ARCHIVUM MATHEMATICUM (BRNO)
Tomus 54 (2018), 165–176

BEST PROXIMITY POINT FOR PROXIMAL BERINDE NONEXPANSIVE MAPPINGS ON STARSHAPED SETS

Nuttawut Bunlue and Suthep Suantai

Abstract. In this paper, we introduce the new concept of proximal mapping, namely proximal weak contractions and proximal Berinde nonexpansive mappings. We prove the existence of best proximity points for proximal weak contractions in metric spaces, and for proximal Berinde nonexpansive mappings on starshape sets in Banach spaces. Examples supporting our main results are also given. Our main results extend and generalize some of well-known best proximity point theorems of proximal nonexpansive mappings in the literatures.

1. Introduction

Fixed point theory plays an important role in solving nonlinear equations arising in different areas such as difference and differential equations, discrete and continuous dynamic systems, variational analysis, physics, engineering and economics.

These problems can be modeled as fixed point equation of the form $x = Tx$ where $T\colon A \to X$ is a nonlinear mapping from a subset A of X. In the case that $A \cap T(A) = \emptyset$, the fixed point equation $x = Tx$ has no solution because $d(x, Tx) > 0$ for all $x \in A$. Under this circumstance, it is of interest to determine an approximate solution x such that the distance between x and Tx is minimum. For more precisely, suppose $T\colon A \to B$ where A, B are subsets of a metric space (X, d). It noted that $d(x, Tx) \geq D(A, B)$, where $D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. It natural to ask the question of finding x such that $d(x, Tx) = D(A, B)$, such point x is known as a best proximity point of T. It is clear that if T is a self-mapping, a best proximity point is a fixed point, that is, $x = Tx$.

Existence of best proximity point of nonself-mappings have been studied by many authors, see [2, 5, 6, 13, 15, 17, 18, 19, 21] and [22]. Best proximity point theorems can be applied to study equilibrium point in economics, see [10]–[12], so this topic attracts attentions of many mathematicians.

Basha [1] introduced a new concept of proximal contraction which can be reduced to a contraction in the case of self-mappings.

2010 *Mathematics Subject Classification*: primary 47H10; secondary 47H09.

Key words and phrases: best proximity point, proximal weak contraction mapping, proximal Berinde nonexpansive mapping, starshaped set.

Received January 25, 2018. Editor A. Pultr.

DOI: 10.5817/AM2018-3-165

Definition 1 ([1])**.** Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T\colon A \to B$ is said to be a proximal contraction if there exists constant number $\alpha \in (0, 1)$ such that

$$\left.\begin{array}{l} d(u_1, Tx_1) = D(A, B) \\ d(u_2, Tx_2) = D(A, B) \end{array}\right\} \Longrightarrow d(u_1, u_2) \leq \alpha d(x_1, x_2)$$

for all x_1, x_2, u_1, $u_2 \in A$.

In 2013, Gabeleh [7] introduced a new concept of proximal nonexpansive mappings and proved existence of best proximity point of such mapping when (A, B) is a pair of nonempty closed convex subsets of X and A is a compact set.

Definition 2 ([7])**.** Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T\colon A \to B$ is said to be a proximal nonexpansive if

$$\left.\begin{array}{l} d(u_1, Tx_1) = D(A, B) \\ d(u_2, Tx_2) = D(A, B) \end{array}\right\} \Longrightarrow d(u_1, u_2) \leq d(x_1, x_2)$$

for all x_1, x_2, u_1, $u_2 \in A$.

Motivated by weak contraction of Berinde [3] and Suzuki [20], in 2014, Gabeleh [9] introduced a new classes of proximal contractions which is called Berinde weak proximal contraction.

Definition 3 ([9])**.** Let (A, B) be a pair of nonempty closed subsets of a metric space (X, d). A mapping $T\colon A \to B$ is said to be a Berinde weak proximal contraction if there exist $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$ such that for all $x, y, u, v \in A$ with $d(u, Tx) = D(A, B) = d(v, Ty)$, we have

$$\frac{1}{1+\alpha+\beta} d^*(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(u, v) \leq \alpha d(x, y) + \beta d^*(Tx, y)\,,$$

where $d^*(x, Tx) = d(x, Tx) - D(A, B)$.

Chen [4] proved an interesting existence theorem of proximity points for proximal nonexpansive mappings under starshape sets A and B.

Theorem 1.1 ([4])**.** *Let (A, B) be a pair of nonempty, closed subsets of a Banach space X such that A is a p-starshaped set, B is a $q-$starshaped set, and $\|p - q\| = D(A, B)$. Suppose A is compact, (B, A) is a semi-sharp proximinal pair. Assume that $T\colon A \to B$ satisfies the following conditions:*

(1) *T is a proximal nonexpansive,*

(2) *$T(A_0) \subseteq B_0$.*

Then there exists an elements x^* in A_0 such that

$$\|x^* - Tx^*\| = D(A, B)\,.$$

Motivated by above results, we aim to introduce new concept of generalized proximal contraction and proximal nonexpansive mapping, called proximal weak contraction and proximal Berinde nonexpansive, respectively, and prove existence of best proximity point of such mappings under certain conditions. We also give an example supporting our main results.

2. Preliminaries

Let (A,B) be a pair of nonempty subsets of metric space (X,d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x,y) = D(A,B) \quad \text{for some} \quad y \in B\},$$

$$B_0 = \{y \in B : d(x,y) = D(A,B) \quad \text{for some} \quad x \in A\},$$

where

$$D(A,B) = \inf\{d(x,y) : x \in A, y \in B\}.$$

A nonempty subset A of a linear space X is called a p-starshape set if there exist a point p in A such that $\alpha p + (1-\alpha)x \in A$, for all $x \in A$, $\alpha \in [0,1]$, and p is called a center of A. It is easy to see that each convex set C is a p-starshaped set for each $p \in C$.

Notice that in a normed space $(X, \|\cdot\|)$, if both of A and B are closed and A_0 is nonempty, then A_0 is a closed set. Consider on starshape set, if A is a p-starshape set, B is a q-starshaped set and $\|p-q\| = D(A,B)$, implies that A_0 is a p-starshape set and B_0 is a q-starshaped set.

Definition 4 ([14])**.** Let (A,B) be a pair of nonempty subsets of a metric space (X,d). The pair (A,B) is said to be a semi-sharp proximinal pair if for each x in A (respectively, in B) there exists at most one x^* in B (respectively, in A) such that $d(x,x^*) = D(A,B)$.

Definition 5 ([8])**.** Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A \neq \emptyset$. Then the pair (A,B) is said to have the weak P-property if for all x_1, $x_2 \in A_0$ and y_1, $y_2 \in B_0$,

$$\left.\begin{array}{l} d(x_1,y_1) = D(A,B) \\ d(x_2,y_2) = D(A,B) \end{array}\right\} \Longrightarrow d(x_1,x_2) \leq d(y_1,y_2).$$

In Definition 5, if $d(x_1,x_2) = d(y_1,y_2)$, we said (A,B) have P-property [16]. It is clear that the weak P-property is weaker than the P-property and (A,B) has the P-property if and only if both (A,B) and (B,A) have the weak P-property. Moreover, if a pair (A,B) has the weak P-property then (B,A) must be a semi-sharp proximinal pair. Obviously a semi-sharp proximinal pair (A,B) is not necessarily to have the weak P-property.

3. Main results

3.1. **Proximity point for the proximal weak contraction.** We begin this section by giving definition and proving a theorem on the existence of best proximity points for proximal weak contraction in metric spaces.

Definition 6. Let (A,B) be a pair of nonempty subset of a metric space (X,d). A mapping $T\colon A \to B$ is said to be *proximal weak contraction* if there exist $\alpha \in (0,1)$ and $L \geq 0$ such that

$$\left.\begin{array}{l} d(u_1,Tx_1) = D(A,B) \\ d(u_2,Tx_2) = D(A,B) \end{array}\right\} \Longrightarrow d(u_1,u_2) \leq \alpha d(x_1,x_2) + Ld(x_2,u_1)$$

for all $x_1, x_2, u_1, u_2 \in A$.

For self-mapping, we see that the proximal weak contraction reduces to the weak contraction mapping introduced by Berinde in [3].

Theorem 3.1. *Let (X, d) be a complete metric space and (A, B) a pair of nonempty subsets of X such that A_0 is nonempty and closed. Suppose that $T\colon A \to B$ is a proximal weak contraction and $T(A_0) \subseteq B_0$. Then*

(1) *there exists $x^* \in A$ such that $d(x^*, Tx^*) = D(A, B)$, and if $1 - \alpha - L > 0$, then x^* is unique,*

(2) *the sequence $\{x_n\}$, defined by x_0, $x_1 \in A_0$ and*

$$d(x_{n+1}, Tx_n) = D(A, B), \quad \text{for all} \quad n \in \mathbb{N},$$

converges to x^*.

Proof. Let $x_0 \in A_0$. Then there exist $x_1 \in A_0$ such that

$$d(x_1, Tx_0) = D(A, B)$$

because $Tx_0 \in T(A_0) \subseteq B_0$. Continuing this process, we get a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = D(A, B)\,, \quad \text{for all} \quad n \in \mathbb{N}\,.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence and its limit is a best proximity point of T. From

$$d(x_n, Tx_{n-1}) = D(A, B) \quad \text{and} \quad d(x_{n+1}, Tx_n) = D(A, B)\,, \quad \text{for all} \quad n \in \mathbb{N}\,,$$

by proximal weak contractiveness of T, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\le \alpha d(x_n, x_{n-1}) + L d(x_n, x_n) \\ &\le \alpha d(x_n, x_{n-1})\,. \end{aligned}$$

Therefore, for each $p \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+p}, x_n) &\le d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + d(x_{n+1}, x_n) \\ &\le \alpha^{n+p-1} d(x_1, x_0) + \alpha^{n+p-2} d(x_1, x_0) + \cdots + \alpha^n d(x_1, x_0) \\ &\le \frac{\alpha^n}{1-\alpha} d(x_1, x_0)\,. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in A. Since X complete and A_0 closed, there exists $x^* \in A_0$ such that $x_n \to x^*$. By the assumption $T(A_0) \subseteq B_0$ again, there exists $u \in A_0$ such that

$$d(u, Tx^*) = D(A, B)\,.$$

Since $d(x_{n+1}, Tx_n) = D(A, B)$ for all $n \in \mathbb{N}$, by proximal weak contractiveness of T, we have

$$d(x_{n+1}, u) \le \alpha d(x_n, x^*) + L d(x^*, x_{n+1})\,.$$

It implies that $d(x_{n+1}, u) \to 0$. Therefore $x_n \to u$ and hence $u = x^*$. That is

$$d(x^*, Tx^*) = D(A, B)\,.$$

Finally, we show that if $1-\alpha-L>0$ then the best proximity point of T is unique. Suppose there exists $x^{**}\in A_0$ such that

$$d(x^{**},Tx^{**})=D(A,B)\,.$$

Since T is proximal weak contraction, we get

$$d(x^*,x^{**})\leq \alpha d(x^*,x^{**})+Ld(x^*,x^{**})\,,$$

which implies

$$(1-\alpha-L)d(x^*,x^{**})\leq 0\,.$$

Hence x^* and x^{**} are same point. □

An immediate consequence of Theorem 3.1 is the following.

Corollary 3.2 ([4]). *Let (X,d) be a complete metric space and (A,B) a pair of nonempty closed subsets of X such that $A_0\neq\emptyset$. Suppose that $T\colon A\to B$ satisfies the following conditions:*

(1) *T is a proximal contraction,*

(2) *$T(A_0)\subseteq B_0$.*

Then there exists a unique $x^*\in A$ such that $d(x^*,Tx^*)=D(A,B)$.

Example 3.3. Let $X=\mathbb{R}^2$ with the usual metric,
$A_1=\{(x,0):x\in[0,1)\}$,
$A_2=\{(1,y):y\in[-1,0]\}$,
$B_1=\{(x,y):x^2+y^2=1,x\in[-1,0]\}$,
$B_2=\{(x,1):x\in(0,1]\}$,
$A=A_1\cup A_2$,
$B=B_1\cup B_2$,
and let $T\colon A\to B$ be given by

$$T(x,y)=\begin{cases}(\frac{x^2}{2},1)\,, & \text{if}\quad (x,y)\in A_1\,,\\ (\frac{y}{2},-\sqrt{1-\frac{y^2}{4}})\,, & \text{if}\quad (x,y)\in A_2\,.\end{cases}$$

To show that T is proximal weak contraction, let $x_1=(x_1',y_1')$, $x_2=(x_2',y_2')\in A$ and u_1, $u_2\in A$ be such that $d(u_1,Tx_1)=d(u_2,Tx_2)=D(A,B)$. We will consider the following 5 cases.

Case 1. x_1, $x_2\in A_1$. Then $u_1=\big(\frac{x_1^{'2}}{2},0\big)$, $u_2=\big(\frac{x_2^{'2}}{2},0\big)$. This implies

$$d(u_1,u_2)=d\Big(\Big(\frac{x_1^{'2}}{2},0\Big)\Big(\frac{x_2^{'2}}{2},0\Big)\Big)\leq\frac{1}{2}d(x_1,x_2)\leq\frac{1}{2}d(x_1,x_2)+Ld(x_2,u_1)$$

for any $L\geq 0$.

Case 2. x_1, $x_2\in A_2-\{(1,0)\}$. Then $u_1=u_2=(0,0)$. This implies

$$d(u_1,u_2)\leq\frac{1}{2}d(x_1,x_2)\leq\frac{1}{2}d(x_1,x_2)+Ld(x_2,u_1)$$

for any $L\geq 0$.

Case 3. Let $x_1 \in A_1$, $x_2 \in A_2$. Then $u_1 = (\frac{x_1'^2}{2}, 0)$, $u_2 \in \{(0,0),(1,-1)\}$. If $x_1' \in \left[0, \frac{1}{2}\right]$ and $u_2 = (0,0)$, we have

$$d(u_1,u_2) \le \frac{1}{8} \le \frac{1}{2}d(x_1,x_2) \le \frac{1}{2}d(x_1,x_2) + Ld(x_2,u_1)$$

for any $L \ge 0$.
If $x_1' \in \left[0, \frac{1}{2}\right]$ and $u_2 = (1,-1)$, then $x_2 = (1,0)$. This implies

$$d(u_1,u_2) \le \sqrt{2},\ d(x_1,x_2) \ge \frac{1}{2} \quad \text{and} \quad d(x_2,u_1) \ge \frac{7}{8}\,.$$

Thus

$$d(u_1,u_2) \le \sqrt{2} \le \frac{1}{2}d(x_1,x_2) + Ld(x_2,u_1)$$

where $L \ge \frac{8}{7}\left(\sqrt{2} - \frac{1}{2}\right)$.
If $x_1' \in \left(\frac{1}{2}, 1\right)$ and $u_2 = (0,0)$, we get

$$d(u_1,u_2) \le \frac{1}{2},\ d(x_1,x_2) \ge 0 \quad \text{and} \quad d(x_2,u_1) \ge \frac{1}{2}\,.$$

Thus

$$d(u_1,u_2) \le \frac{1}{2} \le \frac{1}{2}d(x_1,x_2) + Ld(x_2,u_1)$$

where $L \ge 1$.
If $x_1' \in (\frac{1}{2}, 1)$ and $u_2 = (1,-1)$, then $x_2 = (1,0)$. This implies

$$d(u_1,u_2) \le \frac{\sqrt{113}}{8}\,,\ d(x_1,x_2) \ge 0 \quad \text{and} \quad d(x_2,u_1) \ge \frac{1}{2}\,.$$

Thus

$$d(u_1,u_2) \le \frac{\sqrt{113}}{8} \le \frac{1}{2}d(x_1,x_2) + Ld(x_2,u_1)$$

where $L \ge \frac{\sqrt{113}}{4}$.

Case 4. $x_1 \in A_2, x_2 \in A_1$. Then $u_1 \in \{(0,0),(1,-1)\}$, $u_2 = (\frac{x_2'^2}{2}, 0)$. If $x_2' \in \left[0, \frac{1}{2}\right]$ and $u_1 = (0,0)$, then

$$d(u_1,u_2) \le \frac{1}{8} \quad \text{and} \quad d(x_1,x_2) \ge \frac{1}{2}\,.$$

Thus

$$d(u_1,u_2) \le \frac{1}{2}d(x_1,x_2) \le \frac{1}{2}d(x_1,x_2) + Ld(x_2,u_1)$$

for any $L \ge 0$.
If $x_2' \in \left[0, \frac{1}{2}\right]$ and $u_1 = (1,-1)$, then $x_1 = (1,0)$. This implies

$$d(u_1,u_2) \le \sqrt{2}\,,\ d(x_1,x_2) \ge \frac{1}{2} \quad \text{and} \quad d(x_2,u_1) \ge \frac{\sqrt{5}}{2}\,.$$

Thus

$$d(u_1,u_2) \le \sqrt{2} \le \frac{1}{2}d(x_1,x_2) + Ld(x_2,u_1)$$

where $L \geq \frac{2}{\sqrt{5}}(\sqrt{2}-\frac{1}{2})$.
If $x_2' \in \left(\frac{1}{2},1\right)$ and $u_1=(0,0)$, then

$$d(u_1,u_2) \leq \frac{1}{2}\,,\ d(x_1,x_2) \geq 0 \quad \text{and} \quad d(x_2,u_1) \geq \frac{1}{2}\,.$$

Thus

$$d(u_1,u_2) \leq \frac{1}{2} \leq \frac{1}{2}d(x_1,x_2)+Ld(x_2,u_1)$$

where $L \geq 1$.

If $x_2' \in (\frac{1}{2},1)$ and $u_1=(1,-1)$, then $x_2=(1,0)$. This implies

$$d(u_1,u_2) \leq \frac{\sqrt{113}}{8}\,,\ d(x_1,x_2) \geq 0 \quad \text{and} \quad d(x_2,u_1) \geq 1\,.$$

Thus

$$d(u_1,u_2) \leq \frac{\sqrt{113}}{8} \leq \frac{1}{2}d(x_1,x_2)+Ld(x_2,u_1)$$

where $L \geq \frac{\sqrt{113}}{8}$.

Case 5. $x_1=x_2=(1,0)$. Then u_1, $u_2 \in \{(0,0),(1,-1)\}$. Suppose that $u_1 \neq u_2$, we have

$$d(u_1,u_2)=\sqrt{2}\,,\ d(x_1,x_2)=0 \quad \text{and} \quad d(x_2,u_1)=1\,.$$

Therefore

$$d(u_1,u_2) \leq \frac{1}{2}d(x_1,x_2)+Ld(x_2,u_1)$$

where $L \geq \sqrt{2}$.

We can conclude from all of above cases that T is proximal weak contraction with $\alpha=\frac{1}{2}$ and $L=\frac{\sqrt{113}}{4}$. We also note that the point $x=(0,0)$ is a best proximity point of T.

We remark that T is neither a Berinde weak proximal contraction nor a proximal contraction. Let $x=(1,0)$, $y=(1,-1)$. Then $u=(1,-1)$, $v=(0,0)$ are such that $d(u,Tx)=d(v,Ty)=1=D(A,B)$. We obtain

$$d(u,v)=\sqrt{2}\,,\quad d(x,y)=1\,,\quad d^*(y,Tx)=0 \quad \text{and} \quad d^*(x,Tx)=\sqrt{2}-1\,.$$

This implies for each $\alpha \in (0,1)$, $\beta \in (0,\infty)$,

$$\frac{1}{1+\alpha+\beta}d^*(x,Tx) \leq d^*(x,Tx) \leq d(x,y)\,,$$

but

$$d(u,v)=\sqrt{2} \geq \alpha d(x,y)=\alpha d(x,y)+\beta d^*(y,Tx)\,.$$

3.2. **Proximity point for the proximal Berinde nonexpansive.** In this section, we first introduce a new concept of proximal nonexpansive mapping, called proximal Berinde nonexpansive mapping. This concept motivated by weak contraction of Berinde.

Definition 7. Let (A,B) be a pair of nonempty subsets of a metric space (X,d). A mapping $T\colon A\to B$ is said to be *proximal Berinde nonexpansive* if there exist $L\geq 0$ such that

$$\left.\begin{aligned} d(u_1,Tx_1)&=D(A,B)\\ d(u_2,Tx_2)&=D(A,B)\end{aligned}\right\}\Longrightarrow d(u_1,u_2)\leq d(x_1,x_2)+Ld(x_2,u_1)$$

for all x_1, x_2, u_1, $u_2\in A$.

It is obvious that every proximal nonexpansive mappings is proximal Berinde nonexpansive with $L=0$.

Theorem 3.4. *Let X be a Banach space, (A,B) be a pair of nonempty, closed subsets of X such that A is a p-starshaped set, B is a q-starshaped set, and $\|p-q\|=D(A,B)$. Assume A_0 is compact, (B,A) is a semi-sharp proximinal pair. Suppose that $T\colon A\to B$ satisfies the following conditions:*

(1) *T is a proximal Berinde nonexpansive,*

(2) *$T(A_0)\subseteq B_0$.*

Then there exists x^* in A_0 such that

$$\|x^*-Tx^*\|=D(A,B)\,.$$

Proof. For each $n\in\mathbb{N}$, define $T_n\colon A_0\to B_0$ by

$$T_nx=(1-a_n)Tx+a_nq,\ x\in A_0\,,$$

where $\{a_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty}a_n=0$. Since B_0 is a $q-$starshaped set and $T(A_0)\subseteq B_0$, we get $T_n(A_0)\subseteq B_0$.

Next, we will show that T_n is proximal weak contraction for each $n\in\mathbb{N}$. Let x_1, x_2, u_1, $u_2\in A_0$ be such that

$$\|u_1-T_nx_1\|=\|u_2-T_nx_2\|=D(A,B)\,. \tag{1}$$

Since Tx_1, $Tx_2\in B_0$, there exist s_1, $s_2\in A_0$ such that

$$\|s_1-Tx_1\|=\|s_2-Tx_2\|=D(A,B)\,.$$

By definition of T, we have

$$\|s_1-s_2\|\leq\|x_1-x_2\|+L\|x_2-s_1\|, \tag{2}$$

for some $L\geq 0$. Now we set

$$v_1=(1-a_n)s_1+a_np \text{ and } v_2=(1-a_n)s_2+a_np\,.$$

Since A_0 is a p-starshaped set, then $v_1,v_2\in A_0$. We note that

$$\begin{aligned} D(A_0,B_0)&\leq\|v_1-T_nx_1\|\\ &=\|(1-a_n)v_1+a_np-(1-a_n)Tx_1-a_nq\|\\ &\leq(1-a_n)\|s_1-Tx_1\|+a_n\|p-q\|\end{aligned}$$

$$= D(A_0, B_0).$$

Therefore $\|v_1 - T_n x_1\| = D(A_0, B_0)$. Since (B, A) is a semi-sharp proximinal pair and equation (1), this implies $v_1 = u_1$. Using the same method, we get $v_2 = u_2$. By proximal Berinde nonexpansiveness and (2), we have

$$\begin{aligned}
\|u_1 - u_2\| &= \|v_1 - v_2\| \\
&= \|(1 - a_n)(s_1 - s_2)\| \\
&\leq (1 - a_n)\|x_1 - x_2\| + (1 - a_n)L\|x_2 - s_1\|\,.
\end{aligned}$$

Thus for each n, T_n is proximal weak contraction with $k_n = 1 - \alpha_n$ and $L'_n = (1 - a_n)L$. By Theorem 3.1, T_n has a best proximity point $x^*_n \in A_0$ such that

$$\|x^*_n - T_n x^*_n\| = D(A_0, B_0) \tag{3}$$

Since A_0 is compact and $\{x^*_n\}$ is a sequence in A_0, without loss of generality, we assume that, there exist $x^* \in A_0$ such that $x^*_n \to x^*$.

Next, let us show that x^* is a best proximity point of T. Since $Tx^*_n \in B_0$ for any n, there exist $x'_n \in A_0$ such that

$$\|x'_n - Tx^*_n\| = D(A, B)\,. \tag{4}$$

From

$$\begin{aligned}
D(A_0, B_0) &\leq \|(1 - a_n)x'_n + a_n p - T_n x^*_n\| \\
&= \|(1 - a_n)x'_n + a_n p - (1 - a_n)Tx^*_n - a_n q\| \\
&\leq (1 - a_n)\|x'_n - Tx^*_n\| + a_n\|p - q\| \\
&= D(A_0, B_0)\,,
\end{aligned}$$

which implies

$$\|(1 - a_n)x'_n + a_n p - T_n x^*_n\| = D(A, B)\,. \tag{5}$$

Since (B, A) is a semi-sharp proximinal pair, we obtain by (3) and (5) that

$$x^*_n = (1 - a_n)x'_n + a_n p\,,$$

which implies

$$\|x^*_n - x'_n\| = a_n\|x'_n - p\| \to 0 \text{ as } \ n \to \infty \quad (\text{because} \quad a_n \to 0)\,.$$

That is $\lim_{n\to\infty} x'_n = \lim_{n\to\infty} x^*_n = x^*$. As we know $T(x^*) \in B_0$, so there exists $u \in A_0$ such that

$$\|u - Tx^*\| = D(A, B)\,. \tag{6}$$

By (4) and (6), we get

$$\|x'_n - u\| \leq \|x^*_n - x^*\| + L\|x^* - x'_n\|\,,$$

which implies $\|x'_n - u\| \to 0$ as $n \to \infty$. Thus $u = x^*$ and then x^* is a best proximity point of T. □

It is clear that if a pair (A, B) has the weak P-property, then (B, A) is a semi-sharp proximinal pair. So we have the following result.

Corollary 3.5. *Let X be a Banach space, (A,B) be a pair of nonempty, closed subsets of X such that A is a p-starshaped set, B is a q-starshaped set, and $\|p-q\| = D(A,B)$. Assume A_0 is compact, (A,B) have the weak P-property. Suppose that $T\colon A \to B$ satisfies the following conditions:*

(1) *T is a proximal Berinde nonexpansive,*

(2) *$T(A_0) \subseteq B_0$.*

Then there exists x^* in A_0 such that

$$\|x^* - Tx^*\| = D(A,B)\,.$$

The following results are directly obtained by Theorem 3.4.

Corollary 3.6 ([4])**.** *Let X be a Banach space, (A,B) be a pair of nonempty, closed subsets of X such that A_0 is a p-starshaped set, B is a q-starshaped set, and $\|p-q\| = D(A,B)$. Assume A is compact, (B,A) is a semi-sharp proximinal pair. Suppose that $T\colon A \to B$ satisfies the following conditions:*

(1) *T is a proximal nonexpansive,*

(2) *$T(A_0) \subseteq B_0$.*

Then there exists x^* in A_0 such that

$$\|x^* - Tx^*\| = D(A,B)\,.$$

Corollary 3.7. *Let X be a Banach space, A be a nonempty compact subsets of X such that A is a p-starshaped set. Suppose $T\colon A \to A$ is Berinde nonexpansive. Then there exists x^* in A such that $x^* = Tx^*$.*

Example 3.8. Let $X = \mathbb{R}^2$ with $\|(x,y)\| = |x| + |y|$,
$A = \{(x,0) : x \in [0,2]\}$,
$B_1 = \{(x,y) : y - x = 1, x \in [-1,0)\}$,
$B_2 = \{(x,1) : x \in [0,\frac{3}{2}]\}$,
$B = B_1 \cup B_2$,
and let $T\colon A \to B$ be defined by

$$T(x,0) = \begin{cases} (x^2,1)\,, & \text{if} \quad x \in \left[0,\frac{1}{2}\right]\,, \\ \left(-\frac{x^2}{2}, -\frac{x^2}{2}+1\right)\,, & \text{if} \quad x \in \left(\frac{1}{2},\frac{3}{4}\right]\,, \\ \left(\frac{x+3}{4},1\right)\,, & \text{if} \quad x \in \left(\frac{3}{4},\frac{3}{2}\right)\,, \\ \left(-\frac{x}{4}+\frac{9}{8},1\right)\,, & \text{if} \quad x \in \left[\frac{3}{2},2\right]\,. \end{cases}$$

We see that the following properties are satisfied:

(1) A is a convex set, B is not convex set but it is a $(0,1)$-starshaped set,

(2) A_0 is a compact set,

(3) $A_0 \subseteq A$, $B_0 = B$ and $T(A_0) \subseteq B_0$,

(4) (B,A) is a semi-sharp proximinal pair, (A,B) is not a semi-sharp proximinal pair because $\|(0,0)-(-1,0)\| = D(A,B) = \|(0,0)-(0,1)\|$ but $(-1,0) \neq (0,1)$.

(5) $(0,0)$ and $(1,0)$ are best proximity points of T.

We can conclude that T is a proximal Berinde nonexpansive mapping with $L = \frac{14}{3}$. It is easy to see that T is not proximal nonexpansive mapping because when $x_1 = (\frac{3}{2}, 0), x_2 = (\frac{5}{4}, 0)$, we get $u_1 = (\frac{3}{4}, 0), u_2 = (\frac{17}{16}, 0)$ and $\|u_1 - u_2\| = \frac{5}{16} \geq \frac{1}{4} = \|x_1 - x_2\|$. We also note that if $x_1 = (\frac{3}{2}, 0), x_2 = (\frac{3}{4}, 0)$, then $u_1 = (\frac{3}{4}, 0), u_2 = (0, 0)$. So we have $\|x_2 - u_1\| = 0$ and $\|u_1 - u_2\| = \frac{3}{4} = \|x_1 - x_2\|$. Which implies that T is not proximal weak contraction.

Acknowledgement. The authors would like to thank the Thailand Research Fund under the project RTA 5780007 and Graduate School Chiang Mai University for the financial support. The first author was supported by Development and Promotion of Science and Technology talents project (DPST) Scholarship.

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Graduate PhD Degree Program in Mathematics,
Chiang Mai University,
Chiang Mai 50200, Thailand
E-mail: `nattawutnet@hotmail.com`

Department of Mathematics, Chiang Mai University,
Chiang Mai 50200, Thailand
E-mail: `suthep.s@cmu.ac.th`