*(U) \cap H = \emptyset$. In particular $\tilde{g}^*$ is a reduced representation of g^* on U. We take a relative compact open subset W of U with $p \in W$ and put

$$M = \inf_{z \in W} H\big(\tilde{g}^*(z)\big) > 0\,.$$

For each $\epsilon > 0$, there exists a positive integer n_0 such that for all $n > n_0$ and $0 \leq k \leq N$ we have $|f_{nk} - g_k^*| \leq \frac{\epsilon M}{2(N+1)^2||H||}$ on W. Therefore, we have

$$\begin{aligned}\sum_{k,l=0}^{N} \frac{|f_{nk}g_l^* - f_{nl}g_k^*|}{\|\tilde{f}_n\| \cdot \|\tilde{g}^*\|} &\leq \sum_{k,l=0}^{N} \frac{|f_{nk}| \cdot |g_l^* - f_{nl}| + |f_{nl}| \cdot |f_{nk} - g_k^*|}{\|\tilde{f}_n\| \cdot \|\tilde{g}^*\|} \\ &\leq \sum_{k,l=0}^{N} \frac{\frac{\epsilon M}{2(N+1)^2\|H\|}(|f_{nk}| + |f_{nl}|)}{\|\tilde{f}_n\| \cdot \|\tilde{g}^*\|} \leq \epsilon\end{aligned}$$

on W, where the last inequality follows from that $|f_{nk}| < \|\tilde{f}\|$, $|f_{nl}| \leq \|\tilde{f}\|$ and $\|\tilde{g}^*\| \geq \frac{|H(\tilde{g}^*)|}{\|H\|} \geq \frac{M}{\|H\|}$. Hence, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to g^* on W as holomorphic mappings.

Therefore $\{f_n\}_{n=1}^{\infty}$ converges holomorphically to g^* on Δ with $g^*|_{\Delta^*} = g$. The lemma is proved. □

Lemma 14. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic mappings from $\Delta = \{z\ ;\ |z| < r\}$ into $\mathbb{P}^N(\mathbb{C})$. Suppose that f_n meromorphically converges on $\Delta^* = \{z\ ;\ 0 < |z| < r\}$ to a holomorphic mapping g. Let $\{H_n\}$ be a sequence of hypersurfaces and H be a hypersurface in $\mathbb{P}^N(\mathbb{C})$ of the same degree Ω. Assume that $g(\Delta^*) \not\subset H$ and $\{H_n\}$ converges to H when we regard H_n and H as two points in $\mathbb{P}^N(\mathbb{C})^*$. Then the following assertions hold:*

(a) *If $f_n(\Delta) \cap H_n = \emptyset$ $\forall n \geq 1$ then $\{f_n\}_{n=1}^{\infty}$ holomorphically converges to a holomorphic mapping g^* on Δ such that $g^*|_{\Delta^*} = g$.*

(b) *If $\{\nu_{f_n,H_n}\}_{n=1}^{\infty}$ converges to a divisor ν on Δ then $\{f_n\}_{n=1}^{\infty}$ meromorphically converges to a holomorphic mapping g^* on Δ such that $g^*|_{\Delta^*} = g$.*

Proof. (1) Assume that, f_n has a reduced representation $\tilde{f} = (f_{n0}, \ldots, f_{nN})$ on Δ for each $n = 1, 2, \ldots$, and g has a reduced representation $\tilde{g} = (g_0, \ldots, g_N)$ on Δ^*. We define F_n $(n \geq 1)$ (resp. G) the holomorphic mapping from Δ (resp. Δ^*) into $\mathbb{P}^{N+1}(\mathbb{C})$ having the reduced representation $\tilde{F}_n = (f_{n0}^d, \ldots, f_{nN}^d, H_n(\tilde{f}_n))$ (resp. $\tilde{G} = (g_0^d, \ldots, g_N^d, H(\tilde{g})))$ in a homogeneous coordinates system $(W_0 : \cdots : W_{N+1})$ of $\mathbb{P}^{N+1}(\mathbb{C})$. We easily see that $\{F_n\}_{n=1}^{\infty}$ meromorphically converges to G. Moreover, if we denote by P the hyperplane of $\mathbb{P}^{N+1}(\mathbb{C})$ defined by the linear form

$$W_{N+1} = 0\,,$$

then $F_n(\Delta) \cap P = \emptyset$ $(n \geq 1)$ and $G(\Delta^*) \cap P = \emptyset$. Thus, by Lemma 13, G has a holomorphic extension G^* on Δ and $\{F_n\}_{n=1}^{\infty}$ holomorphic conveges to G^* on Δ. We denote by σ the holomorphic mapping from $\mathbb{P}^N(\mathbb{C})$ into $\mathbb{P}^{N+1}(\mathbb{C})$ defined by

$$\sigma(\omega_0 : \cdots : \omega_N) = \big(\omega_0^d : \cdots : \omega_N^d : H(\omega_0, \ldots, \omega_N)\big)\,.$$

Since $\mathbb{P}^N(\mathbb{C})$ is compact and σ is continuous, $\sigma(\mathbb{P}^N(\mathbb{C}))$ is a close subset of $\mathbb{P}^{N+1}(\mathbb{C})$. Obvious, $g^*(\Delta^*) = G(\Delta^*) \subset \sigma(\mathbb{P}^n(\mathbb{C}))$. Thus $G^*(\Delta) \subset \sigma(\mathbb{P}^n(\mathbb{C}))$.

By the definition, it is clear that $\sigma(\mathbb{P}^N(\mathbb{C})) \cap \bigcap_{i=0}^{N}\{W_i = 0\} = \emptyset$. Hence, there exists a hyperplane, for instance it is $\{W_0 = 0\}$, such that $G^*(0) \notin \{W_0 = 0\}$. Hence there is a disk $\Delta(0, r)$ $(0 < r < 1)$ such that $G^*(\Delta(0, r))$ is contained in a relative conpact open subset of $\mathbb{P}^{N+1}(\mathbb{C}) \setminus \{W_0 = 0\}$. Since $\{F_n\}_{n=1}^{\infty}$ holomorphically conveges to G^*, there exist a positive number $r_1 < r$ and a positive integer n_0 such that $F_n(\Delta(0, r_1)) \subset \mathbb{P}^{N+1}(\mathbb{C}) \setminus \{W_0 = 0\}$ for all $n \geq n_0$. This yields that $f_n(\Delta(0, r_1)) \cap \{\omega_0 = 0\} = \emptyset$ for all $n > n_0$.

Also, since $G^*(0) \notin \{W_0 = 0\}$, $G(\Delta^*(0, r_1)) \not\subset \{W_0 = 0\}$, and hence $g(\Delta^*(0, r_1)) \not\subset \{\omega_0 = 0\}$. Then, applying Lemma 10(a) for the sequence $\{f_n|_{\Delta(0,r_1)}\}_{n=n_0}^{\infty}$, the mapping $g|_{\Delta^*(0,r_1)}$ and the hyperplane $\{\omega_0 = 0\}$ on $\Delta(0, r_1)$, we have that $\{f_n|_{\Delta(0,r_1)}\}_{n=n_0}^{\infty}$ holomorphically converges to a holomorphic mapping g^* on $\Delta(0, r_1)$ with $g^*|_{\Delta^*(0,r_1)} = g|_{\Delta^*(0,r_1)}$. Extending g^* over Δ by setting $g^*(z) = g(z)$ for all $z \in \Delta \setminus \Delta(0, r_1)$, we obtain a holomorphic mapping g^* from Δ into $\mathbb{P}^N(\mathbb{C})$. Obviously, $\{f_n\}_{n=1}^{\infty}$ holomorphically converges to g^* on Δ and $g^*|_{\Delta^*} = g$. The assertion (a) is proved.

(b) We will use the same notations and also repeat the similar argument as above. By Lemma 10, G has a holomorphic extension G^* on Δ and $\{F_n\}_{n=1}^{\infty}$ meromorphically conveges to G^* on Δ. We see that, there exists an index i_0, for instance $i_0 = 0$, such that $G^*(\Delta) \subset \{W_0 = 0\}$. Denote by P the hyperplane $\{W_0 = 0\}$. Then the sequence $\{\nu_{F_n,P}\}_{n=1}^{\infty}$ are compactly bounded. This implies that the sequence $\{\nu_{f_n,P'}\}_{n=1}^{\infty}$ are compactly bounded, where P' is the hyperplane $\{\omega_0 = 0\}$. From Lemma 10(b), $\{f_n\}_{n=1}^{\infty}$ meromorphically converges to g^* on Δ and $g^*|_{\Delta^*} = g$. The assertion (b) is proved. □

Proof of Main theorem. (a) Take an arbitrary sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$. Then, there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ (again denoted by $\{f_n\}_{n=1}^{\infty}$) such that

$$\lim_{n\to\infty} f_n^{-1}(H_i(f_n)) = E_i\,, \quad \forall i \in \{1, 2, \ldots, q\}\,,$$

where all E_i are discrete subset of Ω. Set $E = \bigcup_{i=1}^{q} E_i$. Since $\inf\{D(H_1(f), \ldots, H_q(f)) : f \in \mathcal{F}\} > 0$, the degrees of $H_i(f)$ $(1 \leq i \leq q)$ for $f \in \mathcal{F}$ are bounded from above. Then there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ (again denoted by $\{f_n\}_{n=1}^{\infty}$) such that

$$\deg H_i(f_n) = d_i > 0 \quad \text{and} \quad \lim H_i(f_n) = H_i \ \forall 1 \leq i \leq q\,.$$

We also note that $D(H_1, \ldots, H_q) \geq \inf\{D(H_1(f), \ldots, H_q(f)) : f \in \mathcal{F}\} > 0$, i.e., $\{H_1, \ldots, H_q\}$ are in general position.

Firstly, we will prove that we may extract from $\{f_n\}$ a subsequence which converges compactly on $\Omega \setminus E$. Indeed, for any $z_0 \in \Omega \setminus E$, there exist a relatively compact neighborhoood U_{z_0} of z_0 in $\Omega \setminus E$ and $n_0 \in \mathbb{N}^*$ such that for any $n \geq n_0$ we have

$$f_n^{-1}(H_i(f_n)) \cap U_{z_0} = \emptyset \quad \forall i \in \{1, 2, \dots, q\}.$$

Then $\{f_n|_{U_{z_0}}\}_{n=n_0}^\infty \subset \mathrm{Hol}(U_{z_0}, \mathbb{P}^N(\mathbb{C}))$. Suppose that the family $\{f_n|_{U_{z_0}}\}_{n=n_0}^\infty$ is not holomorphically normal family. By Lemma 8, there exist a subsequence of $\{f_n|_{U_{z_0}}\}_{n=n_0}^\infty$ (denoted again by $\{f_n|_{U_{z_0}}\}_{n=n_0}^\infty$), $p_0 \in U_{z_0}$, $\{p_n\} \subset U_{z_0}$ with $p_n \to p_0$, $\{\rho_n\} \subset (0, \infty)$ with $\rho_n \searrow 0$ such that the sequence of holomorphic maps

$$g_n(z) = f_n(p_n + \rho_n z) : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$$

converges compactly on $\mathbb{C}$ to a nonconstant holomorphic map $g\colon \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$. Hence $\{g_n\}_{n\geq n_0}$ meromorphically converges to g on $\mathbb{C}$.

Then for each $p \in \mathbb{C}$, there exists a small enough open neighborhood U of p in $\mathbb{C}$, on which g_n has a reduced representation $\tilde{g}_n = (g_{n0}, \dots, g_{nN})$ for all n large enough satisfying that $\{g_{nk}\}$ converges compactly to a holomorphic function g_k and $\tilde{g} = (g_0, \dots, g_N)$ is a representation of g on U. This implies that the sequence $\{(H_k(f_n))(\tilde{g}_n)\}$ converges compactly to $H_k(\tilde{g})$ on U. Thus, by the Hurwitz's theorem, one of the following two assertions holds:

1. $H_k(\tilde{g}) \equiv 0$ on U, i.e., $g(U) \subset H_k$, and hence $g(\mathbb{C}) \subset H_k$.
2. $H_k(\tilde{g})$ is nowhere vanishing on U, i.e., $g(U) \cap H_k = \emptyset$.

Then, we must have that either $g(\mathbb{C}) \subset H_k$ or $g(\mathbb{C}) \cap H_k = \emptyset$ for all $1 \leq k \leq q$. Hence, there exists a subset $I \subset \{1, 2, \dots, q\}$ such that $g(\mathbb{C}) \subset (\bigcap_{i\in I} H_i) \setminus (\bigcup_{i\notin I} H_i)$. By [9, Corollary 1.4 (ii)], all irreducible component of $(\bigcap_{i\in I} H_i) \setminus (\bigcup_{i\notin I} H_i)$ is hyperbolic, and hence g is constant. This is contraction.

Thus $\{f_n\}_{n=n_0}^\infty$ is holomorphically normal on U_{z_0}. Therefore, by the usual diagonal argument, we can find a subsequence (again denoted by $\{f_n\}_{n=1}^\infty$) which converges compactly on $\Omega \setminus E$ to a holomorphic mapping f.

Now, we will show that f is holomorphically extendable over E. For each $z \notin E$, we see that there exist an open neighborhood U of z in $\Omega \setminus E$ and a positive integer n_0 such that $f_n(U) \cap H_i(f_n) = \emptyset$ $(1 \leq i \leq q)$ for all $n \geq n_0$. By using the same argument as above, we will have that either $f(U) \subset H_i$ or $f(U) \cap H_i = \emptyset$ for all $1 \leq i \leq q$. Hence, there exists a subset $I' \subset \{1, 2, \dots, q\}$ such that $f(\Omega \setminus E) \subset (\bigcap_{i\in I'} H_i) \setminus (\bigcup_{i\notin I'} H_i)$ and $\sharp I' = l \leq N$. By again [9, Corollary 1.4 (ii)], all irreducible component of $(\bigcap_{i\in I'} H_i) \setminus (\bigcup_{i\notin I'} H_i)$ is hyperbolic. Then f is holomorphically extendable over E, by [8, Corollary 1.2.3].

This conclusion yields that $\mathcal{F}$ is a weak normal family. The assertion (a) is proved.

(b) In order to prove the assertion (b), it suffices for us to prove that from the last sequence $\{f_n\}_{n=1}^\infty$ obtained in the part (a) we may extract a subsequence which meromorphically converges to f on an open neighborhood of each point z for all $z \in E$ except for at most p_{N+1} points. We see that there is an index $i_0 \in \{1, \dots, N+1\}$ such that $f(\Omega) \not\subset H_{i_0}$. Now, we set

$$S_1 = \{z \in E; \forall \epsilon > 0, \forall M \in \mathbb{Z}^+, \exists n', \forall n \geq n', \exists z' \in \Delta(z, \epsilon), \nu_{f_n, H_i(f_n)}(z') > M\},$$

and $S_2 = E \setminus S_1$. We will show that $\sharp S_1 \le p_{i_0}$. Indeed, suppose contrarily that there exist $z_1, \dots, z_{p_{i_0}+1}$ in S_1. Take a compact neighborhood K of the set $\{z_1, \dots, z_{p_{i_0}+1}\}$ in Ω. Then there exists a integer constant $c_K > 0$ such that

$$\sup_{f \in \mathcal{F}} \sharp\{z \in K \ ; \ \nu_{f, H_{i_0}(f)}(z) \ge c_k\} \le p_{i_0} \, . \tag{15}$$

Take a positive number $\epsilon > 0$ such that $\Delta(z_k, \epsilon) \subset K$. Then for all z_k $(1 \le k \le p_{i_0} + 1)$, there exist an n large enough, $z'_k \in \Delta(z_k, \epsilon) \subset K$ such that $\nu_{f_n, H_{i_0}(f)}(z'_k) \ge c_K$. Hence

$$\sharp\{z \in K \ ; \ \nu_{f_n, H_{i_0}(f_n)}(z) > c_k\} \ge p_{i_0} + 1 \, .$$

This contradicts (15). Therefore, $\sharp S_1 \le p_{i_0}$.

We now only remains prove that: for each point $p \in S_2$, we may extract a subsequence of $\{f_n\}_{n=1}^{\infty}$ which converges compactly to f on an open neighbourhood of p.

Take a fixed point $p \in S_2$. Then, there exist $\epsilon > 0$, $M \in \mathbb{Z}^+$, and a subsequence of $\{f_n\}_{n=1}^{\infty}$ (we denote again by $\{f_n\}_{n=1}^{\infty}$) satisfying

(1) $\Delta(p, \epsilon) \cap E = \{p\}$,

(2) $\nu_{f_n, H_{i_0}(f_n)}(z) \le M, \ \forall z \in \Delta(p, \epsilon)$.

From (2) and the assumption of the assertion (a), the sequence of divisors $\{\nu_{f_n, H_{i_0}(f_n)}\}$ is bounded conpactly on $\Delta(p, \epsilon)$. Then there is a subsequence of $\{f_n\}_{n=1}^{\infty}$ (we denote again by $\{f_n\}_{n=1}^{\infty}$) converging to a divisor on $\Delta(p, \epsilon)$. Then by Lemma 10, $\{f_n\}_{n=1}^{\infty}$ meromorphically converges to f on $\Delta(p, \epsilon)$.

Then by using the diagonal argument, we get a subsequence of $\{f_n\}_{n=1}^{\infty}$ meromorphically converges to f on Ω except for a set of at most $p_{i_0} \le p_{N+1}$ points.

Therefore $\mathcal{F}$ is meromorphically quasinormal of order at most p_{N+1}. The assertion (b) is proved.

(c) Similarly as above, we will show that the last sequence $\{f_n\}_{n=1}^{\infty}$ obtained in the part (a) has a subsequence which converges compactly to f on an open neighborhood of each point z for all $z \in E$ except for at most p_{N+1} points. Take an index $i_0 \in \{1, \dots, N+1\}$ such that $f(\Omega) \not\subset H_{i_0}$. Since $\sharp\{z \in \Omega \ ; \ f_n(z) \in H_{i_0}\} \le p_{i_0}$ for all $n \ge 1$, $\sharp E_{i_0} \le p_{i_0} \le p_{N+1}$.

It suffices for us to prove that: for an arbitrary point $p \in E \setminus E_{i_0}$, we may extract a subsequence of $\{f_n\}_{n=1}^{\infty}$ which converges compactly to f on an open neighborhood p. Indeed, there exist $\epsilon > 0$ and a subsequence of $\{f_n\}_{n=1}^{\infty}$ (we denote again by $\{f_n\}_{n=1}^{\infty}$) satisfying

(1) $\Delta(p, \epsilon) \cap E = \{p\}$,

(2) $f_n(\Delta(p, \epsilon)) \cap H_{i_0}(f_n) = \emptyset$.

From (2) and by Lemma 13, $\{f_n\}_{n=1}^{\infty}$ holomorphically converges to f on $\Delta(p, \epsilon)$. Hence by using the diagonal argument, we get a subsequence of $\{f_n\}_{n=1}^{\infty}$ holomorphically converges to f on $\Omega \setminus E_{i_0}$, where $\sharp E_{i_0} \le p_{N+1}$.

Therefore $\mathcal{F}$ is holomorphically quasinormal of order at most p_{N+1}. The assertion (c) is proved. $\square$

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